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Green's Functions for Sturm-Liouville Problems on Directed Tree Graphs

Funciones de Green para problemas de Sturm-Liouville en árboles direccionales

Jorge M. Ramirez

Universidad Nacional de Colombia, Medellín, Colombia

Abstract. Let Γ be geometric tree graph with m edges and consider the second order Sturm-Liouville operator $\mathcal{L}[u] = (-pu')' + qu$ acting on functions that are continuous on all of Γ, and twice continuously differentiable in the interior of each edge. The functions p and q are assumed continuous on each edge, and p strictly positive on Γ. The problem is to find a solution $f : \Gamma \to \mathbb{R}$ to the problem $\mathcal{L}[f] = h$ with $2m$ additional conditions at the nodes of Γ. These node conditions include continuity at internal nodes, and jump conditions on the derivatives of f with respect to a positive measure ρ. Node conditions are given in the form of linear functionals $l_1, \dots, l_{2m}$ acting on the space of admissible functions. A novel formula is given for the Green's function $G : \Gamma \times \Gamma \to \mathbb{R}$ associated to this problem. Namely, the solution to the semi-homogenous problem $\mathcal{L}[f] = h$, $l_i[f] = 0$ for $i = 1, \dots, 2m$ is given by $f(x) = \int_\Gamma G(x, y)h(y)\, d\rho$.

Key words and phrases. Problema Sturm-Liouville en grafo, función de Green.

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Resumen. Sea Γ un grafo tipo árbol con m aristas y considere el operador de Sturm-Liouville $\mathcal{L}[u] = (-pu')' + qu$ definido en el espacio de funciones continuas en Γ y continuamente diferenciables dos veces al interior de cada arista de Γ. Las funciones p y q se suponen continuas en cada arista, y p es estrictamente positiva en todo Γ. El problema consiste en hallar la solución $f : \Gamma \to \mathbb{R}$ al problema dado por $\mathcal{L}[f] = h$ mas $2m$ condiciones en los nodos de Γ: en los nodos internos se especifican continuidad de f y condiciones de salto para las derivadas de f con respecto a una medida ρ. Estas condiciones de nodo se expresan en la forma de funcionales lineales $l_1, \dots, l_{2m}$ actuando sobre el espacio de funciones admisibles para $\mathcal{L}$. Se presenta una nueva fórmula para la función de Green $G : \Gamma \times \Gamma \to \mathbb{R}$ asociada con este problema. Es decir, se expresa la solución del problema semi-homogéneo $\mathcal{L}[f] = h$, $l_i[f] = 0$ para $i = 1, \dots, 2m$ como $f(x) = \int_\Gamma G(x, y)h(y)\, d\rho$.

Palabras y frases clave. Sturm-Liouville problems on graphs, Green's function.

1. Introduction

The Sturm-Liouville differential operator

$$\mathcal{L}[f] = -(pf')' + qf \tag{1}$$

on an interval, appears in the analysis of many different types of models in the natural sciences. The problem $\mathcal{L}[f] = h$ or $\mathcal{L}[f] = \nu qf$, together with appropriate boundary conditions, arises when considering Kirchoff's law in electrical circuits, the balance of tension in a elastic string, or the steady state temperature in a heated rod (see for example [4, 3, 2]). A more complete review of the mathematical theory can be found in [10].

The extension of operator (1) to the case of a domain composed of intervals arranged in a graph has received recent attention (see for example [5, 9, 1]). A complete bibliographical review with historical notes and applications can be found in [6]. The particular work presented here is motivated by the following problem in mathematical ecology: the stability of populations of organisms in river networks where the dispersion of individuals is governed by an advection–diffusion operator.

is motivated by an applied problem in mathematical ecology, that of the stability for populations of organisms in river networks [8].

Let Γ be a tree graph in $\mathbb{R}^2$, that is, Γ is a collection of edges joined with nodes, where each edge can be treated as a finite open interval (a complete description of the notation and assumptions for Γ is laid out in Section 2.1). We are interested in the Storm-Liuville operator of the form

$$\mathcal{L}[f]_e = -(pf_e')' + qf_e, \qquad e \in \Gamma, \tag{2}$$

where the subscript e denotes the restriction of a function to edge e. In Equation (2), the functions p and q are assumed continuous and bounded, with p being also uniformly bounded away from zero.

Given a function $h : \Gamma \to \mathbb{R}$, one is interested in solving the problem

$$\mathcal{L}[f] = h, \qquad f \in \mathcal{E}(\Gamma) \tag{3}$$

where $\mathcal{E}(\Gamma)$ denotes the set of functions that are twice continuously differentiable inside each edge, and satisfy a given set of node conditions. It can easily be shown that problem (3) has a unique solution if and only if it is non-degenerate, namely, the only solution to the homogeneous problem $\mathcal{L}[f] = 0$, $f \in \mathcal{E}(\Gamma)$, is $f \equiv 0$.

Let $\mathcal{G}$ be some right inverse mapping of $\mathcal{L}$ such that $\mathcal{L}\big[\mathcal{G}[f]\big] = f$ for all admissible f. Then $\mathcal{G}$ is the Green's operator for problem problem (3). Moreover, it will be shown that:

Theorem 1. *If problem* (3) *is non-degenerate, it has a unique Green's function. Namely, there exists a function* $G : \Gamma \times \Gamma \to \mathbb{R}$ *such that the solution to* (3), $f = \mathcal{G}[h]$, *is* $f(x) = \int_\Gamma G(x,y)h(y)\,dy$.

The proof of the existence of the Green's function G defined in Theorem 1 follows from standard arguments. Here, the proof is obtained by simply giving an explicit formula for $G(x,y)$. Uniqueness of the Green's function follows from the hypothesis of non-degeneracy.

In [7] the authors provide a proof of existence, uniqueness and a formula for G for general graphs. The goal here is to present a new formula, both simpler and less expensive to compute, for the Green's function in the case Γ is a tree graph. The techniques here are elementary and based on the classical Lagrange's method for Sturm-Liuoville problems (see for example [2]).

The organization is as follows. The next Section settles the notation and defines the class of Sturm-Liouville problems to be considered. Finally, Section 3 is devoted to the construction and the formula for the the Green's function.

2. Sturm-Liouville Problems on Tree Graphs

2.1. Tree Graphs and Functions

By a tree graph we understand a finite collection of edges embedded in $\mathbb{R}^2$, joined with nodes and containing no loops. That is, for any two points x, y in the graph, there exists one single path through the graph joining them. We assume that each edge e of the graph allows a sufficiently smooth parametrization, contains no self-intersections, and is finite, therefore can be considered as the interval $e = (0, l_e)$. The collection of all edges is denoted by Γ. At each endpoint of an edge it is located a node of Γ. The set of nodes is $N(\Gamma)$ and boldface is used to denote individual nodes. The graph, including its nodes, is denoted as $\overline{\Gamma} := \Gamma \cup N(\Gamma)$.

Points in Γ are denoted by the pair (e, x) with $0 < x < l_e$, or by single letters if specification of the edge is not crucial. If $\boldsymbol{n}$ is a node, let $i(\boldsymbol{n})$ denote the set of incident edges at $\boldsymbol{n}$, namely those for which $\boldsymbol{n}$ is an endpoint. Boundary nodes are those $\boldsymbol{n}$ with $\#i(\boldsymbol{n}) = 1$. The set of all boundary nodes of Γ is $\partial\Gamma$. The set of internal nodes is $I(\Gamma) = N(\Gamma) \setminus \partial\Gamma$. Node $\boldsymbol{n}$ has $\#i(\boldsymbol{n}) \geq 1$ possible representations: for each $e \in i(\boldsymbol{n})$, one either has $\boldsymbol{n} = (e, 0)$ or $\boldsymbol{n} = (e, l_e)$. The representation of points in $\overline{\Gamma}$ is therefore dependent on the parametrization direction of its edges.

The value of a function $f : \Gamma \to \mathbb{R}$ at a point in Γ is denoted as $f_e(x) = f(e, x)$. That is, f_e is the restriction of f to the edge e. For a node $\boldsymbol{n} \in \partial\Gamma$

located at the endpoint of some edge e, $f(\boldsymbol{n})$ denotes the appropriate one-sided limit of f_e. For an internal node $\boldsymbol{n}$ with $i(\boldsymbol{n}) = \{e_1, \dots, e_n\}$ the value $f_{e_1}(\boldsymbol{n})$ denotes the one–sided limit of f_{e_i} as x approaches the endpoint of e_i at which $\boldsymbol{n}$ is located, $i = 1, \dots, n$. If all these limits coincide, f is said to be continuous at $\boldsymbol{n}$, and $f(\boldsymbol{n})$ is defined as the common value.

We must also differentiate functions given on Γ. For a point $(e, x) \in \Gamma$ with $0 < x < l_e$, the derivative $f'(e, x) = f'_e(x)$ is computed as the usual derivative of the restriction f_e at x according to the particular parametrization direction of e. A change in the orientation of the parametrization of the edge implies a sign change on f'_e. Note that the sign of f''_e or $(pf'_e)'$ remains unchanged. For a node $\boldsymbol{n}$ located at an endpoint of edge e, we introduce the *boundary derivative* f'^b_e as the derivative "out of node $\boldsymbol{n}$ into edge e", as if the parametrization of e has $\boldsymbol{n} = (e, 0)$. Boundary derivatives are useful because they make the following equality hold

$$\int_e (pf')'\, dx = p(l_e) f'^b(l_e) - p(0) f'^b(0), \tag{4}$$

regardless of whether the integral is computed from 0 to l_e, or from l_e to 0.

The space of functions that are n times continuously differentiable in Γ is denoted by $\mathcal{C}^n(\Gamma)$, $n = 0, 1, 2, \dots$; $\mathcal{C}(\Gamma) := \mathcal{C}^0(\Gamma)$. Clearly, such spaces are identifiable with direct sums of the form $\bigoplus_e \mathcal{C}^n(0, l_e)$. The set $\mathcal{C}\big(\overline{\Gamma}\big)$ is composed of functions in $\mathcal{C}(\Gamma)$ that are also continuous at each node.

2.2. Sturm-Liouville Operators

Let $p, q \in \mathcal{C}(\Gamma)$ be bounded with $\inf_\Gamma p(x) > 0$. The object of this study is the differential operator

$$\mathcal{L}[f] := -(pf')' + qf, \qquad f \in \mathcal{D}^2_p(\Gamma), \tag{5}$$

where $\mathcal{D}^2_p(\Gamma)$ denotes the space of functions $f \in \mathcal{C}(\Gamma)$ such that $(pf')' \in \mathcal{C}(\Gamma)$.

If Γ contains m edges, then the dimension of $\mathcal{N}(\mathcal{L}) := \big\{u \in \mathcal{D}^2_p(\Gamma) : \mathcal{L}[u] = 0\big\}$ is $2m$. A basis $\{\varphi_1, \dots, \varphi_{2m}\}$ for $\mathcal{N}(\mathcal{L})$ can be found as follows. Let e be the i-th edge and define φ_{2i-1} and φ_{2i} as the solutions to $-(p_e f')' + q_e f = 0$ on e satisfying

$$\varphi_{2i-1}(0) = 1, \quad \varphi'_{2i-1}(0) = 0, \quad \varphi_{2i}(0) = 0, \quad \varphi'_{2i}(0) = 1,$$

extended to all of Γ via $\varphi_{2i-1}(\widetilde{e}, x) = \varphi_{2i}(\widetilde{e}, x) = 0$, $0 < x < l_{\widetilde{e}}$ for all $\widetilde{e} \neq e$. Consider now a collection of $2m$ linear functionals $\{l_i, \dots, l_{2m}\}$ defined on $\mathcal{D}^2_p(\Gamma)$. These functionals play the role of the linear conditions at the nodes of Γ. The problem

$$f \in \mathcal{D}^2_p(\Gamma), \quad \mathcal{L}[f] = h, \quad l_i[f] = c_i, \quad i = 1, \dots, 2m \tag{6}$$

will be uniquely solvable if and only if the homogenous problem

$$f \in \mathcal{D}_p^2(\Gamma), \quad \mathcal{L}[f] = 0, \quad l_i[f] = 0, \quad i = 1, \dots, 2m \tag{7}$$

has no solution except the trivial solution $f \equiv 0$. In this case we say that problem (6) is *non-degenerate*.

Non-degeneracy can be characterized as follows. Let Δ be the matrix defined by $\Delta_{i,j} = l_i[\varphi_j]$, $i, j = 1, \dots, 2m$. Non-degeneracy is therefore equivalent to $\det(\Delta) \neq 0$. In this case, the solution to problem (6) can be written explicitly. Let z be some solution to the semi-homogeneous problem

$$z \in \mathcal{D}_p^2(\Gamma), \quad \mathcal{L}[z] = h, \quad l_i[z] = 0, \quad i = 1, \dots, 2m. \tag{8}$$

Then, the solution $f = z + \sum_{i=1}^{2m} a_i \varphi_i$ to problem (6) must satisfy

$$\begin{bmatrix} 1 & \varphi_1 & \cdots & \varphi_{2m} \\ 0 & & & \\ \vdots & & \boxed{\Delta} & \\ 0 & & & \end{bmatrix} \begin{bmatrix} f \\ -a_1 \\ \vdots \\ -a_{2m} \end{bmatrix} = \begin{bmatrix} z \\ -c_1 \\ \vdots \\ -c_{2m} \end{bmatrix}. \tag{9}$$

Hence, Cramer's rule gives the useful formula

$$f = \frac{1}{\det(\Delta)} \det \begin{bmatrix} z & \varphi_1 & \cdots & \varphi_{2m} \\ -c_1 & & & \\ \vdots & & \boxed{\Delta} & \\ -c_{2m} & & & \end{bmatrix}. \tag{10}$$

2.3. The Physical Problem

Motivated by physical applications, we now specialize to semi-homogenous Sturm-Liouville problems where the collections of functionals $\{l_i : i = 1, \dots, 2m\}$ corresponds to a particular choice of conditions at the nodes of $\overline{\Gamma}$.

Consider the operator $\mathcal{L}[f]$ acting on the set

$$\mathcal{E}(\Gamma) = \mathcal{D}_p^2(\Gamma) \cap \mathcal{C}(\overline{\Gamma}) \cap F_\rho(\Gamma) \cap B_D(\Gamma), \tag{11}$$

where $B_D(\Gamma)$ and $F_\rho(\Gamma)$ contain the boundary and weighted flux matching conditions respectively:

$$B_D = \{f \in \mathcal{C}(\overline{\Gamma}) : f(\boldsymbol{n}) = 0,\ \boldsymbol{n} \in \partial\Gamma\}, \tag{12}$$

$$F_\rho(\Gamma) = \left\{ f \in \mathcal{C}^1(\Gamma) : \sum_{e \in (\boldsymbol{n})} \rho_e f_e'^b(\boldsymbol{n}) = 0,\ \boldsymbol{n} \in I(\Gamma) \right\}. \tag{13}$$

The function ρ in (12) is assumed constant on edges and strictly positive. Other linear boundary conditions types than Dirichlet can be considered without major changes to the arguments that follow.

The conditions encoded in $\mathcal{E}(\Gamma)$ can be cast in terms of linear functionals: let $\boldsymbol{n} \in I(\Gamma)$ with $i(\boldsymbol{n}) = \{e_1, \dots, e_k\}$, and define the functionals

$$\widetilde{l}_{\boldsymbol{n},i}[f] = f_{e_{i+1}}(\boldsymbol{n}) - f_{e_i}(\boldsymbol{n}), \qquad i = 1, \dots, k-1, \tag{14}$$

$$\widetilde{l}_{\boldsymbol{n},k}[f] = \sum_{i=1}^{k} \rho_{e_i} f'^{b}_{e_i}(\boldsymbol{n}). \tag{15}$$

For a boundary node $\boldsymbol{n}$ located at the endpoint of edge e, define simply

$$\widetilde{l}_{\boldsymbol{n}}[f] = f_e(\boldsymbol{n}). \tag{16}$$

Relabeling gives a collection of $2m$ functionals $\{l_1, \dots, l_{2m}\}$, such that the problem of finding $f \in \mathcal{E}(\Gamma)$ satisfying $\mathcal{L}[f] = h$ can be written as the semi-homogenous problem (8).

3. Construction of the Green's Function

The goal is to arrive to a formula for the solution to problem (8). The solution to the associated non-homogenous problem will then follow from (10).

Definition 2. A Green's function for operator $\mathcal{L}$ is a function $G : \Gamma \times \Gamma \to \mathbb{R}$ such that for all $h \in \operatorname{Ran}(\mathcal{L})$, the solution to problem (8) is given by

$$f(x) = \int_\Gamma G(x, y) h(y)\, dy. \tag{17}$$

The operator $\mathcal{G} : h \mapsto \int_\Gamma G(x, y) h(y)\, dy$ is called the Green's operator.

The first step is elementary and consists on verifying properties of the Wronskian of functions on Γ. For $f, g \in \mathcal{C}^1(\Gamma)$ the Wronskian $W[f, g]$ is defined on an edge e of Γ as

$$W[f, g]_e = f_e g'_e - g_e f'_e. \tag{18}$$

Lemma 3. *Let f, g, h be functions in $\mathcal{C}^1(\Gamma)$.*

a) If $f, g \in \mathcal{D}^2_p(\Gamma)$ then Lagrange's identity holds on each edge e,

$$f_e \mathcal{L}[g]_e - g_e \mathcal{L}[f]_e = -\frac{d}{=}\, dx\big(p_e W[f, g]_e\big). \tag{19}$$

b) If $f, g \in B_D$, then $W[f, g] \in B_D(\Gamma)$.

c) *If* $f, g \in \mathcal{C}(\overline{\Gamma}) \cap F_\rho(\Gamma)$*, then* $\sum_{e\in i(\boldsymbol{n})} \rho_e W[f,g]_e(\boldsymbol{n}) = 0$ *for all* $\boldsymbol{n} \in I(\Gamma)$*. Here, the derivatives in the definition* W *at* $\boldsymbol{n}$ *are replaced by boundary derivatives.*

d) $hW[f,g] - fW[h,g] = gW[f,h]$.

e) *If* $f, g \in \mathcal{D}^2_p(\Gamma)$ *with* $\mathcal{L}[f] = \mathcal{L}[g] = 0$ *on some edge, then* $pW[f,g]$ *is constant there.*

***Proof*.** Statement a) follows from a simple calculation and b) is obvious. For c), it suffices to use continuity and change derivatives to boundary derivatives,

$$\sum_{e\in i(\boldsymbol{n})} \rho_e W[f_e, g_e](\boldsymbol{n}) = f(\boldsymbol{n}) \sum_{e\in i(\boldsymbol{n})} \rho_e g_e'^b(\boldsymbol{n}) + g(\boldsymbol{n}) \sum_{e\in i(\boldsymbol{n})} \rho_e f_e'^b(\boldsymbol{n}) = 0.$$

d) is obtained by rearranging terms. To prove e), compute $(p_e W[f,g]_e)' = f_e(p_e g_e')' - g_e(p_e f_e')'$, use $(p_e f_e')' = -q_e f_e$ and $(p_e g_e')' = -q_e g_e$ to finally get $(p_e W[f,g]_e)' = 0$. ☑

For the following definition, and subsequent formulas, assume without loss of generality that the parametrization of Γ is such that for all $\boldsymbol{n} \in \partial\Gamma$, we have $\boldsymbol{n} = (e, l_e)$, where e is the edge to which n belongs.

Definition 4. Refer to Figure 1. Let $(e,x) \in \Gamma$,

(1) The two connected components of $\overline{\Gamma} \setminus \{(e,x)\}$ are denoted $\overline{\Gamma}(e,x)$ and $\overline{\Lambda}(e,x)$ respectively. The point (e,x) is adjoined as a boundary node to $\overline{\Gamma}(e,x)$ and $\overline{\Lambda}(e,x)$. By convention, $\overline{\Gamma}(e,x)$ is taken as the tree that contains the node $(e,0)$. There is an edge denoted by e in both $\overline{\Gamma}(e,x)$ and $\overline{\Lambda}(e,x)$; it is parametrized as the intervals $(0,x)$ and (x,l_e) respectively.

(2) For an edge e, $\overline{\Gamma}(e) := \cup_{x\in(0,l_e)} \overline{\Gamma}(e,x)$, and $\overline{\Lambda}(e) := \cup_{x\in(0,l_e)} \overline{\Lambda}(e,x)$.

(3) As in the case of the full tree, $\Gamma(e,x)$ – without the bar – denotes the collection of points inside edges of $\overline{\Gamma}(e,x)$. Similarly for $\Lambda(e,x), \Gamma(e), \Lambda(e)$.

(4) $\mathcal{E}_0(\Gamma(e))$ is the set of functions $f \in \mathcal{D}^2_p(\Gamma(e)) \cap \mathcal{C}(\overline{\Gamma}(e)) \cap F_\rho(\Gamma(e))$ such that $f(\boldsymbol{n}) = 0$ for $\boldsymbol{n} \in \partial\Gamma(e) \setminus \{(e,l_e)\}$.

(5) Similarly, $\mathcal{E}_0(\Lambda(e))$ is comprised of functions $f \in \mathcal{D}^2_p(\Lambda(e)) \cap \mathcal{C}(\overline{\Lambda}(e)) \cap F_\rho(\Lambda(e))$ such that $f(\boldsymbol{n}) = 0$ for $\boldsymbol{n} \in \partial\Lambda(e) \setminus \{(e,0)\}$.

Lemma 5. *Let* e *be a fixed edge. If problem* (8) *is non-degenerate, then there exist solutions* $\psi^{\Gamma(e)} \in \mathcal{E}_0(\Gamma(e))$, $\psi^{\Lambda(e)} \in \mathcal{E}_0(\Lambda(e))$ *to* $\mathcal{L}\psi^{\Gamma(e)} = 0$ *on* $\Gamma(e)$*, and* $\mathcal{L}\psi^{\Lambda(e)} = 0$ *on* $\Lambda(e)$*. These functions can further be chosen such that* $pW[\psi^{\Gamma(e)}, \psi^{\Lambda(e)}] = -1$ *on* e*.*

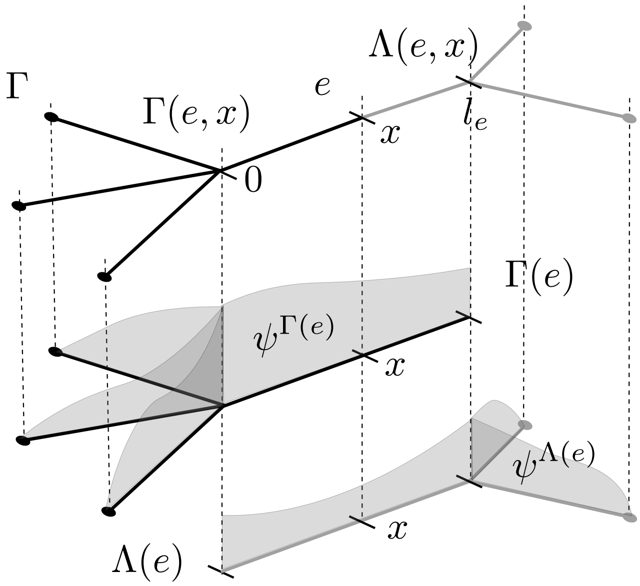

FIGURE 1. Schematic representation of a tree Γ with six edges. For e and x as shown, the black sub-tree on the upper figure in $\Gamma(e,x)$, the gray sub-tree is $\Lambda(e,x)$. The middle and lower figures depict trees $\Gamma(e)$ and $\Lambda(e)$ respectively with a schematization of the functions $\psi^{\Gamma(e)}$ and $\psi^{\Lambda(e)}$.

Proof. If e contains no nodes in $\partial\Gamma$, choose a node $\boldsymbol{n}' \notin \partial\Gamma(e)$ and let e' be its edge. If e contains a node in $\partial\Gamma$, make $\boldsymbol{n}'$ equal to that node, and $e' = e$. Rearrange the basis $\{\varphi_1, \ldots, \varphi_{2m}\}$ such that φ_1 and φ_2 are supported on e', and the collection of functionals $\{l_1, \ldots, l_{2m}\}$ such that $l_1[f] = f(\boldsymbol{n}')$. Let Δ be the matrix defined in Section (2.2). By the nondegeneracy of problem (8), there exists a solution $a = (a_1, \ldots, a_{2m})$ to $\Delta a = \varepsilon^{(1)}$, where $\varepsilon^{(1)}$ denotes the $\mathbb{R}^{2m}$ vector that has one in the first coordinate, and zero elsewhere. The function $\psi = \sum_{i=1}^{2m} a_i \varphi_i$ is a solution in $\mathcal{D}_p^2(\Gamma) \cap \mathcal{C}(\overline{\Gamma}) \cap F_\rho(\Gamma)$ to $\mathcal{L}[\psi] = 0$ on all of Γ and such that $\psi(\boldsymbol{n}) = 0$ for $\boldsymbol{n} \in \partial(\Gamma) \setminus \{\boldsymbol{n}'\}$. The restriction of ψ to $\Gamma(e)$ serves as the required function $\psi^{\Gamma(e)}$. A similar construction applies for $\psi^{\Lambda(e)}$. By Lemma 3, $pW\left[\psi^{\Gamma(e)}, \psi^{\Lambda(e)}\right]$ is constant on e, and the desired normalization can be achieved if this constant is not zero. Assume on the contrary that $pW\left[\psi^{\Gamma(e)}, \psi^{\Lambda(e)}\right] = 0$ on e. Since the Wronskian vanishes, there is $k \neq 0$ such that $\psi_e^{\Gamma(e)} = k\psi_e^{\Lambda(e)}$. Then, the function $f := \psi^{\Gamma(e)} \mathbb{1}_{\Gamma(e)} + k\psi^{\Lambda(e)} \mathbb{1}_{\Gamma(e)^c}$ would be a solution to the homogenous problem (7) violating the assumption of non-degeneracy. ☑

***Remark* 6.** The computation of the $\psi^{\Lambda(e)}, \psi^{\Gamma(e)}$ can be performed quite inexpensively. For a boundary node $\boldsymbol{n}' = (e', l_{e'}) \in \partial\Gamma$, the solution ψ constructed in the proof of Lemma 5 can be restricted to define $\psi^{\Gamma(e)}$ for all nodes e such that either $e = e'$ or e' does not belong to $\Gamma(e)$. Similarly, it can be used to define $\psi^{\Lambda(e)}$ for all nodes e such that e' does not belong to $\Lambda(e)$. This implies that the linear system $\Delta a = \varepsilon^{(1)}$ has to be solved only $\#\partial\Gamma$ times.

The specific form of the Green's function can now be written.

Theorem 7. *Assume problem* (8) *is non-degenerate. The following function is a Green's function for operator* $\mathcal{L}$

$$G(x,y) = \frac{1}{\rho_e} \times \begin{cases} \psi^{\Gamma(e)}(y)\, \psi^{\Lambda(e)}(x), & y \in \Gamma(e,x) \\ \psi^{\Lambda(e)}(y)\, \psi^{\Gamma(e)}(x), & y \in \Lambda(e,x) \end{cases}, \qquad x \in e. \tag{20}$$

Moreover, this function is unique in the class of continuous functions on Γ *that are continuous with respect to the first variable.*

***Proof*.** Let $h \in \operatorname{Ran}(\mathcal{L})$, and $f \in \mathcal{E}(\Gamma)$ a solution to $\mathcal{L}[f] = h$. Fix an edge e, and $x \in e$. Applying Lagrange's identity (19) for $\psi^{\Gamma(e)}$ and f and integrating over $\Gamma(e,x)$ with respect to the measure ρ gives

$$\int\limits_{\Gamma(e,x)} \psi^{\Gamma(e)} h \, d\rho = - \sum_{a \subset \Gamma(e,x)} \Big(p_a W \big[\psi^{\Gamma(e)}, f\big]_a \, \rho_a \Big) \Big|_0^{l_a},$$

where the sum on the right hand side is taken over all edges a of $\Gamma(e,x)$. Parts b) and c) of Lemma 3 ensure that all terms in the sum cancel except for the value at (e,x),

$$\int\limits_{\Gamma(e,x)} \psi^{\Gamma(e)} h \, d\rho = -p_e(x) \rho_e W \big[\psi^{\Gamma(e)}, f\big]_e(x). \tag{21}$$

Similarly, Lagrange's identity for $\psi^{\Lambda(e)}$ and f, gives

$$\int\limits_{\Lambda(e,x)} \psi^{\Lambda(e)} h \, d\rho = p_e(x) \rho_e W \big[\psi^{\Lambda(e)}, f\big]_e(x). \tag{22}$$

Multiply equations (21) and (22) by $\psi^{\Lambda(e)}(x)$ and $\psi^{\Gamma(e)}(x)$ respectively, add the resulting equations, and apply part d) of Lemma 3 to the right hand side of the result. Finally, since $pW[\psi^{\Gamma(e)}, \psi^{\Lambda(e)}] = -1$ on e,

$$\int_{\Gamma(e,x)} \psi^{\Lambda(e)}(x)\, \psi^{\Gamma(e)}(y) h(y)\, d\rho(y) + \int_{\Lambda(e,x)} \psi^{\Gamma(e)}(x)\, \psi^{\Lambda(e)}(y) h(y)\, d\rho(y) = f_e(x)\rho_e. \quad (23)$$

Since Γ is a disjoint union of $\Gamma(e,x)$ and $\Lambda(e,x)$, the function $G(x,y)$ defined in (20) satisfies Definition (2). Let $h \in C(\Gamma)$ be arbitrary. It will be established now that $h \in \operatorname{Ran}(\mathcal{L})$ simply by showing that $f := \mathcal{G}h$ solves $\mathcal{L}[f] = h$. Write

$$f(x) = \psi^{\Lambda(e)}(x) \int_{\Gamma(e)\smallsetminus e} \psi^{\Gamma(e)} h\, d\rho + \psi^{\Gamma(e)}(x) \int_{\Lambda(e)\smallsetminus e} \psi^{\Lambda(e)} h\, d\rho + \psi^{\Lambda(e)}(x) \int_0^x \psi^{\Gamma(e)} h\, d\rho + \psi^{\Gamma(e)}(x) \int_x^{l_e} \psi^{\Lambda(e)} h\, d\rho. \quad (24)$$

Applying $\mathcal{L}$ to the first two terms in (24) gives zero since $\mathcal{L}\big[\psi^{\Lambda(e)}\big] = \mathcal{L}\big[\psi^{\Lambda(e)}\big] = 0$. A routine calculation finally shows that

$$\mathcal{L}[f] = -hpW\big[\psi^{\Gamma(e)}, \psi^{\Lambda(e)}\big]_e + \mathcal{L}\big[\psi^{\Lambda(e)}\big] \int_0^x \psi^{\Gamma(e)} h\, d\rho + \mathcal{L}\big[\psi^{\Gamma(e)}\big] \int_x^{l_e} \psi^{\Lambda(e)} h\, d\rho$$

which yields $\mathcal{L}[f] = h$. Lastly, the non-degeneracy of problem (8) and the fact that $\operatorname{Ran}(\mathcal{L}) = \mathcal{C}(\Gamma)$, imply the uniqueness of G as stated in the theorem. ☑

***Remark* 8.** The construction of the Green's function in Theorem 7 has one particular important advantage over the one proposed by [7]. In that work, $G(x,y)$ is given as

$$G(x,y) = H(x,y) - \sum_{i=1}^{2m} l_i\big[H(\cdot,y)\big]\eta_i(x) \quad (25)$$

where $H(x,y)$ is equal to the Green's function of operator $\mathcal{L}$ on $(0,l_e)$ if $x, y \in e$, and equal to zero whenever x and y belong to different edges. The functions η_i are solutions to $\mathcal{L}[\eta_i] = 0$, $l_j[\eta_i] = \delta_{ij}$. Note that this formula requires solving $\Delta a = \varepsilon^{(1)}$ a total of $2m$ times to compute $G(x,y)$ at any given single pair of points (e_x, x), (e_y, y) of Γ. Via formula (20), one needs only the functions $\psi^{\Lambda(e_x)}$ and $\psi^{\Gamma(e_x)}$ and therefore, the system $\Delta a = \varepsilon^{(1)}$ must be solved only twice. On the other hand, formula (25) has the advantage of using H, which is a diagonal fundamental solution to $\mathcal{L}[f] = h$.

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Escuela de Matemáticas
Universidad Nacional de Colombia,
sede Medellín
Calle 59A No 63-20, Bloque 43,
Medellín, Colombia.
e-mail: jmramirezo@unal.edu.co