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On the Two-Parabolic Subgroups of $\mathbf{SL(2,\mathbb{C})}$

Sobre los subgrupos dos-parabólicos de $\mathbf{SL(2,\mathbb{C})}$

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Abstract. We consider homomorphisms H_t from the free group F of rank 2 onto the subgroup of $\mathrm{SL}(2,\mathbb{C})$ that is generated by two parabolic matrices. Up to conjugation, H_t depends only on one complex parameter t. We study the possible relators, that is, the words $w \in F$ with $w \neq 1$ such that $H_t(w) = I$ for some $t \in \mathbb{C}$.

We find several families of relators. Of particular interest here are relators connected with 2-bridge knots, which we consider in a purely algebraic setting. We describe an algorithm to determine whether a given word is a possible relator.

Key words and phrases. Representation, Parabolic, Wirtinger presentation, Two-generated groups, Homomorphism, Longitude.

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Resumen. Consideramos homomorfismos H_t del grupo libre F de rango 2 sobre el subgrupo de $\mathrm{SL}(2,\mathbb{C})$ que es generado por dos matrices parabólicas. Salvo conjugación, H_t depende sólo de un parámetro complejo t. Estudiamos los posibles relatores, esto es, las palabras $w \in F$ con $w \neq 1$ tal que $H_t(w) = I$ para algún $t \in \mathbb{C}$.

Encontramos varias familias de relatores. De particular interés aquí son los relatores asociados con nudos de 2 puentes, los cuales consideramos de forma puramente algebraica. Describimos un algoritmo para determinar cuándo una palabra dada es un posible relator.

Palabras y frases clave. Representación, parabólico, presentación de Wirtinger, grupos dos-generados, homomorfismos, longitud.

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1. Introduction

The subgroup of $\mathrm{SL}(2,\mathbb{C})$ generated by $A_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has been studied by many mathematicians, for instance by R. Riley [16, 17], J. Gilman [4] and P. Waterman [6]. It is of particular interest in knot theory [12, Chapter 4][2].

In terms of the corresponding Moebius transformations α and β it is, up to conjugation, the only subgroup of $\mathrm{PSL}(2,\mathbb{C})$ generated by two parabolic transformations with distinct fixed points. Indeed, we may assume that $\alpha(0) = 0$, $\beta(\infty) = \infty$, moreover that $\beta(z) = z + 1$. Writing $\alpha(z) = (az + b)/(cz + d)$ with $ad - bc = 1$, we may also assume that $\operatorname{tr}\alpha = a + d = 2$. Since $b = 0$ it follows that $a = d = 1$ and $c = t \in \mathbb{C} \setminus \{0\}$ remains as a free parameter. It is convenient to allow $t = 0$.

Let F be the free group $\langle x, y\rangle$. We consider the homomorphisms $H_t : F \to \mathrm{SL}(2,\mathbb{C})$ with $H_t(x) = A_t$ and $H_t(y) = B$. For clarity we distinguish between the abstract group F and its image in the matrix group $\mathrm{SL}(2,\mathbb{C})$. Our main interest is to study the set of possible relators, that is the sets

$$R^{\pm} = \{r \in F, r \neq 1 \mid \text{there is } s \in \mathbb{C} \text{ with } H_s(r) = \pm I\}.$$

If $r \in R^+$ then $H_s(F)$ has a presentation $\langle x, y;\ r_1, r_2, \ldots\rangle$ with $r_1 = r$ and perhaps other relators $r_2, \ldots$

We shall exhibit various families of relators, some old, some new. An important family of relators comes from the presentations $\langle x, y; xw = wy\rangle$ of 2-bridge knots. Riley introduced the automorphism $w \in F \to \widetilde{w} \in F$ induced by $x \to x^{-1}$ and $y \to y^{-1}$. Our group has the special property that $r \in R^{\pm}$ implies $\widetilde{r} \in R^{\pm}$. At the end we give 8 examples to illustrate the results and show their scope.

We try to give a systematic account of the theory including some folklore results. We will use several results from Combinatorial Group Theory [13, 10] and stress the connections to Knot Theory [2]. We have not been able to elucidate the role of palindromes, that is, words of F that read the same way forwards and backwards [8, 5].

We will not discuss the set orthogonal to R^+, namely

$$\{s \in \mathbb{C} \mid \text{there exists } r \neq 1 \text{ such that } H_s(r) = I\},$$

the set of s where H_s is not injective. This set and its closure has been studied in [16, 4, 6] and, in a more general context, in [18, 14].

2. Groups and Homomorphisms

Let $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$ be the groups with elements of the forms

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \gamma(z) = \frac{az+b}{cz+d}, \qquad (a,b,c,d \in \mathbb{C}, ad-bc=1)$$

respectively. For $t \in \mathbb{C}$ we write

$$A_t = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

While the group $\mathrm{PSL}(2,\mathbb{C})$ of Moebius transformations is perhaps more important in analysis and geometry, the group $\mathrm{SL}(2,\mathbb{C})$ is more convenient for computations.

Let F be the abstract free group $\langle x, y \rangle$. There are [11, Theorem 8.04] unique homomorphisms

$$H_t : F \to \mathrm{SL}(2,\mathbb{C}) \qquad \text{with} \qquad H_t(x) = A_t, H_t(y) = B. \tag{2}$$

$$h_t : F \to \mathrm{PSL}(2,\mathbb{C}) \qquad \text{with} \qquad h_t(x) = \alpha_t, h_t(y) = \beta, \tag{3}$$

where $\alpha_t(z) = z/(tz+1)$ and $\beta(z) = z+1$ are both parabolic.

Every word $w \neq 1$ in F can be uniquely written as

$$w = x^{e_0} y^{e_1} \cdots x^{e_{m-1}} y^{e_m}, \qquad e_\mu \in \mathbb{Z} \setminus \{0\} \quad (\mu = 1, \dots, m-1) \tag{4}$$

with $e_0, e_m \in \mathbb{Z}$ and $m \in \mathbb{N}$. The exponent sums

$$\sigma_x(w) = e_0 + e_2 + \cdots + e_{m-1}, \qquad \sigma_y(w) = e_1 + e_3 + \cdots + e_m \tag{5}$$

are invariant under conjugations in F. As in [17, p.206] we write $\widetilde{1} = 1$ and

$$\widetilde{w} = x^{-e_0} y^{-e_1} \cdots x^{-e_{m-1}} y^{-e_m}. \tag{6}$$

This defines an automorphism of F. In formulas we write $(\cdot)\widetilde{}$.

Now suppose that $w \in F$ is not conjugate in F to x^k or y^k with $k \in \mathbb{Z}$. Then the process of cyclic reduction shows that w is conjugate to a word u of the form

$$u = y^{k_1} x^{j_1} \cdots y^{k_n} x^{j_n}, \qquad k_\nu, j_\nu \in \mathbb{Z} \setminus \{0\} \quad (\nu = 1, \dots, n) \quad \text{with} \quad n \in \mathbb{N}. \tag{7}$$

Our standard form (7) allows us to arrange all words of F up to conjugation in sequences. Let (j_n) and (k_n) be any given sequences with $j_n, k_n \in \mathbb{Z} \setminus \{0\}$. Then we define $u_0 = 1$ and

$$u_n = y^{k_1} x^{j_1} \cdots y^{k_n} x^{j_n}, \qquad \text{for} \qquad n \in \mathbb{N}. \tag{8}$$

A conjugate of every $w \in F$ will appear in many such sequences. No u_n with $n \geq 1$ is conjugate to x^k or y^k.

Now we turn to the matrix elements. We often write

$$H_t(u) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \tag{9}$$

the elements of $H_t(u_n)$ will be $a_n, \dots, d_n$.

The following proposition is folklore.

Proposition 1. *Let* (u_n) *be given by* (8). *Then, for* $n \geq 0$,

$$a_{n+1} = j_{n+1}k_{n+1}ta_n + a_n + j_{n+1}tb_n, \tag{10}$$

$$b_{n+1} = k_{n+1}a_n + b_n, \tag{11}$$

$$c_{n+1} = j_{n+1}k_{n+1}tc_n + c_n + j_{n+1}td_n, \tag{12}$$

$$d_{n+1} = k_{n+1}c_n + d_n. \tag{13}$$

For $n \geq 1$, *the* $a_n, \dots, d_n$ *are polynomials over* $\mathbb{Z}$ *of the forms*

$$a_n = j_1k_1 \cdots j_nk_nt^n + \cdots, \qquad b_n = j_1k_1 \cdots j_{n-1}k_{n-1}k_nt^{n-1} + \cdots, \tag{14}$$

$$c_n = j_1j_2k_2 \cdots j_nk_nt^n + \cdots, \quad d_n = j_1j_2k_2 \cdots j_{n-1}k_{n-1}k_nt^{n-1} + \cdots. \tag{15}$$

The trace $\operatorname{tr} H_t(u_n) = a_n + d_n$ *is non-constant for* $n \geq 1$.

Proof. By (8), (1) and (2) we have

$$\begin{aligned} H_t(u_{n+1}) &= H_t(u_n y^{k_{n+1}} x^{j_{n+1}}) = H_t(u_n) B^{k_{n+1}} A_t^{j_{n+1}} \\ &= \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} 1 & k_{n+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ j_{n+1}t & 1 \end{bmatrix} \end{aligned}$$

and (10)–(13) follow by (9); we have $a_0 = d_0 = 1$ and $b_0 = c_0 = 0$.

Now we prove the other assertions by induction. In each of the recursion formulas (10)–(13), all coefficients are in $\mathbb{Z}$. Furthermore, the degree of the first term is always, by induction hypothesis, higher than the degree of the other terms. Hence, by (14), a_{n+1} begins with $j_{n+1}k_{n+1}t \cdot j_1k_1 \cdots j_nk_nt^n$ and b_{n+1} begins with $k_{n+1} \cdot j_1k_1 \cdots j_nk_nt^n$, similarly for c_{n+1} and d_{n+1}. The statement about the trace follows from (14) and (15). ☑

Theorem 2. *Let* u_n *satisfy* (8) *with* $|j_\nu| = |k_\nu| = 1$. *Then the coefficients* $a_{n,m}$ *of* $a_n(t)$ *and so on satisfy*

$$|a_{n,m}| \leq \binom{n+m}{2m} (0 \leq m \leq n), \qquad |b_{n,m}| \leq \binom{n+m}{2m+1} (0 \leq m \leq n-1),$$

$$|c_{n,m}| \leq \binom{n+m-1}{2m-1} (1 \leq m \leq n), \quad |d_{n,m}| \leq \binom{n+m-1}{2m} (0 \leq m \leq n-1).$$

If $u_n = (yx)^n$ *then equality holds without taking absolute values.*

Compare [16, p. 233] for the last statement. The values for the case $v_n = (yx)^n$ were found by the method of generating functions. We have for instance

$$\sum_{n=0}^{\infty} a_n(t) z^n = \frac{1-z}{(1-z)^2 - tz}.$$

Proof. Let $n \geq 0$ and $m \geq 0$. Writing $a_{n,-1} = \cdots = d_{n,-1} = 0$, we obtain from (10)–(13) that

$$\begin{aligned}
a_{n+1,m} &= a_{n,m} + j_{n+1}k_{n+1}a_{n,m-1} + j_{n+1}b_{n,m-1},\\
b_{n+1,m} &= k_{n+1}a_{n,m} + b_{n,m},\\
c_{n+1,m} &= c_{n,m} + j_{n+1}k_{n+1}c_{n,m-1} + j_{n+1}d_{n,m-1},\\
d_{n+1,m} &= k_{n+1}c_{n,m} + d_{n,m}.
\end{aligned}$$

Now we verify the assertions by induction on n. The case $n = 0$ is clear because $H_t(1) = I$. We repeatedly use that $\binom{\alpha}{\beta} + \binom{\alpha}{\beta-1} = \binom{\alpha+1}{\beta}$.

Since $|j_{n+1}| = |k_{n+1}| = 1$ the above recursion formulas show that

$$\begin{aligned}
|a_{n+1,m}| &\leq |a_{n,m}| + |a_{n,m-1}| + |b_{n,m-1}|\\
&\leq \binom{n+m}{2m} + \binom{n+m-1}{2m-2} + \binom{n+m-1}{2m-1}\\
&= \binom{n+m}{2m} + \binom{n+m}{2m-1} = \binom{n+1+m}{2m},
\end{aligned}$$

$$|b_{n+1,m}| \leq |a_{n,m}| + |b_{n,m}| \leq \binom{n+m}{2m} + \binom{n+m}{2m+1} = \binom{n+1+m}{2m+1}$$

$$\begin{aligned}
|c_{n+1,m}| &\leq |c_{n,m}| + |c_{n,m-1}| + |d_{n,m-1}|\\
&\leq \binom{n+m-1}{2m-1} + \binom{n+m-2}{2m-3} + \binom{n+m-1}{2m-2} = \binom{n+m}{2m-1},
\end{aligned}$$

$$|d_{n+1,m}| \leq |c_{n,m}| + |d_{n,m}| \leq \binom{n+m-1}{2m-1} + \binom{n+m-1}{2m} = \binom{n+m}{2m}.$$

If $u_n = (yx)^n$ then $j_{n+1} = k_{n+1} = 1$ and all quantities are non-negative. Hence we have equality in all the above inequalities. ☑

Our homomorphism has an important property with respect to the automorphism $u \to \widetilde{u}$ defined in (6). The following proposition is well-known.

Proposition 3. *Let $u \in F$, $H_t(u) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $Q = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. Then*

$$H_t(\widetilde{u}) = QH_t(u)Q^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}. \tag{16}$$

Proof. It is easy to see that

$$Q\begin{bmatrix} a & b \\ c & d \end{bmatrix}Q^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}. \tag{17}$$

Hence it follows from (1) and (2) that

$$QA_tQ^{-1} = \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} = A_t^{-1} = H_t(x^{-1}) = H_t(\widetilde{x}),$$

$$QBQ^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = B^{-1} = H_t(y^{-1}) = H_t(\widetilde{y}).$$

This implies (16) because $(uv)\widetilde{\ } = \widetilde{u}\widetilde{v}$, $(u^{-1})\widetilde{\ } = \widetilde{u}^{-1}$ and H_t is a homomorphism. ☑

3. Relators

Our main interest is in the set of words

$$R^{\pm} = \{r \in F, r \neq 1 \mid \text{ there exists } s \in \mathbb{C} \text{ with } H_s(r) = \pm I\}, \tag{18}$$

$$R = \{r \in F, r \neq 1 \mid \text{ there exists } s \in \mathbb{C} \text{ with } h_s(r) = \mathrm{id}\}. \tag{19}$$

Let $N(r)$ denote the normal closure of r, that is the smallest normal subgroup of F with $r \in N(r)$.

It follows from Proposition 3 that $H_s(u) = I$ implies $H_s(\widetilde{u}) = I$; see (6) for the definition of $\widetilde{u}$. Hence, for $r \in R^+$ or $r \in R$, the normal closure of $\{r, \widetilde{r}\}$ also belongs to R^+ or R for the same s. Thus we have

$$u = v_1 r_1 v_1^{-1} \cdots v_m r_m v_m^{-1} \in R^+ \qquad \text{or} \qquad u \in R \tag{20}$$

where $v_\mu \in F$ and $r_\mu \in \{r, r^{-1}, \widetilde{r}, \widetilde{r}^{-1}\}$ for $\mu = 1, \dots, m$ and $m \in \mathbb{N}$. It follows that the exponent sums (5) for $u \in \mathbb{N}$ satisfy

$$\sigma_x(u) = \lambda\sigma_x(r), \quad \sigma_y(u) = \lambda\sigma_y(r) \qquad \text{for some} \qquad \lambda \in \mathbb{Z}. \tag{21}$$

If $r \in R$ and thus $h_s(r) = \mathrm{id}$ for some $s \in \mathbb{C}$ then, by the first isomorphism theorem, there is a homomorphism

$$h_{r,s} : \langle x, y; r\rangle \xrightarrow{onto} h_s(F) \tag{22}$$

defined by $h_{r,s}(w) := h_s(u)$ for any $w \in uN(r)$. Now we show that $h_{r,s}$ is in general not an isomorphism so that the representation of $\langle x, y\ ;\ r\rangle$ is not faithful. See Example 1.

Proposition 4. *If $h_{r,s}$ is an isomorphism then $\widetilde{r}$ is conjugate to r or r^{-1}.*

Note that, if $\widetilde{r}$ is conjugate to r, then $\sigma_x(r) = \sigma_y(r) = 0$. This result is related to [3, Theorem 3.1]. On the other hand, it is easy to see that even $\widetilde{r} = r^{-1}$ holds if r is a palindrome, that is, the word r reads forwards the same as backwards. See [3, Propositions 3.4 and 3.2] for a fuller description.

Proof. Let $h_{r,s}$ be an isomorphism. Since $h_s(\widetilde{r}) = \mathrm{id}$ it follows that $\widetilde{r} \in N(r)$. Hence the normal closure $N(\widetilde{r})$ of $\widetilde{r}$, the smallest normal subset of F containing $\widetilde{r}$, satisfies $N(\widetilde{r}) \subset N(r)$. Furthermore $\widetilde{r} = v_1 r^{\pm 1} v_1^{-1} \cdots v_m r^{\pm 1} v_m^{-1}$ and therefore

$$r = (\widetilde{r})\widetilde{\ } = \widetilde{v}_1 \widetilde{r}^{\pm 1} \widetilde{v}_1^{-1} \cdots \widetilde{v}_m \widetilde{r}^{\pm 1} \widetilde{v}_m^{-1} \in N(\widetilde{r}).$$

Hence $N(r) \subset N(\widetilde{r})$ so that r and $\widetilde{r}$ have the same normal closure $N(r)$. It follows [13, p. 261] [10, Proposition. 5.8, p. 106] that $\widetilde{r}$ is conjugate to r or r^{-1}. ☑

Now we present a general method to obtain relators. See Examples 2 and 3.

Theorem 5. *Let $u \in F$ not be conjugate to x^k or y^k with $k \in \mathbb{Z}$. Then*

$$u\widetilde{u} \in R^+, \qquad u^2 \in R^-, \tag{23}$$

$$u^n \in R^+ \cap R^-, \qquad \textit{for} \qquad n \geq 3, \tag{24}$$

thus $u\widetilde{u} \in R$ and $u^n \in R$ for $n \geq 2$.

For $u\widetilde{u}$ we have to exclude the case that u is a palindrome because then $u\widetilde{u} = 1$, compare (18) and (19). Note that $u\widetilde{u} = 1$ holds if and only if u is a palindrome.

Proof.

(a) We obtain from (9) and (16) that

$$H_t(u\widetilde{u}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} = \begin{bmatrix} 1 + a(a-d) & -b(a-d) \\ c(a-d) & 1 - d(a-d) \end{bmatrix}.$$

Since $a - d$ is non-constant by Proposition 1, it follows that $a(s) - d(s) = 0$ for some $s \in \mathbb{C}$. Hence $H_s(u\widetilde{u}) = I$.

(b) First let $n \geq 2$. There exists s such that $\operatorname{tr} H_s(u) = 2\cos(\pi/n)$. Then $\operatorname{tr} H_s(u) \neq \pm 2$. Hence $H_s(u)$ is conjugate to $\operatorname{diag}(e^{i\pi/n}, e^{-i\pi/n}) \neq I$. It follows that $H_s(u^n) = -I$.

Now let $n \geq 3$. There exists s such that $\operatorname{tr} H_s(u) = 2\cos(2\pi/n)$ so that, again, $\operatorname{tr} H_s(u) \neq \pm 2$. Hence $H_s(u)$ is conjugate to $\operatorname{diag}(e^{2\pi i/n}, e^{-2\pi i/n}) \neq I$ so that $H_s(u^n) = I$. ☑

Proposition 6. *Let $u \in F$ have the form* (7). *If $s \neq 0$ and $H_s(u) = I$ then s is an algebraic number of degree $\leq (n-1)/2$, and if $H_s(u) = -I$ then s is an algebraic number of degree $\leq n/2$. If $|j_\nu| = |k_\nu| = 1$ for $\nu = 1, \dots, n$ then s is an algebraic integer.*

This is common knowledge except for the sharp bounds $(n-1)/2$ and $n/2$ for the degrees, see Example 4. Note that $H_s(F) \subset \mathrm{SL}(2, \mathbb{Z}[s])$. We have shown that, up to conjugation, u has the form (7) whenever u is not conjugate to x^k or y^k.

Proof. We use the notation (9). If $H_s(u) = \pm I$ then $c(s) = 0$. Hence, by (15), s is an algebraic number which is an algebraic integer if $|j_\nu| = |k_\nu| = 1$. Now let $p(t)$ be the minimal polynomial of s. First let $H_s(u) = I$ and $s \neq 0$. We write

$$a + d - 2 = -(a-1)(d-1) + bc.$$

Since $a(s) = d(s) = 1$ and $b(s) = c(s) = 0$, we conclude that p divides $(a-1)$, $(d-1), b$ and c. Hence p^2 divides $a + d - 2$. Since $s \neq 0$, it follows that tp^2 divides $a + d - 2$, which is a polynomial of degree n by (14). Thus s has degree $\leq (n-1)/2$.

Now let $H_s(u) = -I$. We write

$$a + d + 2 = (a+1)(d+1) - bc.$$

Now p divides $(a+1), (d+1), b$ and c. Hence p^2 divides the polynomial $a+d+2$ of degree n. Thus s has degree $\leq n/2$. ☑

Now we describe an *algorithm to determine* whether $u \in F$ belongs to R^+ or R^-. This is not the case if u is conjugate to x^k or y^k. Therefore we may assume that u has the form (7). We use the notation (9).

First we check whether it is possible that $b = c = 0$ for some $t \in \mathbb{C}$. To do this we calculate the polynomial

$$q_0 := \gcd(b, c) \in \mathbb{Z}[t]. \tag{25}$$

If $\deg q_0 = 0$ then $u \notin R^+ \cup R^-$. If however $\deg q_0 > 0$ then we calculate the polynomials

$$q^\pm := \gcd(a \mp 1, q_0). \tag{26}$$

If $\deg q^+ = 0$ then $u \notin R^+$, if $\deg q^- = 0$ then $u \notin R^-$.

Now if $\deg q^\pm > 0$ then there is $s \in \mathbb{C}$ such that $a(s) = \pm 1$. It follows from (25) and (26) that $b(s) = c(s) = 0$ so that $1 = a(s)d(s) - b(s)c(s) = \pm d(s)$. Therefore we have $H_s(u) = \pm I$ and thus $u \in R^\pm$. Additionally we may factorize $q^\pm$ into irreducible polynomials over $\mathbb{Z}$. If s is a zero of a factor then all other zeros t of this factor satisfy $H_t(u) = \pm I$. The main computational difficulty of this algorithm is that very large integer coefficients may occur during the calculation of (25) and (26).

4. The Wirtinger Relators and the Longitude

Let K be a knot in $\mathbb{R}^3$, see e.g. [2, 9]. The complement $\Omega = \overline{\mathbb{R}^3 \setminus V(K)}$, where $V(K)$ is a tubular neighborhood of K, is a multiply connected domain. The fundamental group $\Pi_1(\Omega)$ is an important invariant of K though it does not completely determine the equivalence class of K, although the prime knots are determined by their knot group [7]. A very well understood family of knots are the so called 2-bridge knots and links [20, 2, 15, 19].

The fundamental group of a 2-bridge knot admits a presentation $\langle x, y;\ xw_n = w_n y\rangle$ where

$$\begin{aligned} w_n &= y^{k_n} x^{k_{n-1}} \cdots y^{k_1} x^{k_1} y^{k_2} \cdots y^{k_{n-1}} x^{k_n}, &\quad k_\nu \in \{1, -1\}, &\quad n \text{ odd} \\ w_n &= y^{k_n} x^{k_{n-1}} \cdots y^{k_2} x^{k_1} y^{k_1} \cdots y^{k_{n-1}} x^{k_n}, &\quad k_\nu \in \{1, -1\}, &\quad n \text{ even} \end{aligned} \tag{27}$$

for $n \in \mathbb{N}$, where $k_\nu, \nu = 1, \ldots, n$, satisfy some additional conditions [2, 1]. We inverted the usual order of exponents in order to have a recursive definition. On the following we leave the context of knot theory and call any word of the form (27) a *Wirtinger word*.

It follows from (27) that, with $\widetilde{\ }$ defined in (6),

$$w_{n+1} = \big(y^{-k_{n+1}} \widetilde{w}_n x^{-k_{n+1}}\big)^{\sim} \tag{28}$$

Instead of (9) we now write

$$W_n = H_t(w_n) = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}. \tag{29}$$

It follows from (28), (1) and Proposition 3 that, with $k = k_{n+1}$,

$$\begin{aligned} W_{n+1} &= Q B^{-k} W_n^{-1} A^{-k} Q^{-1} \\ &= Q \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_n & -b_n \\ -c_n & a_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -kt & 1 \end{bmatrix} Q^{-1}. \end{aligned}$$

Using also (17) and $k^2 = 1$, we obtain

$$W_{n+1} = \begin{bmatrix} ta_n + ktb_n + kc_n + d_n & ka_n + b_n \\ kta_n + c_n & a_n \end{bmatrix}. \tag{30}$$

Since $b_0 = c_0 = 0$ we deduce by induction the well-known formula [12, p. 141]

$$c_n = tb_n. \tag{31}$$

Hence we obtain from (30) the recursion formulas

$$a_{n+1} = ta_n + 2k_{n+1} t b_n + a_{n-1}, \qquad b_{n+1} = k_{n+1} a_n + b_n.$$

Now (27) is a special case of (7). Hence the estimates of Theorem 2 apply also with the new notation.

Now we drop the index n and write

$$W_t := H_t(w) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}. \tag{32}$$

Theorem 7. *If w satisfies the Wirtinger condition* (27) *then*

$$H_s(xw) = H_s(\widetilde{w}y^{-1}) \tag{33}$$

holds if and only if $a(s) + 2b(s) = 0$. Thus $wy\widetilde{w}^{-1}x \in R^+$.

Proof. By (16) and (31), the condition (33) is equivalent to

$$\begin{bmatrix} a & b \\ t(a+b) & tb+d \end{bmatrix} = A_t W_t = \widetilde{W_t} B^{-1} = \begin{bmatrix} a & -a-b \\ -tb & tb+d \end{bmatrix},$$

and this condition holds if and only if t satisfies $a(t) + 2b(t) = 0$. The non-constant polynomial $a + 2b$ has a root s. Hence

$$r = wy\widetilde{w}^{-1}x = x^{-1}(xw)(\widetilde{w}y^{-1})^{-1}x \in R^+. \qquad \boxtimes$$

In knot theory the condition (33) is replaced by

$$H_s(xw) = H_s(wy), \qquad r = wy^{-1}w^{-1}x \in R^+. \tag{34}$$

which holds if and only if $a(s) = 0$, see e.g. [12, p. 141].

By Proposition 3 and conjugation, we see that (34) implies

$$H_s(x\widetilde{w}) = H_s(\widetilde{w}y), \qquad \widetilde{r} = \widetilde{w}y\widetilde{w}^{-1}x^{-1} \in R^+. \tag{35}$$

Condition (34) induces a homomorphism $h_{r,s}$ from $\langle x, y;\ r\rangle$ into $\mathrm{PSL}(2,\mathbb{C})$; see (22). Now (35) says that it automatically induces a homomorphism from $\langle x, y;\ r, \widetilde{r}\rangle$. In the case of a 2-bridge knot it is known [2, 1] that there exists a faithful discrete $\mathrm{SL}(2,\mathbb{C})$-representation of a 2-bridge knot of type (p, q) with $q \not\equiv \pm 1$, so that $xw = wy$ implies $x\widetilde{w} = \widetilde{w}y$. But this is not true in general, see Example 5.

The situation is different for (33) because $\widetilde{r} = \widetilde{w}y^{-1}w^{-1}x^{-1}$ is conjugate to r^{-1} so that r and $\widetilde{r}$ have the same normal closure; compare Proposition 4. Hence they induce the same group.

For 2-bridge knots the group $G = \langle x, y;\ xw = wy\rangle$ and its peripheral subgroup are important concepts to distinguish equivalence classes of knots. This subgroup is generated by a meridian, say y, and the *longitude* $l = w^{-1}\widetilde{w}$

(see [17, p. 206]). We omitted Riley's factor $y^{2\sigma}$. It is easy to check that $(r = 1, \widetilde{r} = 1)$ is equivalent to $(r = 1, ly = yl)$.

Now we study the longitude $l = w^{-1}\widetilde{w}$ in a more general context. We do not assume that the word w comes from knot theory and we do not assume the consequence (31) of the Wirtinger condition. For $w \in F$ we obtain from (32) and (16) that

$$H_t(l) = W_t^{-1}\widetilde{W_t} = \begin{bmatrix} ad + bc & -2bd \\ -2ac & ad + bc \end{bmatrix}. \tag{36}$$

We note that $l = w^{-1}\widetilde{w}$ is a palindrome.

Theorem 8. *Let w satisfy* (7) *with $|j_\nu| = |k_\nu| = 1$ and let $a(s) = 0$. Then*

$$L_s := H_s(l) = \begin{bmatrix} -1 & -2b(s)d(s) \\ 0 & -1 \end{bmatrix}. \tag{37}$$

If $a = c + d$ then $b(s)d(s) = 1$. If the polynomial a is irreducible and if $a \neq c + d$ then $b(s)d(s) \notin \mathbb{Q}$ and L_s and B generate a free abelian group of rank 2.

Formulas similar to (37) follow from (36) if $b(s) = 0, c(s) = 0$ or $d(s) = 0$. The 2-bridge knots of type $(2n + 1, 1)$ have the Wirtinger word $w = (yx)^n$. It follows from Theorem 2 that $a = c + d$ so that $b(s)d(s) = 1$. See Examples 6, 7 and 8.

Proof. Since $ad - bc = 1$ we can write $ad + bc = -1 + 2ad$. Hence (37) follows from (36). If $a = c + d$ then $c(s) = -d(s)$ and thus $b(s)d(s) = -b(s)c(s) = 1$ because $a(s) = 0$.

Now let $q := b(s)d(s)$ and suppose that $q \in \mathbb{Q}$. Since $a(s) = 0$, it follows from Proposition 1 that s is an algebraic integer so that $q \in \mathbb{Z}$. It follows from (14) and (15) that

$$f(t) := qc(t) + d(t) = q\lambda t^n + \cdots, \qquad \lambda = \pm 1. \tag{38}$$

Since $a(s) = 0$ implies $b(s)c(s) = -1$ we have

$$b(s)f(s) = qb(s)c(s) + b(s)d(s) = -q + q = 0$$

so that $f(s) = 0$. Hence the irreducible polynomial $a(t)$ divides $f(t)$. Since $a(t) = \lambda t^n + \cdots$ with the same λ, we conclude from (38) that $q = 1$ and therefore $a = c + d$. If $a \neq c + d$ we therefore have $-2b(s)d(s) \notin \mathbb{Q}$ so that L_s and B are free abelian generators. ☑

5. Examples

The words of F in the following examples are generated by

$$z_0 = yx, \qquad z_1 = yx^{-1}, \qquad z_2 = y^{-1}x, \qquad z_3 = y^{-1}x^{-1}.$$

All polynomials will be written as the product of irreducible factors in $\mathbb{Z}[t]$. The factorization used the program Kash3 developed by M. Pohst and his group, `www.math.tu-berlin.de/~kant`.

Example 1. The following two words

$$\begin{aligned} r_1 &= z_0^2 z_1 z_3^2, & \sigma_x(r_1) &= -1, & \sigma_y(r_1) &= 1,\\ r_2 &= z_0^{10}, & \sigma_x(r_2) &= 10, & \sigma_y(r_2) &= 10 \end{aligned}$$

are relators with the same minimal polynomial $1+3t+t^2$. The normal closures satisfy $r_1 \notin N(r_2)$ and $r_2 \notin N(r_1)$ because the exponent sums do not satisfy (21). It follows that no homomorphism $h_{r,s}$ with $s = -1/2 \pm \sqrt{5}/2$ can be injective, see (22).

Example 2. Let $u = z_0^2 z_2$ and $r = u\widetilde{u} = z_0^2 z_2 z_3^2 z_1$. The polynomials for u are $a(t) = 1 + 4t - t^2 - t^3$ and $d(t) = 1 - t - t^2$. Now part (a) of the proof of Theorem 5 shows that $H_t(r) = I$ if and only if $a(s) - d(s) = s(5 - s^2) = 0$. Hence $r \in R^+$.

Example 3. Let $r = z_0^2$. Then

$$H_t(r) = \begin{bmatrix} 1+3t+t^2 & 2+t \\ 2t+t^2 & 1+t \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{for} \qquad t \in \mathbb{C}$$

so that $r \notin R^+$. But (23) shows that $r \in R^-$.

Example 4. The following words belong to $R^+ \cap R^-$. Their minimal polynomials

$$\begin{aligned} u = z_0^5 : \qquad & p^+(t) = 5 + 5t + t^2, \qquad & p^-(t) = 1 + 3t + t^2,\\ u = z_0^6 : \qquad & p^+(t) = 3 + 4t + t^2, \qquad & p^-(t) = 2 + 9t + 6t^2 + t^3 \end{aligned}$$

have the smallest degrees possible by Proposition 6.

Example 5. The Wirtinger word $w = z_0 z_1 z_1 z_0$ does not come from a 2-bridge knot. Its relator is $r = wy^{-1}w^{-1}x$ with $\sigma_x(r) = 1$, $\sigma_y(r) = -1$. Furthermore $\widetilde{r} = \widetilde{w}y\widetilde{w}^{-1}x^{-1}$ with $\sigma_x(\widetilde{r}) = -1$, $\sigma_y(\widetilde{r}) = 1$ so that $\widetilde{r}$ is not conjugate to r. Now r^{-1} contains $y^{-1}x^{-1}y^{-1}x^{-1}$ whereas no conjugate of $\widetilde{r}$ contains this word. Hence r is not conjugate to r^{-1} either. Thus it follows from Proposition 4 that, with $H_s(r) = I$, the homomorphism

$$\langle x, y;\ r, \widetilde{r} \rangle = \langle x, y;\ xw = wy, x\widetilde{w} = \widetilde{w}y \rangle \to \mathrm{SL}(2, \mathbb{C})$$

is not injective.

Example 6. The Wirtinger word of the 2-bridge knot of type $(9, 1)$ is $w = z_0^4$ and

$$a(t) = (1 + t)(1 + 9t + 6t^2 + t^3)$$

is reducible. It satisfies $a = c + d$ and thus $b(s)d(s) = 1$ by Theorem 8.

Example 7. Let $w = z_0 z_3 z_2 z_0$. This is not a Wirtinger word because $c \neq tb$. It satisfies

$$\begin{aligned} a(t) &= (1 + t)p(t), \quad p(t) = -1 + t + 2t^2 + t^3, \\ b(t)d(t) - 1 &= (1 + t)(-1 - t - t^2 + 2t^3 + 2t^4 + t^5), \\ b(t)d(t) + 1 &= (-1 + t + t^2 + t^3)p(t). \end{aligned}$$

Hence $b(-1)d(-1) = 1$ whereas $b(s)d(s) = -1$ if $p(s) = 0$. Thus, in Theorem 8, the assumption that $a(t)$ is irreducible can not be omitted.

Example 8. The 2-bridge knot of type $(5, 3)$ has $w = z_1 z_2$ and $a(t) = 1-t+t^2$. This gives $s = (1 + i\sqrt{3})/2$ and $b(s)d(s) = \pm i\sqrt{3}$.

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