

# On a Problem of Krasnosel'skii and Rutickii

Sobre un problema de Krasnosel'skii y Rutickii

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ABSTRACT. In [5, p. 30], M. A. Krasnosel'skii and Ya. B. Rutickii proposed a problem, which can be reformulated as follows. Let  $f$  be an  $N$ -function such that  $f(ts) \leq f(t)f(s)$ ,  $s, t \geq 1$ . Is there another  $N$ -function  $F$  such that  $F(st) \leq F(t)F(s)$ ,  $s, t > 0$  and equivalent to  $f$  on  $[1, \infty)$ ? We give a necessary and sufficient condition for a positive and constructive solution.

*Key words and phrases.* Orlicz Functions,  $N$ -Functions, Submultiplicative Functions, Matuszewska–Orlicz indices.

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RESUMEN. En [5, p. 30], M. A. Krasnosel'skii y Ya. B. Rutickii proponen un problema, el cual puede ser reformulado de la siguiente manera: Sea  $f$  una  $N$ -función tal que  $f(ts) \leq f(t)f(s)$  para todo  $t, s \geq 1$ . ¿Existe alguna función  $F$  tal que  $F(st) \leq F(s)F(t)$ ,  $s, t \geq 0$ , que sea equivalente a  $f$  en  $[1, \infty)$ ? En este artículo, damos una condición necesaria y suficiente para una solución positiva y constructiva.

*Palabras y frases clave.* Funciones de Orlicz,  $N$ -funciones, funciones submultiplicativas, índices de Matuszewska–Orlicz.

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### 1. Introduction and Preliminaries

In [1] J. Alexopoulos answers negatively the question posed by Krasnosel'skii and Rutickii, in the case when  $f$  has a complementary function  $f^*$  such that,  $(f^*(x))^2 \leq f^*(kx)$ . This rises the problem: characterize those  $N$ -function for which the answer to the question is positive.

Let us begin with some auxiliary definitions and results.

Let  $\mathbb{R}^+ := (0, \infty)$ . Let  $I \subset \mathbb{R}^+$  be such that  $I^2 \subset I$ . A function  $f : I \rightarrow \mathbb{R}^+$  is *submultiplicative* if

$$f(ts) \leq f(t)f(s), \quad t, s \in I.$$

If the reverse inequality holds, then we say that  $f$  is *supermultiplicative*.

We shall assume that the functions involved are not of the form  $f(t) = t^p g(t)$ , with  $g(t)$  bounded and bounded away from zero on  $I$ .

A function  $f : [0, \infty) \rightarrow [0, \infty)$  is an *Orlicz function*, if it is convex, increasing,  $f(0) = 0$ , and  $f(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . If, furthermore,  $\lim_{t \rightarrow 0} \frac{f(t)}{t} = 0$  and  $\lim_{t \rightarrow \infty} \frac{f(t)}{t} = \infty$ , then we say that  $f$  is an  $N$ -function.

An  $N$ -function is said to be  $\delta_2$  if there are positive constants  $M, a$ , such that  $f(2t) \leq Mf(t)$ ,  $t \in (0, \frac{a}{2})$ . If  $f(2t) \leq Mf(t)$ ,  $a \leq t < \infty$ , then we say that  $f$  is  $\Delta_2$ . If this inequality holds for any  $t \in \mathbb{R}^+$ , then we say that  $f$  is  $(\delta_2, \Delta_2)$ .

Let  $I = [0, \lambda]$ . Let  $f_1, f_2$  be positive functions defined on  $I$ . If there are positive constants  $a, b$  such that  $af_1(t) \leq f_2(t) \leq bf_1(t)$ ,  $t \in I$  then we say that  $f_1$  and  $f_2$  are *equivalent at zero*. If this inequality holds for  $t \geq \lambda$ , then we say that  $f_1$  and  $f_2$  are *equivalent at infinity*.

The solution to the problem considered herein depends on some numerical parameters, the so called Matuszewska-Orlicz indices [7, 8, 9, 10, 11, 12, 13]. There are various ways to define these indices. We shall use the version given by Lindenstrauss and Tzafriri [8, 9, 10].

Let  $f : [0, \infty) \rightarrow [0, \infty)$  be an Orlicz function. Define  $\alpha_f^0, \alpha_f^\infty, \beta_f^0, \beta_f^\infty$  by

$$\begin{aligned} \alpha_f^0 &:= \sup \left\{ p > 0 : \sup_{1 \leq s, t < \infty} \frac{f(st)}{s^p f(t)} < \infty \right\}, \\ \beta_f^0 &:= \inf \left\{ p > 0 : \inf_{1 \leq s, t < \infty} \frac{f(st)}{s^p f(t)} > 0 \right\}, \\ \alpha_f^\infty &:= \sup \left\{ p > 0 : \sup_{1 \leq s, t < \infty} \frac{f(s)t^p}{f(st)} < \infty \right\}, \\ \beta_f^\infty &:= \inf \left\{ p > 0 : \inf_{1 \leq s, t < \infty} \frac{f(s)t^p}{f(st)} > 0 \right\}. \end{aligned}$$

Let  $f : [0, 1] \rightarrow \mathbb{R}^+$  be a submultiplicative and  $\delta_2$  Orlicz-function. It is known that  $\alpha_f^0 = \lim_{t \rightarrow 0} \frac{\log f(t)}{\log t}$  [11, 12, 13]. Similarly, if  $f : [1, \infty) \rightarrow \mathbb{R}^+$  then  $\beta_f^\infty = \lim_{t \rightarrow 0} \frac{\log f(t)}{\log t}$ .

The following result is well known.

**Theorem 1.** *Let  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a submultiplicative function. Write  $\alpha := \alpha_f^0$ ,  $\beta := \beta_f^\infty$ . We have that*

$$-\infty < \alpha \leq \beta < \infty.$$

Also,  $f$  can be represented as follows:

$$f(t) = \begin{cases} t^\alpha g_1(t), & 0 < t \leq 1; \\ t^\beta g_2(t), & t > 1. \end{cases}$$

This representation is unique in the sense that, for each  $\varepsilon > 0$ ,  $t^\varepsilon g_1(t) \rightarrow 0$  as  $t \rightarrow 0$  and  $t^{-\varepsilon} g_2(t) \rightarrow 0$ ,  $t \rightarrow \infty$ . Also,  $g_1(t) \geq 1$ , and  $g_2(t) \geq 1$ .

The subadditive version of this theorem appears in [6, p. 410]. See also [4, p. 244].

We end this section with a few remarks.

The problem posed by Krasnosel'skii and Rutickii, which hereafter will be called "*the problem*", is as follows: Given an  $N$ -function  $f : [0, \infty) \rightarrow [0, \infty)$  such that, for  $c > 0$ ,  $t_0 > 0$ , we have that  $f(ts) \leq cf(t)f(s)$ , for all  $s, t \geq t_0$ , find an  $N$ -function  $F$ , submultiplicative on  $[0, \infty)$  and equivalent to  $f$  on  $[t_0, \infty)$ .

For any such function we have that the function

$$\bar{f}_{t_0}(t) := \sup_{t_0 \leq u \leq \infty} \frac{f(tu)}{f(u)}, \quad t \geq 1,$$

with  $t_0 > 1$ , is submultiplicative on  $[1, \infty)$ ,  $\bar{f}_{t_0}(1) = 1$  and equivalent to  $f$  on  $[t_0, \infty)$ . Therefore, we shall assume that the function  $f$  itself, has these properties.

From Theorem 1 we can draw some conclusions: Let  $f$  be an  $N$ -function, submultiplicative on  $[1, \infty)$ , let  $F$  be a positive solution to *the problem*, then we must have that  $\alpha_F^0 > 1$ ; in fact, if  $\alpha_F^0 = 1$ , then, according to Theorem 1, for any  $t \in [0, 1]$  we have that  $F(t) = tg(t)$ , with  $g(t) \geq 1$ , and so,  $\lim_{t \rightarrow 0} \frac{F(t)}{t} \neq 0$ .

Also, since  $f$  and  $F$  are equivalent on  $[t_0, \infty)$  and the Matuszewska-Orlicz indices are invariant under equivalence of functions, we must have that  $\beta_f^\infty = \beta_F^\infty$ . Since  $\alpha_F^0 \leq \beta_F^\infty$ , we have that  $\beta_f^\infty > 1$ .

T. Andó [2] has furnished a negative solution to *the problem*. This author proved that the  $N$ -function  $f(t) = (1+t) \log(1+t) - t$ , which is submultiplicative on  $[e^2 - 1, \infty)$  can not be extended submultiplicatively to  $[0, \infty)$ .

Now, for this function we have that  $\beta_f^\infty = 1$ .

This is a simple proof that  $f$  cannot be extended to a submultiplicative  $N$ -function on  $[0, \infty)$ , equivalent to  $f$  on  $[e^2 - 1, \infty)$ .

## 2. The Solution

Matuszewska and Orlicz [11] defined the indices in the following way: Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a  $(\delta_2, \Delta_2)$  Orlicz function. Define

$$\begin{aligned}\bar{f}(t) &:= \sup_{0 < u \leq 1} \frac{f(tu)}{f(u)}, & \underline{f}(t) &:= \inf_{0 < u \leq 1} \frac{f(tu)}{f(u)}, \\ \bar{g}(t) &:= \sup_{1 < u < \infty} \frac{f(tu)}{f(u)}, & \underline{g}(t) &:= \inf_{1 \leq u < \infty} \frac{f(tu)}{f(u)}.\end{aligned}$$

Also, define

$$\begin{aligned}\alpha_0 &:= \lim_{t \rightarrow 0} \frac{\log \bar{f}(t)}{\log t}, & \beta_0 &:= \lim_{t \rightarrow 0} \frac{\log \underline{f}(t)}{\log t}, \\ \alpha_\infty &:= \lim_{t \rightarrow \infty} \frac{\log \bar{g}(t)}{\log t}, & \beta_\infty &:= \lim_{t \rightarrow \infty} \frac{\log \underline{g}(t)}{\log t}.\end{aligned}$$

These numbers are all finite [11].

We prove now that  $\alpha_f^0 = \alpha_0$ ,  $\beta_f^0 = \beta_0$ ,  $\alpha_f^\infty = \alpha_\infty$  and,  $\beta_f^\infty = \beta_\infty$ . Since all functions involved are non negative, then for  $p > 0$  we have that

$$\begin{aligned}\sup_{0 < u, t \leq 1} \frac{f(tu)}{f(u)t^p} &= \sup_{0 < t \leq 1} \left\{ \sup_{0 < u \leq 1} \frac{f(tu)}{f(u)t^p} \right\} \\ &= \sup_{0 < t \leq 1} \left\{ \frac{\bar{f}(t)}{t^p} \right\} \\ &= \sup_{0 < t \leq 1} \left\{ \sup_{0 < u \leq 1} \frac{\bar{f}(tu)}{\bar{f}(u)t^p} \right\} \\ &= \sup_{0 < u, t \leq 1} \frac{\bar{f}(tu)}{\bar{f}(u)t^p},\end{aligned}$$

whence,  $\alpha_f^0 = \alpha_0$ . All other cases are dealt with in the same way.

We mention, by passing, that the approach of Matuszewska-Orlicz, in considering the above limits is particularly useful. Indeed, when we consider the problem of finding these limits for other classes of functions, one is confronted with the existence of functions which are submultiplicative on  $(0, 1]$  and  $[1, \infty)$  separately but not on  $(0, \infty)$ . Others are submultiplicative on  $[0, 1]$  and supermultiplicative on  $[1, \infty)$ , and so on, and we cannot rely on Theorem 1 alone (see [3]).

We split the proof into several lemmas which have an interest of their own.

**Lemma 2.** *Let  $f : [1, \infty) \rightarrow [1, \infty)$  be a  $\Delta_2$  function such that  $\frac{f(t)}{t}$ ,  $t \geq 1$ , is increasing. Then, the function  $F(t)$ ,  $t \in (0, 1]$ , defined by*

$$F(t) := \sup_{1 \leq tu} \frac{f(tu)}{f(u)}$$

satisfies  $F(t_1 t_2) \leq F(t_1)F(t_2)$ ,  $t_1, t_2 \in (0, 1]$ , and  $\frac{F(t)}{t}$  is increasing. Also, the function  $W(t)$  defined by

$$W(t) := \int_0^t \omega(\tau) d\tau, \quad \text{where } \omega(\tau) = \begin{cases} \frac{F(\tau)}{\tau}, & 0 < \tau \leq 1; \\ \frac{f(\tau)}{\tau}, & \tau \geq 1 \end{cases}$$

is convex and satisfies the  $(\delta_2, \Delta_2)$  condition.

**Proof.** For any  $t_1, t_2$  in  $(0, 1)$  we have

$$\begin{aligned} F(t_1 t_2) &= \sup_{1 \leq t_1 t_2 u} \frac{f(t_1 t_2 u)}{f(u)} \\ &\leq \sup_{1 \leq t_1 t_2 u} \frac{f(t_1 t_2 u)}{f(t_1 u)} \sup_{1 \leq t_1 t_2 u} \frac{f(t_1 u)}{f(u)} \\ &\leq \sup_{1 \leq t_1 u} \frac{f(t_1 u)}{f(u)} \sup_{1 \leq t_2 s} \frac{f(t_2 s)}{f(s)} \\ &= F(t_1)F(t_2) \end{aligned}$$

Also, for  $\alpha, t$  in  $(0, 1)$ , we have

$$F(\alpha t) = \sup_{1 \leq \alpha t u} f(\alpha t u)/f(u) \leq \alpha \sup_{1 \leq t u} f(t u)/f(u) = \alpha F(t);$$

that is,  $F(t)/t$  is increasing.

As the function  $f(t)$  is  $\delta_2$  for  $t \leq 1$ , then

$$F(2t) = \sup_{1 \leq 2tu} \frac{f(2tu)}{f(u)} = \sup_{1 \leq tv} \frac{f(tv)}{f(\frac{v}{2})} \leq M \sup_{1 \leq tv} \frac{f(tv)}{f(v)} = MF(t), \quad 0 < t \leq 1/2,$$

for some  $M > 0$ .

The function  $\omega(s)$  is clearly  $\delta_2$  for  $t \leq 1/2$  and  $\Delta_2$  for  $t \geq 1$ , since so are  $F$  and  $f$  respectively. Let  $t$  be in  $(1/2, 1)$ . Then

$$F(2t) = \sup_{1 \leq 2tu} \frac{f(2tu)}{f(u)} \geq \sup_{2 \leq u} \frac{f(tu)}{f(u)} \geq \frac{f(2t)}{f(2)};$$

that is,  $f(2t) \leq f(2)F(t)$ ,  $1/2 \leq t \leq 1$ , and consequently  $\omega(s)$  is  $(\delta_2, \Delta_2)$ . Since  $\omega(s)$  is also increasing, then  $W(t)$  is convex and  $(\delta_2, \Delta_2)$ .  $\square$

**Lemma 3.** Let  $f$  and  $F$  be as in the previous lemma. Assume further that  $f(st) \leq f(s)f(t)$  for all  $s, t \geq 1$ . Then the function  $\bar{f}(t)$  defined by

$$\bar{f}(t) := \begin{cases} F(t), & 0 < t \leq 1; \\ f(t), & 1 \leq t < \infty, \end{cases}$$

satisfies the inequality  $\overline{f}(st) \leq \overline{f}(s)\overline{f}(t)$ ,  $s, t > 0$  and the function  $W(t)$  defined by

$$W(t) := \int_0^t \frac{\overline{f}(\tau)}{\tau} d\tau$$

is  $(\delta_2, \Delta_2)$ , convex, and, for some  $k > 0$ ,  $W(st) < kW(s)W(t)$ ,  $s, t > 0$ .

**Proof.** According to Lemma 2,  $W(t)$  is convex and  $(\delta_2, \Delta_2)$ . Let us now prove that  $\overline{f}(t)$  is submultiplicative on  $(0, \infty)$ . We need only to prove that  $\overline{f}(st) \leq \overline{f}(s)\overline{f}(t)$  with  $t < 1$  and  $s > 1$ .

If  $st < 1$ , then

$$\begin{aligned} \overline{f}(st) = F(st) &= \sup_{1 \leq stu} \frac{f(stu)}{f(u)} \leq \sup_{1 \leq stu} \frac{f(tu)}{f(u)} \\ &\leq f(s) \sup_{1 \leq tu} \frac{f(tu)}{f(u)} = f(s)F(t) = \overline{f}(s)\overline{f}(t). \end{aligned}$$

If, on the other hand,  $st > 1$ , then  $f(st) = \overline{f}(st)$ , and since  $f(s)F(t) \geq f(st)$ ,  $st > 1$ , then  $\overline{f}(st) \leq \overline{f}(s)\overline{f}(t)$ . It now follows that there exists  $k > 0$  such that  $W(st) \leq kW(s)W(t)$ ,  $s, t > 0$ . □

**Remark 4.** Let  $f$  be as above. Let  $g(t), t \in (0, 1]$ , be such that the function  $G(t)$  defined by

$$G(t) := \begin{cases} g(t), & 0 < t \leq 1; \\ f(t), & 1 \leq t < \infty \end{cases}$$

is submultiplicative, then  $g(t) \geq F(t)$ , for all  $t \in (0, 1)$ ; in fact, we have from  $G(st) \leq G(s)G(t)$  that  $f(st)/f(s) < g(t)$ , and so  $F(t) = \sup_{1 \leq st} f(st)/f(s) \leq g(t)$ .

**Lemma 5.** Let  $f : [0, \infty) \rightarrow [0, \infty)$  be a  $(\delta_2, \Delta_2)$   $N$ -function. Let  $F$  be defined as in Lemma 2. Then  $\alpha_f^\infty = \alpha_F^0$ .

**Proof.** Recall that  $\alpha_f^\infty = \lim_{t \rightarrow \infty} \log \underline{g}(t) / \log t$ , with  $\underline{g}(t) = \inf_{1 \leq s < \infty} f(st)/f(t)$ ,  $t \geq 1$ . Now, since for each  $t \in (0, 1]$  we have that

$$F(t) = \sup_{1 \leq st} \frac{f(st)}{f(s)} = \frac{1}{\inf_{1 \leq u < \infty} \frac{f(u/t)}{f(u)}} = \frac{1}{g(1/t)},$$

then  $\alpha_F^0 = \alpha_f^\infty$ . □

**Theorem 6.** Let  $f : [0, \infty) \rightarrow [0, \infty)$  be an  $N$ -function, submultiplicative on  $[1, \infty)$ . There is a submultiplicative  $N$ -function  $F : [0, \infty) \rightarrow [0, \infty)$ , equivalent to  $f$  on  $[1, \infty)$ , if and only if,  $\alpha_f^\infty > 1$ .

**Proof.** Let  $F$  be defined as in Lemma 2. Then, as we have seen, the function  $W(t) = \int_0^s \bar{f}(\tau)\tau^{-1} dr$ , with  $\bar{f}(\tau)$  as in Lemma 3, is convex and for some  $k > 0$ , we have that  $W(st) \leq kW(s)W(t)$ . Also,  $W(t) \leq F(t)$  and since, from the hypothesis, we must have  $\alpha_F^0 = \alpha_f^\infty > 1$ , then writing  $F(t) = t^{\alpha_F^0}h(t)$ , we have  $0 \leq \lim_{t \rightarrow 0} W(t)/t^{\alpha_F^0}h(t) = 0$ .

From the hypothesis that  $f$  is an  $N$ -function, we deduce that  $\lim_{t \rightarrow \infty} W(t)/t = \infty$ ; this proves that  $W(t)$  is an  $N$ -function.

It is apparent that  $W$  is equivalent to  $f$  at infinity. Finally, the function  $\bar{W}(t) = \sup_{0 < s < \infty} W(st)/W(s)$  is a submultiplicative  $N$ -function, equivalent to  $f$  at infinity.

If, on the other hand,  $\alpha_f^\infty = 1$  then for any submultiplicative function  $G(t)$ , equivalent to  $f$  on  $[1, \infty)$ , we have that  $F(t) \leq G(t)$ ,  $t \in [0, 1]$ , according to Remark 4. Therefore  $1 = \alpha_F^0 = \alpha_f^0 \geq \alpha_G^0$  and  $G$  cannot be an  $N$ -function.  $\square$

Notice that for any submultiplicative  $N$ -function  $f$  on  $[0, \infty)$  we have, according to the last argumentation, that  $\alpha_f^\infty \geq \alpha_f^0$ .

### References

- [1] J. Alexopoulos, *De la Vallée Poussin's Theorem and Weakly Compact Sets in Orlicz Spaces*, QM **17** (1994), 231–248.
- [2] T. Andô, *On Some Properties of Convex Functions*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys **8** (1960), 413–418.
- [3] C. E. Finol and M. Wójtowicz, *Multiplicative Properties of Real Functions with Applications to Classical Functions*, Aequationes Math **59** (2000), 134–149.
- [4] E. Hille and R. S. Phillips, *Functional Analysis and Semi-groups*, Amer-Math. Soc, Providence, United States, 1957.
- [5] M. A. Krasnosiel'skii and Ya B. Rutickii, *Convex Functions and Orlicz Spaces*, Noordhoff Ltd, Groningen, Holland, 1961.
- [6] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*, P.W.N, Warszawa-Kraków, Polska, 1985.
- [7] K. Lindberg, *On Subspaces of Orlicz Sequence Spaces*, Studia Math **45** (1973), 119–146.
- [8] J. Lindenstrauss and L. Tzaferi, *On Orlicz Sequence Spaces II*, Israel J Math **10** (1971), no. 3, 379–390.

- [9] ———, *On Orlicz Sequences Spaces III*, Israel J Math **10** (1973), no. 4, 368–389.
- [10] ———, *Classical Banach Spaces I*, Ergebnisse der Math. und ihrer Grenzgebiete, Springer-Verlag, Berlin, Germany, 1977.
- [11] W. Matuszewska and W Orlicz, *On Certain Properties of  $\varphi$ -Functions*, Bull Acad. Polon. Sci. Sér. Math. Astronom Phys **8** (1960), 439–443.
- [12] ———, *On Some Classes of Funtions with Regard to the their Order of Growth*, Studia Math **26** (1965), 11–24.
- [13] W. Orlicz, *On spaces of  $\varphi$ -integrable functions*, Proc. of the International Symp. on Linear Spaces held at the Hebrew University of Jerusalem (Jerusalem, Israel), Academic Press, July, 5-12 1960, pp. 357–365.

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