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On the Solvability of Commutative Power-Associative Nilalgebras of Nilindex 4

Sobre la solubilidad de nilágebras conmutativas de potencias asociativas de nilíndice 4

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Abstract. Let A be a commutative power-associative nilalgebra. In this paper we prove that when A (of characteristic $\neq 2$) is of dimension ≤ 10 and the identity $x^4 = 0$ is valid in A, then $((y^2)x^2)x^2 = 0$ for all y, x in A and $((A^2)^2)^2 = 0$. That is, A is solvable.

Key words and phrases. Commutative, Power-associative, Nilalgebra, Solvable, Nilpotent.

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Resumen. Sea A una nilágebra conmutativa de potencias asociativas. En este trabajo demostramos que cuando A (de característica $\neq 2$) es de dimensión ≤ 10 y la identidad $x^4 = 0$ es válida en A, entonces $((y^2)x^2)x^2 = 0$ para todo y, x en A y $((A^2)^2)^2 = 0$. Es decir, A es soluble.

Palabras y frases clave. Conmutativa, potencias asociativas, nilálgebra, soluble, nilpotente.

1. Preliminaries

In this section A is a commutative algebra over a field K. If x is an element of A, we define $x^1 = x$ and $x^{k+1} = x^k x$ for all $k \geq 1$ A is called power-associative, if the subalgebra of A generated by any element $x \in A$ is associative. An element $x \in A$ is called nilpotent, if there is an integer $r \geq 1$ such that $x^r = 0$. If

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any element in A is nilpotent, then A is called a nilalgebra. Now A is called a nilalgebra of nilindex $n \geq 2$, if $y^n = 0$ for all $y \in A$ and there is $x \in A$ such that $x^{n-1} \neq 0$.

If B, D are subspaces of A, then BD is the subspace of A spanned by all products bd with $b \in B$, $d \in D$. Also we define $B^1 = B$ and $B^{k+1} = B^k B$ for all $k \geq 1$. If there exists an integer $n \geq 2$ such that $B^n = 0$ and $B^{n-1} \neq 0$, then B is nilpotent of index n.

A is called solvable in case $A^{(k)} = 0$ for some integer k, where $A^{(1)} = A$ and $A^{(n+1)} = \left(A^{(n)}\right)^2$ for all $n \geq 1$.

A is a Jordan algebra, if it satisfies the Jordan identity $x^2(yx) = (x^2y)x$ for all $x, y \in A$. It is known that any Jordan algebra (of characteristic $\neq 2$) is power-associative and also that any finite-dimensional Jordan nilalgebra is nilpotent (see [9]).

If the identity $x^3 = 0$ is valid in A, then A is a Jordan algebra (see [11, p. 114]). Therefore, if A is a finite dimensional, then A is nilpotente and hence solvable.

We will denote by $\langle a_1, \ldots, a_j \rangle$ the subspace of A generated over K by the elements $a_1, \ldots, a_j \in A$. In the following a greek letter indicates an element of the field K.

The problem of nilpotence in a commutative power-associative nilalgebra is known as Albert's problem [1]: Is every commutative finite dimensional power-associative nilalgebra nilpotent?

In [10], D. Suttles constructs (as a counterexample to a conjecture due to A. A. Albert) a commutative power-associative nilalgebra of nilindex 4 and dimension 5, which is solvable and is not nilpotent. In [4] (Theorem 3.3), we prove that this algebra is the unique commutative power-associative nilalgebra of nilindex 4 and dimension 5, which is not Jordan algebra.

At present there exists the following conjecture: Any commutative finite dimensional power-associative nilalgebra is solvable. The solvability of these algebras for dimension 4, 5 and 6, are proved in [8], [4] and [2] respectively.

Let A be a commutative power-associative nilalgebra. In [6], is proved that when A is of nilindex n and dimension $\leq n+2$, then A is solvable. In [5], we prove that if A is of nilindex 4 and dimension ≤ 8, then A is solvable. In [7], is proved that if A is of nilindex 5 and dimension 8, then A is solvable. Therefore, if A is of dimension ≤ 8, then A is solvable.

We will use the following results which we demonstrated in [5]:

Theorem 1. *Let A be a commutative power-associative nilalgebra (of characteristic $\neq 2, 3$) such that $x^4 = 0$ for all x in A.*

a) If exist elements $y, x \in A$ such that $(yx^2)x^2 \neq 0$, then $y, yx, (yx)x, \big((yx)x\big)x, (yx^2)x^2, x, x^2, x^3, yx^2$ are linearly independent.

b) If A is of dimension ≤ 8, then $\big((A^2)^2\big)^2 = 0$. That is, A is solvable.

2. Solvability

In this section, A is a commutative power-associative algebra over a field K with characteristic $\neq 2, 3$ such that the identity $x^4 = 0$ is valid in A. Linearizing the identities $(x^2)^2 = 0$ and $x^4 = 0$, we obtain that for all $y, x, z, v \in A$:

$$(yx)x^2 = 0, \qquad 2(xy)^2 + x^2y^2 = 0 \tag{1}$$

$$(yz)x^2 + 2(yx)(zx) = 0, \qquad (yx^2)(vx^2) = 0 \tag{2}$$

$$(xy)(zv) + (xz)(yv) + (xv)(yz) = 0 \tag{3}$$

$$2\big((yx)x\big)x + (yx^2)x + yx^3 = 0 \tag{4}$$

$$2\big((yx)x\big)z + 2\big((zx)x\big)y + 2\big((yz)x\big)x + (yx^2)z + (zx^2)y + 2\big((yx)z\big)x + 2\big((zx)y\big)x = 0 \tag{5}$$

It is known that the following identities are valid in A:

$$4\Big(\big((yx)x\big)x\Big)x = (yx^2)x^2 = -2(yx)x^3 \tag{6}$$

$$\Big(\big(((yx)x)x\big)x\Big)x = 0 \tag{7}$$

$$\Big(\big(\cdots(yx^{m_t})\cdots\big)x^{m_2}\Big)x^{m_1} = 0 \tag{8}$$

where $m_1, \ldots, m_t$ are positive integers such that $m_1 + \cdots + m_t \geq 5$. This last identity is proved in [3].

Lemma 1. *If there exist elements $y, x \in A$ such that $(y^3x^2)x^2 \neq 0$, then A is of dimension ≥ 11.*

Proof. We consider the subspace U of A generated by $y^3, y^3x, (y^3x)x, \big((y^3x)x\big)x$, $(y^3x^2)x^2, x, x^2, x^3, y^3x^2$. By Theorem 1(a), U is a subspace of dimension 9.

Using (8) and (6) we get that $(Ux)x$ is generated by $(y^3x)x$, $\big((y^3x)x\big)x$, $(y^3x^2)x^2, x^3, \big((y^3x^2)x\big)x$.

We observe that using (2), (1) and (8) we get

$$\big((yx)x\big)\big((y^2x)x\big) = -\frac{1}{2}\big((yx)(y^2x)\big)x^2 = \frac{1}{4}(y^3x^2)x^2,$$

$$\big(((yx)x)x\big)\big((y^2x)x\big) = \frac{1}{4}\big(((yx)y^2)x^2\big)x^2 = 0,$$

$$\big((y^3x)x\big)\big((y^2x)x\big) = \frac{1}{4}(y^5x^2)x^2 = 0,$$

$$\big(((y^3x)x)x\big)\big((y^2x)x\big) = \frac{1}{4}\big(((y^3x)y^2)x^2\big)x^2 = 0,$$

$$\big((yx^2)x^2\big)x^3 = 0 \qquad \text{and}$$

$$\big(((y^3x^2)x)x\big)\big((y^2x)x\big) = \frac{1}{4}\big(((y^3x^2)y^2)x^2\big)x^2 = 0.$$

Using the previous relations, we obtain that $\big((Ux)x\big)\big((y^2x)x\big)$ is generated by $\big((y^3x^2)x^2\big)\big((y^2x)x\big)$.

Let $\alpha y+\beta yx\in U$. We will prove that $\alpha=\beta=0$.

For this we see that $\Big(\big((\alpha y+\beta yx)x\big)x\Big)\big((y^2x)x\big)\in\big((Ux)x\big)\big((y^2x)x\big)$ and hence $\alpha\big((yx)x\big)\big((y^2x)x\big)+\beta\Big(\big((yx)x\big)x\Big)\big((y^2x)x\big)=\gamma\big((y^3x^2)x^2\big)\big((y^2x)x\big)$. Therefore $\frac{1}{4}\alpha(y^3x^2)x^2=\gamma\big((y^3x^2)x^2\big)\big((y^2x)x\big)$. If we suppose that $\alpha\neq 0$, then $\gamma\neq 0$ and so we obtain that $uv=\frac{1}{4}\alpha\gamma^{-1}u$ where $u=(y^3x^2)x^2$ and $v=(y^2x)x$. Using (7), we get that $(\frac{1}{4}\alpha\gamma^{-1})^5u=0$, which is a contradiction. Therefore $\alpha=0$.

We have now that $\beta yx\in U$ and hence $\beta yx=\alpha_1y^3+\alpha_2y^3x+\alpha_3(y^3x)x+\alpha_4\big((y^3x)x\big)x+\alpha_5(y^3x^2)x^2+\alpha_6x+\alpha_7x^2+\alpha_8x^3+\alpha_9y^3x^2$. Multiplying by x^2 and using (1) we obtain $\alpha_1y^3x^2+\alpha_6x^3+\alpha_9(y^3x^2)x^2=0$, which implies that $\alpha_1=\alpha_6=\alpha_9=0$. Therefore $\beta yx=\alpha_2y^3x+\alpha_3(y^3x)x+\alpha_4\big((y^3x)x\big)x+\alpha_5(y^3x^2)x^2+\alpha_7x^2+\alpha_8x^3$ and so $\beta(yx)x=\alpha_2(y^3x)x+\alpha_3\big((y^3x)x\big)x+\frac{1}{4}\alpha_4(y^3x^2)x^2+\alpha_7x^3$. Multiplying by $(y^2x)x$, we get that $\frac{1}{4}\beta(y^3x^2)x^2=\frac{1}{4}\alpha_4\big((y^3x^2)x^2\big)\big((y^2x)x\big)$. Using the same argument earlier we conclude that $\beta=0$ and therefore $\dim_K(A)\geq 11$. ☑

Theorem 2. *If A is of dimension ≤ 10, then $(y^2x^2)x^2=0$ is an identity in A.*

Proof. Suppose that there exist $y,x\in A$ such that $(y^2x^2)x^2\neq 0$. By Lemma 1 we can suppose that $(y^3x^2)x^2=0$ and Theorem 1(a) implies that $y^2, y^2x, (y^2x)x, \big((y^2x)x\big)x, (y^2x^2)x^2, x, x^2, x^3, y^2x^2$ are linearly independent.

Let $\alpha yx+\beta(yx)x=\alpha_1y^2+\alpha_2y^2x+\alpha_3(y^2x)x+\alpha_4\big((y^2x)x\big)x+\alpha_5(y^2x^2)x^2+\alpha_6x+\alpha_7x^2+\alpha_8x^3+\alpha_9y^2x^2$. Multiplying by x^2 we obtain $\alpha_1y^2x^2+\alpha_6x^3+\alpha_9(y^2x^2)x^2=0$, which implies $\alpha_1=\alpha_6=\alpha_9=0$.

Now we have $\alpha yx+\beta(yx)x=\alpha_2y^2x+\alpha_3(y^2x)x+\alpha_4\big((y^2x)x\big)x+\alpha_5(y^2x^2)x^2+\alpha_7x^2+\alpha_8x^3$. Multiplying by x we get $\alpha(yx)x+\beta\big((yx)x\big)x=\alpha_2(y^2x)x+\alpha_3\big((y^2x)x\big)x+\alpha_4\Big(\big((y^2x)x\big)x\Big)x+\alpha_7x^3$ and therefore $\big(\alpha(yx)x-\alpha_2(y^2x)x\big)^2=\Big(-\beta\big((yx)x\big)x+\alpha_3\big((y^2x)x\big)x+\alpha_4\Big(\big((y^2x)x\big)x\Big)x+\alpha_7x^3\Big)^2$. Using the identities (1), (2) and (8), we obtain $\frac{1}{4}\alpha^2(y^2x^2)x^2-\frac{1}{2}\alpha\alpha_2(y^3x^2)x^2=0$. Since $(y^3x^2)x^2=0$, then $\alpha=0$.

Now $\beta(yx)x=\alpha_2y^2x+\alpha_3(y^2x)x+\alpha_4\big((y^2x)x\big)x+\alpha_5(y^2x^2)x^2+\alpha_7x^2+\alpha_8x^3$. Multiplying three times by x we obtain that $\alpha_2=0$. Now $\big(\beta(yx)x-\alpha_3(y^2x)x\big)^2=\Big(\alpha_4\big((y^2x)x\big)x+\alpha_5(y^2x^2)x^2+\alpha_7x^2+\alpha_8x^3\Big)^2=\Big(\alpha_4\big((y^2x)x\big)x+4\alpha_5\Big(\big((y^2x)x\big)x\Big)x+\alpha_7x^2+\alpha_8x^3)^2$, implies that $\frac{1}{4}\beta^2(y^2x^2)x^2=0$ and so $\beta=0$. Therefore dimension of $A\geq 11$, which is a contradiction. ☑

Lemma 2. *If $(y^2x^2)x^2=0$ for all $x,y\in A$, then the following identities are valid in A:*

$$x^2\big((xz)(yu)\big)+(xz)\big((yu)x^2\big)=0,\qquad (yx)^3=0 \tag{9}$$

$$\big(((xy)^2(uv)^2)x^2\big)(uv)^2 = 0, \qquad \big(((xy)^2(uv)^2)(uv)\big)(xy)^2 = 0 \tag{10}$$
$$\big(((xy)(uv))(xy)\big)(uv)^2 = 0, \qquad \big(((xy)(uv))(uv)\big)(xy)^2 = 0 \tag{11}$$
$$\big(((xy)(uv)^2)(xy)\big)(uv)^2 = 0, \qquad \big(((xy)(uv)^2)(uv)\big)(xy)^2 = 0.$$

Proof. Linearizing the identity $(y^2x^2)x^2 = 0$ we obtain that $x^2\big((xz)(yu)\big) + (xz)\big((yu)x^2\big) = 0$. Substituting z by y, u by y and using (1), we obtain $(yx)^3 = 0$. So we obtain (9).

Replacing x by $(uv)^2$, y by $(xy)^2$, z by x^2 in (5) and using (1) and (2) we get the next expressions $\big(((xy)^2(uv)^2)x^2\big)(uv)^2 = -\big(x^2(uv)^2(xy)^2\big)(uv)^2 = \frac{1}{2}\big(x^2(uv)^2\big)(x^2y^2)(uv)^2 = 0$.

Replacing x by $(xy)^2$, y by $(uv)^2$, z by uv in (5) and using (1) we get that $\big(((xy)^2(uv)^2)(uv)\big)(xy)^2 = -\big(((xy)^2(uv))(uv)^2\big)(xy)^2 = 0$. Therefore, we get (10).

Replacing x by uv, z by xy, u by x in (9) and using (2) and (9), we get that $\big(((xy)(uv))(xy)\big)(uv)^2 = -\big((xy)(uv)\big)\big((xy)(uv)^2\big) = \frac{1}{2}(xy)^2(uv)^3 = 0$. Similarly, we prove that $\big(((xy)(uv))(uv)\big)(xy)^2 = 0$.

Replacing x by $(uv)^2$, y by xy and z by xy in (5), we get the next expression $\big(((xy)(uv)^2)(xy)\big)(uv)^2 = 0$.

Replacing x by xy, z by $(uv)^2$, y by v in (9) and using (3), (1) and (9), we obtain the next expression $\big(((xy)(uv)^2)(uv)\big)(xy)^2 = -\big((xy)(uv)^2\big)\big((uv)(xy)^2\big) = \big((xy)(uv)\big)\big((xy)^2(uv)^2\big) = -2\big((xy)(uv)\big)\big((xy)(uv)\big)^2 = -2\big((xy)(uv)\big)^3 = 0$. So, we prove (11). ☑

Lemma 3. *If the identity $(y^2x^2)x^2 = 0$ is valid in A and $\big((A^2)^2\big)^2 \neq 0$, then there exist elements $y, x, u, v \in A$ such that x^2, y^2, xy, $(xy)^2$, u^2, v^2, uv, $(uv)^2$, $(xy)^2(uv)^2$ are linearly independent. Therefore, A is of dimension ≥ 9.*

Proof. Since $\big((A^2)^2\big)^2 \neq 0$, then there exist elements $y, x, u, v \in A$ such that $(x^2y^2)(u^2v^2) \neq 0$. From (1), $(x^2y^2)(u^2v^2) = 4(xy)^2(uv)^2 = -8\big((xy)(uv)\big)^2 \neq 0$.

We will prove first that $x^2, y^2, xy, (xy)^2, u^2, v^2, uv, (uv)^2$ are linearly independent. Let $\alpha x^2 + \beta y^2 + \gamma xy + \delta(xy)^2 + \alpha_0 u^2 + \beta_0 v^2 + \gamma_0 uv + \delta_0 (uv)^2 = 0$. Multiplying by y^2, afterwards by $u^2v^2 = -2(uv)^2$ and using (1), (2) we obtain $\alpha(x^2y^2)(u^2v^2) = 0$, which implies $\alpha = 0$. Similarly, we get $\beta = \alpha_0 = \beta_0 = 0$. Now we have $\gamma xy + \delta(xy)^2 = -(\gamma_0 uv + \delta_0(uv)^2)$. Using (9), $\big(\gamma xy + \delta(xy)^2\big)^2 = \big(-\big(\gamma_0 uv + \delta_0(uv)^2\big)\big)^2$ implies $\gamma^2(xy)^2 = \gamma_0^2(uv)^2$. Since $(xy)^2, (uv)^2$ are linearly independent, then $\gamma = \gamma_0 = 0$. Now $\delta(xy)^2 = -\delta_0(uv)^2$ implies $\delta = \delta_0 = 0$. We conclude that $x^2, y^2, xy, (xy)^2, u^2, v^2, uv, (uv)^2$ are linearly independent.

Now we will prove that $x^2, y^2, xy, (xy)^2, u^2, v^2, uv, (uv)^2, (xy)^2(uv)^2$ are linearly independent.

Suppose that $(xy)^2(uv)^2 = \alpha x^2 + \beta y^2 + \gamma xy + \delta(xy)^2 + \alpha_0 u^2 + \beta_0 v^2 + \gamma_0 uv + \delta_0(uv)^2$. Multiplying by x^2, afterwards by $(uv)^2 = -\frac{1}{2}u^2v^2$ and using (10), we obtain $\beta(x^2y^2)(uv)^2 = -2\beta(xy)^2(uv)^2 = 0$ and hence $\beta = 0$. Using the same argument and the Identities (10), it is possible to demonstrate that $\alpha = \alpha_0 = \beta_0 = 0$.

Now we have that $(xy)^2(uv)^2 = \gamma xy + \delta(xy)^2 + \gamma_0 uv + \delta_0(uv)^2$. Multiplying by xy, afterwards by $(uv)^2$ and using (10), we get $\gamma(xy)^2(uv)^2 = 0$. Therefore $\gamma = 0$. Similarly we prove that $\gamma_0 = 0$.

Now $(xy)^2(uv)^2 = \delta(xy)^2 + \delta_0(uv)^2$. Multiplying by $(uv)^2$ (also by $(xy)^2$), we obtain that $\delta = \delta_0 = 0$, which is a contradiction. This completes the proof. ☑

Lemma 4. *If the identity $(y^2x^2)x^2 = 0$ is valid in A and $\big((A^2)^2\big)^2 \neq 0$, then there exist elements y, x, u, v in A such that $x^2, y^2, xy, (xy)^2, u^2, v^2, uv, (uv)^2, (xy)^2(uv)^2, (xy)(uv)$ are linearly independent or $x^2, y^2, xy, (xy)^2, u^2, v^2, uv, (uv)^2, (xy)^2(uv)^2, (xy)(uv)^2$ are linearly independent. Therefore A is of dimension ≥ 10.*

Proof. By Lemma 3, we know that there exist x, y, u, v in A such that the subspace U of A generated by $x^2, y^2, xy, (xy)^2, u^2, v^2, uv, (uv)^2, (xy)^2(uv)^2$ has dimension 9.

We will prove that $(xy)(uv) \notin U$ or $(xy)(uv)^2 \notin U$. Suppose that $(xy)(uv)$ and $(xy)(uv)^2$ are elements in U. Then

(a) $(xy)(uv) = \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 xy + \beta_1 u^2 + \beta_2 v^2 + \beta_3 uv + z$ where $z = \alpha_4(xy)^2 + \beta_4(uv)^2 + \lambda_1(xy)^2(uv)^2$, and

(b) $(xy)(uv)^2 = \gamma_1 x^2 + \gamma_2 y^2 + \gamma_3 xy + \delta_1 u^2 + \delta_2 v^2 + \delta_3 uv + \gamma_4(xy)^2 + \delta_4(uv)^2 + \lambda_2(xy)^2(uv)^2$.

Multiplying (a) by xy and afterwards by $(uv)^2 = -\frac{1}{2}u^2v^2$ and using (1), (2), (9) and (10), we obtain that $\alpha_3(xy)^2(uv)^2 = 0$ and therefore $\alpha_3 = 0$. Similarly, multiplying (a) by uv and afterwards by $(xy)^2 = -\frac{1}{2}x^2y^2$, we obtain $\beta_3 = 0$.

Now in (a) we have $(xy)(uv) - z = \alpha_1 x^2 + \alpha_2 y^2 + \beta_1 u^2 + \beta_2 v^2$ and hence $\big((xy)(uv) - z\big)^2 = (\alpha_1 x^2 + \alpha_2 y^2 + \beta_1 u^2 + \beta_2 v^2)^2$. Thus $2\alpha_4\beta_4(xy)^2(uv)^2 + \big((xy)(uv)\big)^2 = 2\alpha_1\alpha_2 x^2y^2 + 2\alpha_1\beta_1 x^2u^2 + 2\alpha_1\beta_2 x^2v^2 + 2\alpha_2\beta_1 y^2u^2 + 2\alpha_2\beta_2 y^2v^2 + 2\beta_1\beta_2 u^2v^2$, which implies (multiplying by $(uv)^2 = -\frac{1}{2}u^2v^2$, multiplying by $(xy)^2$) that $\alpha_1\alpha_2 = 0$ and $\beta_1\beta_2 = 0$. Therefore in (a) we may have the possibilities following:

(i) $(xy)(uv) = \alpha_1 x^2 + \beta_1 u^2 + \alpha_4(xy)^2 + \beta_4(uv)^2 + \lambda_1(xy)^2(uv)^2$,

(ii) $(xy)(uv) = \alpha_1 x^2 + \beta_2 v^2 + \alpha_4 (xy)^2 + \beta_4 (uv)^2 + \lambda_1 (xy)^2 (uv)^2$,

(iii) $(xy)(uv) = \alpha_2 y^2 + \beta_1 u^2 + \alpha_4 (xy)^2 + \beta_4 (uv)^2 + \lambda_1 (xy)^2 (uv)^2$,

(iv) $(xy)(uv) = \alpha_2 y^2 + \beta_2 v^2 + \alpha_4 (xy)^2 + \beta_4 (uv)^2 + \lambda_1 (xy)^2 (uv)^2$.

We will prove that actually $\alpha_1\beta_1 \neq 0$ in (i). If we suppose that $\alpha_1 = 0$, then $(xy)(uv) - \alpha_4(xy)^2 - \lambda_1(xy)^2(uv)^2 = \beta_1 u^2 + \beta_4(uv)^2$. Now $\big((xy)(uv) - \alpha_4(xy)^2 - \lambda_1(xy)^2(uv)^2\big)^2 = \big(\beta_1 u^2 + \beta_4(uv)^2\big)^2$ implies that $\big((xy)(uv)\big)^2 = 0$, which is a contradiction. Similarly, we obtain that $\alpha_1\beta_2 \neq 0$ in (ii), $\alpha_2\beta_1 \neq 0$ in (iii) and $\alpha_2\beta_2 \neq 0$ in (iv).

Multiplying (b) by xy and afterwards by $(uv)^2 = -\frac{1}{2}u^2v^2$ and using (11) and (10) we obtain that $\gamma_3(xy)^2(uv)^2 = 0$ and therefore $\gamma_3 = 0$. Similarly, multiplying (b) by uv and afterwards by $(xy)^2 = -\frac{1}{2}x^2y^2$, we obtain that $\delta_3 = 0$.

Multiplying (b) by v^2 and afterwards by $(xy)^2 = -\frac{1}{2}x^2y^2$, we obtain that $\big(\big((xy)(uv)^2\big)v^2\big)(xy)^2 = \delta_1(u^2v^2)(xy)^2 = -2\delta_1(uv)^2(xy)^2$. Replacing in (9), x by xy, z by $(uv)^2$, y by v, u by v and using (3), we obtain that $\big(\big((xy)(uv)^2\big)v^2\big)(xy)^2 = -\big((xy)(uv)^2\big)\big(v^2(xy)^2\big) = \big((xy)v^2\big)\big((xy)^2(uv)^2\big)$ and hence $\big((xy)v^2\big)\big((xy)^2(uv)^2\big) = -2\delta_1(uv)^2(xy)^2$. Replacing x by $(xy)v^2$ and y by $(xy)^2(uv)^2$ in (7), we obtain that $-32\delta_1^5 = 0$ and so $\delta_1 = 0$. Similarly, multiplying (b) by u^2 and afterwards by $(xy)^2$, we obtain $\delta_2 = 0$.

In (b), we have $(xy)(uv)^2 = \gamma_1 x^2 + \gamma_2 y^2 + \gamma_4(xy)^2 + \delta_4(uv)^2 + \lambda_2(xy)^2(uv)^2$. Multiplying by $(xy)^2$, we get $\delta_4(uv)^2(xy)^2 = 0$ and so $\delta_4 = 0$. Therefore

(c) $(xy)(uv)^2 = \gamma_1 x^2 + \gamma_2 y^2 + \gamma_4(xy)^2 + \lambda_2(xy)^2(uv)^2$ with $\gamma_1\gamma_2 = 0$.

In fact, $\big((xy)(uv)^2 - \gamma_4(xy)^2 - \lambda_2(xy)^2(uv)^2\big)^2 = \big(\gamma_1 x^2 + \gamma_2 y^2\big)^2$ implies that $\gamma_1\gamma_2 x^2y^2 = 0$.

Suppose the case (i), that is,

$$(xy)(uv) = \alpha_1 x^2 + \beta_1 u^2 + \alpha_4(xy)^2 + \beta_4(uv)^2 + \lambda_1(xy)^2(uv)^2$$

with $\alpha_1\beta_1 \neq 0$. Multiplying (i) by y^2 and afterwards by $(uv)^2$, we obtain that $\big(\big((xy)(uv)\big)y^2\big)(uv)^2 = \alpha_1(x^2y^2)(uv)^2 = -2\alpha_1(xy)^2(uv)^2$. But, $\big(\big((xy)(uv)\big)y^2\big)(uv)^2 = -\big((xy)(uv)\big)\big(y^2(uv)^2\big) = \big((xy)(uv)^2\big)\big((uv)y^2\big)$ and therefore $\big((xy)(uv)^2\big)\big((uv)y^2\big) = -2\alpha_1(xy)^2(uv)^2$.

Multiplying (c) by $(uv)y^2$, we get that $\big((xy)(uv)^2\big)\big((uv)y^2\big) = \gamma_1 x^2\big((uv)y^2\big) + \lambda_2\big((xy)^2(uv)^2\big)\big((uv)y^2\big)$ and therefore $\gamma_1 x^2\big((uv)y^2\big) + \lambda_2\big((xy)^2(uv)^2\big)\big((uv)y^2\big) = -2\alpha_1(xy)^2(uv)^2$. If we suppose that $\gamma_1 = 0$, then as $\alpha_1 \neq 0$, we get that $\lambda_2 \neq 0$ and so $\big((xy)^2(uv)^2\big)\big((uv)y^2\big) = -2\alpha_1\lambda_2^{-1}(xy)^2(uv)^2$. Replacing x by $(uv)y^2$ and y by $(uv)^2(xy)^2$ in (7), we obtain that $(-2\alpha_1\lambda_2^{-1})^5 = 0$, which is a contradiction. Therefore $\gamma_1 \neq 0$ and so $\gamma_2 = 0$. Now we have in (c),

$$(xy)(uv)^2 = \gamma_1 x^2 + \gamma_4(xy)^2 + \lambda_2(xy)^2(uv)^2.$$

So we obtain $\gamma_1(xy)(uv) - \alpha_1(xy)(uv)^2 = \gamma_1\beta_1 u^2 + (\gamma_1\alpha_4 - \alpha_1\gamma_4)(xy)^2 + \gamma_1\beta_4(uv)^2 + (\gamma_1\lambda_1 - \alpha_1\lambda_2)(xy)^2(uv)^2$. Since $\big(\gamma_1(xy)(uv) - \alpha_1(xy)(uv)^2 - (\gamma_1\alpha_4 - \alpha_1\gamma_4)(xy)^2 - (\gamma_1\lambda_1 - \alpha_1\lambda_2)(xy)^2(uv)^2\big)^2 = \big(\gamma_1\beta_1 u^2 + \gamma_1\beta_4(uv)^2\big)^2$, then $\gamma_1^2\big((xy)(uv)\big)^2 = -\frac{1}{2}\gamma_1^2(xy)^2(uv)^2 = 0$, a contradiction.

Considering the same argument, it is possible to obtain contradictions in the cases (ii), (iii) and (iv). ☑

Corollary 1. *If A is of dimension ≤ 9, then $\big((A^2)^2\big)^2 = 0$. That is, A is solvable.*

Theorem 3. *If A is of dimension ≤ 10, then $\big((A^2)^2\big)^2 = 0$. That is, A is solvable.*

Proof. Theorem 2, implies that $(y^2x^2)x^2 = 0$ is an identity in A. Suppose that $\big((A^2)^2\big)^2 \neq 0$. By Lemma (4), there exist elements y, x, u, v in A and $w \in \{(xy)(uv), (xy)(uv)^2\}$ such that $\big\{x^2, y^2, (xy), (xy)^2, u^2, v^2, uv, (uv)^2, (xy)^2(uv)^2, w\big\}$ is a basis of A. This implies that $A^2 = A$.

Since $A^2 = A$, then using the identities (1), (2), (9) and the Theorem (2), we obtain that A is generated also by the elements $(xy)^2, (xu)^2, (xv)^2, x^2(uv)$, $x^2(uv)^2$, $x^2\big((xy)^2(uv)^2\big)$, x^2w, $(yu)^2$, $(yv)^2$, $y^2(uv)$, $y^2(uv)^2$, $y^2\big((xy)^2(uv)^2\big)$, y^2w, $(xy)u^2$, $(xy)v^2$, $(xy)(uv)$, $(xy)(uv)^2$, $(xy)\big((xy)^2(uv)^2\big)$, $(xy)w$, $(xy)^2u^2$, $(xy)^2v^2$, $(xy)^2(uv)$, $(xy)^2(uv)^2$, $(uv)^2$, $u^2\big((xy)^2(uv)^2\big)$, u^2w, $v^2\big((xy)^2(uv)^2\big)$, $v^2w, (uv)\big((xy)^2(uv)^2\big), (uv)w$.

Now using the identities of Lemma 2, we obtain that $(xy)^2A$ is generated by $(xy)^2(uv)^2$, $(xy)^2(u^2w)$, $(xy)^2(v^2w)$ and $(uv)^2A$ is generated by $(xy)^2(uv)^2$, $(uv)^2(x^2w)$, $(uv)^2(y^2w)$.

We will prove that $\big((xy)^2A\big)\big((uv)^2A\big) = 0$.

If $w = (xy)(uv)$, then $w^2 = -\frac{1}{2}(xy)^2(uv)^2$ and $\big((xy)^2(u^2w)\big)\big((uv)^2(x^2w)\big) = -\big((xy)^2(uv)^2\big)\big((u^2w)(x^2w)\big) = -w^2\big((u^2x^2)w^2\big) = 0$. In a similar way we prove that the other products are zero.

Now, if $w = (xy)(uv)^2$, then $w^2 = 0$ and hence $\big((xy)^2(u^2w)\big)\big((uv)^2(x^2w)\big) = -\big((xy)^2(uv)^2\big)\big((u^2w)(x^2w)\big) = \frac{1}{2}\big((xy)^2(uv)^2\big)\big((u^2x^2)w^2\big) = 0$. In a similar way we prove that the other products are zero.

We will prove that $\big((xy)^2(uv)^2\big)A = 0$. Observe that it is sufficient to prove that $\big((xy)^2(uv)^2\big)\big(z_1z_2\big) = 0$ for all $z_1, z_2 \in A$. Now $\big((xy)^2(uv)^2\big)\big(z_1z_2\big) = -\big((xy)^2z_1\big)\big((uv)^2z_2\big) - \big((xy)^2z_2\big)\big((uv)^2z_1\big) = 0$. Therefore $J = \langle (xy)^2(uv)^2\rangle$ is an ideal of A. Now $\overline{A} = A/J$ is a commutative power-associative in which $x^4 = 0$ for all x in $\overline{A}$. Corollary 1 and $\dim\big(\overline{A}\big) = 9$ imply that $\overline{A}$ is solvable. Thus $\dim\left(\overline{A}^2\right) < 9$. Finally we conclude that $\overline{A}^2 = A^2/J = A/J = \overline{A}$, which is a contradiction. Therefore $\big((A^2)^2\big)^2 = 0$, as desired. ☑

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