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Cardinality of Sets Associated to B_3 and B_4 Sets

Cardinales de conjuntos asociados a conjuntos B_3 y B_4

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Abstract. A subset $\mathcal{A}$ of a finite abelian group $(G,+)$ is called a B_h set on G if all sums of h elements of $\mathcal{A}$ are different. In this paper we state closed formulas for the cardinality of some sets associated with B_3 and B_4 sets, and we analyze implications for the largest cardinality of a B_h set on G.

Key words and phrases. Sidon sets, B_h sets.

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Resumen. Un subconjunto $\mathcal{A}$ de un grupo abeliano finito $(G,+)$ se llama un conjunto B_h sobre G, si todas las sumas de h elementos de $\mathcal{A}$ son distintas. En este trabajo se establecen fórmulas cerradas para el cardinal de algunos conjuntos asociados a conjuntos B_3 y B_4, y se analizan implicaciones relacionadas con el máximo cardinal que puede tener un conjunto B_h sobre G.

Palabras y frases clave. Conjuntos de Sidon, conjuntos B_h.

1. Introduction and Notation

Let h be a positive integer, $h \geq 2$, and let $\mathcal{A} = \{a_1, a_2, \dots, a_k\}$ be a subset of a finite abelian group $(G,+)$. We say that $\mathcal{A}$ is a B_h *set* on G or $\mathcal{A}$ belongs to the class B_h/G, if all sums of h elements of $\mathcal{A}$ are different in G.

When there is no confusion, we say a B_h set instead of a B_h set on G.

Note that $\mathcal{A} \in B_h$ if and only if the sums

$$a_{i_1} + a_{i_2} + \cdots + a_{i_h}, \tag{1}$$

with $a_{i_1}, a_{i_2}, \dots, a_{i_h} \in \mathcal{A}$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_h \leq k$ are all different.

If A and B are subsets of G, then the sum set and the difference set are defined as usual; i.e.,

$$A + B \coloneqq \{a + b : a \in A,\, b \in B\},$$
$$A - B \coloneqq \{a - b : a \in A,\, b \in B\}.$$

We also define the negative set of A as

$$-A \coloneqq \{-a : a \in A\}.$$

When $B = A$, we write $2A = A+A$. Throughout this paper, for any $r \in \mathbb{Z}^+$, we use the following notation:

$$\begin{aligned}
r \cdot A &\coloneqq \{ra : a \in A\},\\
rA &\coloneqq \{a_1 + \cdots + a_r : a_1, \ldots, a_r \in A\},\\
\widehat{r}A &\coloneqq \{a_1 + \cdots + a_r : a_1, \ldots, a_r \in A, \text{ all of them different}\}.
\end{aligned}$$

In addition to this, if $s \in \mathbb{Z}^+$, we define

$$\begin{aligned}
\widehat{r}A \oplus \widehat{s}A \coloneqq \big\{&(a_1 + \cdots + a_r) + (a_1' + \cdots + a_s') :\ a_i, a_j' \in A,\\
&\big|\{a_1, \ldots, a_r\}\big| = r,\ \big|\{a_1', \ldots, a_s'\}\big| = s,\\
&\{a_1, \ldots, a_r\} \cap \{a_1', \ldots, a_s'\} = \varnothing\big\},\\
\widehat{r}A \ominus \widehat{s}A \coloneqq \big\{&(a_1 + \cdots + a_r) - (a_1' + \cdots + a_s') :\ a_i,\, a_j' \in A,\\
&\big|\{a_1, \ldots, a_r\}\big| = r,\ \big|\{a_1', \ldots, a_s'\}\big| = s,\\
&\{a_1, \ldots, a_r\} \cap \{a_1', \ldots, a_s'\} = \varnothing\big\},\\
r \cdot A \oplus s \cdot A \coloneqq \big\{&(ra) + (sa') :\ a, a' \in A,\ a \neq a'\big\},\\
r \cdot A \ominus s \cdot A \coloneqq \big\{&(ra) - (sa') :\ a, a' \in A,\ a \neq a'\big\},\\
\widehat{r}A \oplus s \cdot A \coloneqq \big\{&(a_1 + \cdots + a_r) + (sa') :\ a_1, \ldots, a_r, a' \in A\\
&\big|\{a_1, \ldots, a_r\}\big| = r,\ a' \notin \{a_1, \ldots, a_r\}\big\},\\
\widehat{r}A \ominus s \cdot A \coloneqq \big\{&(a_1 + \cdots + a_r) - (sa') :\ a_1, \ldots, a_r, a' \in A\\
&\big|\{a_1, \ldots, a_r\}\big| = r,\ a' \notin \{a_1, \ldots, a_r\}\big\}.
\end{aligned}$$

So, if $\mathcal{A}$ is a B_h set, then

$$\big|h\mathcal{A}\big| = \binom{|\mathcal{A}| + h - 1}{h},$$

where $|\mathcal{X}|$ denotes the cardinality of a finite set $\mathcal{X}$.

The main problem in the study of B_h sets consists of stating the maximal cardinality of a B_h set on G. The natural step to follow is to analyze the asymptotic behavior of the function

$$f_h(G) \coloneqq \max\big\{|\mathcal{A}| : \mathcal{A} \in B_h/G\big\} \quad \text{as} \quad |G| \to \infty.$$

If G is a finite group with N elements, then we sometimes write $f_h(G) = f_h(N)$.

With the following counting we state an upper bound for the function $f_h(G)$. In fact, if $\mathcal{A}$ is a subset of G with $|\mathcal{A}| = k$, then there are $\binom{k+h-1}{h}$ sums of the form (1) in G, and if $\mathcal{A}$ is a B_h set on G, all these sums are different, so that

$$\frac{k^h}{h!} \leq \binom{k+h-1}{h} \leq |G|$$

from which we obtain the *trivial upper bound*

$$f_h(G) \leq (h!)^{1/h}|G|^{1/h}.$$

Hence

$$\limsup_{|G|\to\infty} \frac{f_h(G)}{\sqrt[h]{|G|}} \leq (h!)^{1/h}.$$

In this paper we state closed formulas for the cardinality of some sets related to B_3 and B_4 sets, and we analyze implications associated with the functions $f_3(G)$ and $f_4(G)$, being G a finite abelian group.

We start our study with the case $h = 2$, where the sets are well–known as Sidon sets.

2. The Case $h = 2$: Sidon sets

Let G be a finite commutative group and let $\mathcal{A}$ be a Sidon set on G with $|\mathcal{A}| = k$. Associated with $\mathcal{A}$ and $h = 2$, we have the sets

$$\begin{aligned}
2\mathcal{A} = \mathcal{A} + \mathcal{A} &\coloneqq \{a + b : a, b \in \mathcal{A}\},\\
\mathcal{A} - \mathcal{A} &\coloneqq \{a - b : a, b \in \mathcal{A}\},\\
\widehat{2}\mathcal{A} = \mathcal{A} \oplus \mathcal{A} &\coloneqq \{a + b : a, b \in \mathcal{A}, a \neq b\},\\
\mathcal{A} \ominus \mathcal{A} &\coloneqq \{a - b : a, b \in \mathcal{A}, a \neq b\}.
\end{aligned}$$

Since $a + b = c + d \Leftrightarrow a - d = c - b$, the Sidon sets are defined as such sets with the property that all non–zero differences of elements of that set are different. Thus, if $\mathcal{A}$ is a Sidon set, then

$$\begin{aligned}
|\mathcal{A} + \mathcal{A}| &= \binom{k+1}{2} = \binom{k}{2} + k,\\
|\mathcal{A} - \mathcal{A}| &= 2\binom{k}{2} + 1 = k(k-1) + 1,\\
|\mathcal{A} \oplus \mathcal{A}| &= \binom{k}{2},\\
|\mathcal{A} \ominus \mathcal{A}| &= 2\binom{k}{2} = 2|\mathcal{A} \oplus \mathcal{A}|.
\end{aligned}$$

These equalities can be applied to the study of the function $f_2(G)$. In this case we obtain the following result.

Theorem 1. *If $\mathcal{A}$ is a Sidon set on G, with $|\mathcal{A}| = k$ and $|G| = N$, then*

$$k(k-1) \leq N-1.$$

Proof. It follows from $|\mathcal{A} - \mathcal{A}| \leq |G|$. ☑

Observe from Singer's construction in [2] that we can obtain the equality

$$k(k-1) = N-1$$

for infinite values of N and k.

Corollary 2. *The function $f_2(G)$ satisfies*

$$\limsup_{|G|\to\infty} \frac{f_2(G)}{\sqrt{|G|}} \leq 1.$$

3. The Case $h = 3$: B_3 Sets

In this section, let G be a finite commutative group and let $\mathcal{A}$ be a B_3 set on G with $|\mathcal{A}| = k$. Associated with $\mathcal{A}$ and $h = 3$ we have the following sets

$$\begin{aligned}
&\mathcal{A} + \mathcal{A} + \mathcal{A} = 3\mathcal{A},\\
&\mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A} = \widehat{3}\mathcal{A},\\
&\mathcal{A} + \mathcal{A} - \mathcal{A} = 2\mathcal{A} - \mathcal{A},\\
&\mathcal{A} \oplus \mathcal{A} \ominus \mathcal{A} = \widehat{2}\mathcal{A} \ominus \mathcal{A},
\end{aligned}$$

and their negatives $-(3\mathcal{A})$, $-(\widehat{3}\mathcal{A})$, $\mathcal{A} - 2\mathcal{A}$, $\mathcal{A} \ominus \widehat{2}\mathcal{A}$, respectively. The idea is to determine closed formulas for the cardinality of such sets and use them to obtain upper bounds for the function $f_3(G)$.

From the definition of a B_3 set, we state the cardinality of the first two sets

$$\left|3\mathcal{A}\right| = \binom{k+2}{3} \qquad \text{and} \qquad \left|\widehat{3}\mathcal{A}\right| = \binom{k}{3}.$$

The following theorem shows us formulas for the cardinality of the sets $\widehat{2}\mathcal{A} \ominus \mathcal{A}$ and $2\mathcal{A} - \mathcal{A}$.

Theorem 3. *If $\mathcal{A}$ is a B_3 set on G with $|\mathcal{A}| = k$, then*

$$\begin{aligned}
\left|\widehat{2}\mathcal{A} \ominus \mathcal{A}\right| &= \binom{k}{2}(k-2) = 3\binom{k}{3} = 3\left|\widehat{3}\mathcal{A}\right|,\\
\left|2\mathcal{A} - \mathcal{A}\right| &= 3\binom{k}{3} + 2\binom{k}{2} + \binom{k}{1}.
\end{aligned}$$

Corollary 4. *The function $f_3(G)$ satisfies*

$$\limsup_{|G|\to\infty} \frac{f_3(G)}{\sqrt[3]{|G|}} \leq \sqrt[3]{2}.$$

Proof. Since $2\mathcal{A} - \mathcal{A} \subseteq G$, we get that

$$3\binom{k}{3} + 2\binom{k}{2} + \binom{k}{1} = |2\mathcal{A} - \mathcal{A}| \leq |G|,$$

and so,

$$\left(k - \frac{1}{3}\right)^3 \leq k\,(k^2 - k + 2) \leq 2|G|.$$

In particular

$$\left(f_3(G) - \frac{1}{3}\right)^3 \leq 2|G|.$$

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3.1. Proof of Theorem 3

In order to prove Theorem 3, we need some auxiliary lemmas.

Lemma 5. *Let G be a group and let $\mathcal{A}$ be a subset of G. Then*

$$\begin{aligned} 2\mathcal{A} &= \widehat{2}\mathcal{A} \cup 2 \cdot \mathcal{A}, \\ 2\mathcal{A} - \mathcal{A} &= \big(\widehat{2}\mathcal{A} \ominus \mathcal{A}\big) \cup (2 \cdot \mathcal{A} \ominus \mathcal{A}) \cup \mathcal{A}, \\ 3\mathcal{A} &= \widehat{3}\mathcal{A} \cup (2 \cdot \mathcal{A} \oplus \mathcal{A}) \cup 3 \cdot \mathcal{A}. \end{aligned}$$

Proof. Note that the sets of the right hand sides are contained in the sets of the left hand sides. Since the other containments are easy to prove, we just show one case. For example, in the second formula, if $x \in 2\mathcal{A} - \mathcal{A}$, then $x = a + b - c$ for some $a, b, c \in \mathcal{A}$. If a, b, and c are all different, then $x \in \widehat{2}\mathcal{A} \ominus \mathcal{A}$; if $a = b \neq c$, then $x \in 2 \cdot \mathcal{A} \ominus \mathcal{A}$, while in other cases $x \in \mathcal{A}$. ☑

Lemma 6. *If $\mathcal{A}$ is a B_3 set, then the unions in each one of the expressions in Lemma 5 are pairwise disjoint.*

Proof. Since $\mathcal{A}$ is a B_3 set, it follows that $\mathcal{A}$ is a B_2 set, thus $\big(\widehat{2}\mathcal{A}\big) \cap (2 \cdot \mathcal{A}) = \varnothing$. Using the definition of a B_3 set, the proof that the other sets are pairwise disjoint is not difficult, so we just prove one of them. We shall prove, for instance, that $\big(\widehat{2}\mathcal{A} \ominus \mathcal{A}\big) \cap (2 \cdot \mathcal{A} \ominus \mathcal{A}) = \varnothing$. Indeed, if $x \in \big(\widehat{2}\mathcal{A} \ominus \mathcal{A}\big) \cap (2 \cdot \mathcal{A} \ominus \mathcal{A})$, then $x = (a + b) - c = (d + d) - e$ for some $a, b, c \in \mathcal{A}$ all of them different, and $d, e \in \mathcal{A}$ with $d \neq e$. Therefore $a + b + e = d + d + c$, and because $\mathcal{A}$ is a B_3 set, we get that $\{a, b, e\} = \{d, c\}$; that is, $e = c$ and $a = b = d$, which is not possible. ☑

Proof of Theorem 3. Using Lemma 5 and Lemma 6, it is enough to see that if $\mathcal{A}$ is a B_3 set with cardinality k, then

$$\begin{aligned} \left|\widehat{2}\mathcal{A} \ominus \mathcal{A}\right| &= \binom{k}{2}(k-2) = 3\binom{k}{3}, \\ |2 \cdot \mathcal{A} \ominus \mathcal{A}| &= k(k-1) = 2\binom{k}{2}. \end{aligned}$$

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So far we have shown closed formulas for all the sets with $h = 3$ iterations of sum sets and difference sets, related to a B_3 set. Note that by definition of B_3 sets, we have

$$|3\mathcal{A}| = \binom{k+2}{3} = |-3\mathcal{A}|,$$

and using Theorem 3 we get that

$$|2\mathcal{A} - \mathcal{A}| = 3\binom{k}{3} + 2\binom{k}{2} + \binom{k}{1} = |\mathcal{A} - 2\mathcal{A}|.$$

Finally, the last equality of Lemma 5 and the pairwise disjoint of the sets involved there, imply that

$$\begin{aligned} \binom{k+2}{3} = |3\mathcal{A}| &= \left|\widehat{3}\mathcal{A} \cup (2 \cdot \mathcal{A} \oplus \mathcal{A}) \cup 3 \cdot \mathcal{A}\right| \\ &= \left|\widehat{3}\mathcal{A}\right| + |2 \cdot \mathcal{A} \oplus \mathcal{A}| + |3 \cdot \mathcal{A}| \\ &= \binom{k}{3} + 2\binom{k}{2} + \binom{k}{1}. \end{aligned}$$

4. The Case $h = 4$: B_4 Sets

Similarly as before, let G be a finite commutative group and let $\mathcal{A}$ be a B_4 set on G with cardinality k. Associated with $\mathcal{A}$ and $h = 4$ we get the following sets

$$4\mathcal{A}, \qquad \widehat{4}\mathcal{A}, \qquad 2\mathcal{A} - 2\mathcal{A}, \qquad \widehat{2}\mathcal{A} \ominus \widehat{2}\mathcal{A}, \qquad 3\mathcal{A} - \mathcal{A}, \qquad \widehat{3}\mathcal{A} \ominus \mathcal{A}, \tag{2}$$

and their negatives $-(4\mathcal{A})$, $-\big(\widehat{4}\mathcal{A}\big)$, $\mathcal{A} - 3\mathcal{A}$, $\mathcal{A} \ominus \widehat{3}\mathcal{A}$. For each one of them, in this section we show closed formulas for their cardinality and we use them to determine upper bounds for the function $f_4(G)$. Using the definition of a B_4 set, we can state formulas for the cardinality of the sets $4\mathcal{A}$ and $\widehat{4}\mathcal{A}$ immediately as follows:

$$|4\mathcal{A}| = \binom{k+3}{4} \qquad \text{and} \qquad \left|\widehat{4}\mathcal{A}\right| = \binom{k}{4}.$$

In the following theorem we establish closed formulas for the cardinality of the other sets given in (2).

Theorem 7. *If $\mathcal{A}$ is a B_4 set on G with $|\mathcal{A}| = k$, then*

$$\begin{aligned}
|2\mathcal{A} - 2\mathcal{A}| &= 6\binom{k}{4} + 6\binom{k}{3} + 4\binom{k}{2} + 1,\\
|3\mathcal{A} - \mathcal{A}| &= 4\binom{k}{4} + 6\binom{k}{3} + 3\binom{k}{2} + \binom{k}{1},\\
\left|\widehat{2}\mathcal{A} \ominus \widehat{2}\mathcal{A}\right| &= 6\binom{k}{4} = 6\left|\widehat{4}\mathcal{A}\right|,\\
\left|\widehat{3}\mathcal{A} \ominus \mathcal{A}\right| &= 4\binom{k}{4} = 4\left|\widehat{4}\mathcal{A}\right|.
\end{aligned}$$

Corollary 8. *The function $f_4(G)$ satisfies*

$$\limsup_{|G|\to\infty} \frac{f_4(G)}{\sqrt[4]{|G|}} \le \sqrt{2}.$$

Proof. Since $2\mathcal{A} - 2\mathcal{A} \subseteq G$, then

$$6\binom{k}{4} + 6\binom{k}{3} + 4\binom{k}{2} + 1 = |2\mathcal{A} - 2\mathcal{A}| \le |G|.$$

Hence

$$k(k-1)\big(k^2 - k + 6\big) \le 4(|G| - 1),$$

from which we obtain the desired result. ☑

4.1. Proof of Theorem 7

To prove Theorem 7 we need to use the following auxiliary lemmas.

Lemma 9. *Let G be a group and let $\mathcal{A}$ be a subset of G. Then*

$$\begin{aligned}
&2\mathcal{A} - 2\mathcal{A} =\\
&\qquad \big(\widehat{2}\mathcal{A} \ominus \widehat{2}\mathcal{A}\big) \cup \big(\pm\big(\widehat{2}\mathcal{A} \ominus 2\cdot\mathcal{A}\big)\big) \cup (2\cdot\mathcal{A} \ominus 2\cdot\mathcal{A}) \cup (\mathcal{A} \ominus \mathcal{A}) \cup \{0\} \qquad (3)
\end{aligned}$$

$$3\mathcal{A} - \mathcal{A} = \big(\widehat{3}\mathcal{A} \ominus \mathcal{A}\big) \cup (2\cdot\mathcal{A} \oplus \mathcal{A} \ominus \mathcal{A}) \cup (3\cdot\mathcal{A} \ominus \mathcal{A}) \cup (\mathcal{A} \oplus \mathcal{A}) \cup (2\cdot\mathcal{A}),$$

where

$$2\cdot\mathcal{A} \oplus \mathcal{A} \ominus \mathcal{A} := \{2a + b - c : a, b, c \in \mathcal{A}, \text{ all of them different}\}.$$

Furthermore

$$4\mathcal{A} = \widehat{4}\mathcal{A} \cup \big(\widehat{2}\mathcal{A} \oplus 2\cdot\mathcal{A}\big) \cup (\mathcal{A} \oplus 3\cdot\mathcal{A}) \cup (2\cdot\mathcal{A} \oplus 2\cdot\mathcal{A}) \cup (4\cdot\mathcal{A}). \qquad (4)$$

Proof. Clearly, the sets at all right hand sides are contained in the sets of the corresponding left hand sides. The other containments are not hard to prove. In fact, for (3), if $x \in 2\mathcal{A} - 2\mathcal{A}$, then $x = (a+b) - (c+d)$ for some a, b, c and $d \in \mathcal{A}$. We now consider all possible cases for the elements a, b, c, and d.

If a, b, c, and d are all different, then $x \in \widehat{2}\mathcal{A} \ominus \widehat{2}\mathcal{A}$; if $a \neq b$ and $a, b \neq c = d$, then $x \in \widehat{2}\mathcal{A} \ominus 2 \cdot \mathcal{A}$; if $a = b \neq c, d$ with $c \neq d$, then $x \in -\big(\widehat{2}\mathcal{A} \ominus 2 \cdot \mathcal{A}\big)$; if $a = b \neq c = d$, then $x \in 2 \cdot \mathcal{A} \ominus 2 \cdot \mathcal{A}$; if $a \neq b$ and $c \neq d$ with $|\{a, b\} \cap \{c, d\}| = 1$, then $x \in \mathcal{A} \ominus \mathcal{A}$. Finally, if $a = b = c = d$, then $x = 0$, so, the proof of this equality is finished. A similar argument shows the other equalities. ☑

Lemma 10. *If $\mathcal{A}$ is a B_4 set, then the unions in each one of the expressions in Lemma 9 are pairwise disjoint.*

Proof. We show that $\big(\widehat{2}\mathcal{A} \ominus \widehat{2}\mathcal{A}\big) \cap (2 \cdot \mathcal{A} \ominus 2 \cdot \mathcal{A}) = \varnothing$. The other cases are proven in a similar way.

If $x \in \big(\widehat{2}\mathcal{A} \ominus \widehat{2}\mathcal{A}\big) \cap (2 \cdot \mathcal{A} \ominus 2 \cdot \mathcal{A})$, then $x = (a+b) - (c+d) = (e+e) - (f+f)$ for some a, b, c, and d in $\mathcal{A}$ all of them different, and $e, f \in \mathcal{A}$, $e \neq f$. Therefore $a + b + f + f = e + e + c + d$, but $\mathcal{A}$ is B_4, so $\{a, b, f\} = \{e, c, d\}$, which is not possible. ☑

Proof of Theorem 7. Suppose $\mathcal{A} \in B_4$ with cardinality k. By Lemma 9 and Lemma 10 we get that:

$$\begin{aligned}
\big|\widehat{2}\mathcal{A} \ominus \widehat{2}\mathcal{A}\big| &= \binom{k}{2}\binom{k-2}{2} = 6\binom{k}{4}, \\
\big|\widehat{2}\mathcal{A} \ominus 2 \cdot \mathcal{A}\big| &= \binom{k}{2}(k-2) = 3\binom{k}{3} = \big|2 \cdot \mathcal{A} \ominus \widehat{2}\mathcal{A}\big|, \\
|2 \cdot \mathcal{A} \ominus 2 \cdot \mathcal{A}| &= k(k-1) = 2\binom{k}{2} = |\mathcal{A} \ominus \mathcal{A}|.
\end{aligned}$$ ☑

Thus we have given formulas for all the sets with $h = 4$ iterations of sum sets or difference sets, associated with a B_4 set. By the definition of a B_4 set and Theorem 7, we have

$$|4\mathcal{A}| = \binom{k+3}{4} = |-4\mathcal{A}|$$

and

$$\begin{aligned}
|2\mathcal{A} - 2\mathcal{A}| &= 6\binom{k}{4} + 6\binom{k}{3} + 4\binom{k}{2} + 1, \\
|3\mathcal{A} - \mathcal{A}| &= 4\binom{k}{4} + 6\binom{k}{3} + 3\binom{k}{2} + \binom{k}{1},
\end{aligned}$$

respectively. On the other hand, by (3) in Lemma 9 and because of the fact that the unions are pairwise disjoint, we can state a formula for the cardinality of the set $4\mathcal{A}$. That is

$$\begin{aligned}\binom{k+3}{4} = |4\mathcal{A}| &= \left|\widehat{4}\mathcal{A} \cup (2\cdot\mathcal{A} \oplus \widehat{2}\mathcal{A}) \cup (3\cdot\mathcal{A}\oplus\mathcal{A}) \cup (2\cdot\mathcal{A}\oplus 2\cdot\mathcal{A}) \cup (4\cdot\mathcal{A})\right| \\ &= \left|\widehat{4}\mathcal{A}\right| + \left|2\cdot\mathcal{A}\oplus\widehat{2}\mathcal{A}\right| + |3\cdot\mathcal{A}\oplus\mathcal{A}| + |2\cdot\mathcal{A}\oplus 2\cdot\mathcal{A}| + |4\cdot\mathcal{A}| \\ &= \binom{k}{4} + \binom{k}{2}(k-2) + k(k-1) + \binom{k}{2} + \binom{k}{1} \\ &= \binom{k}{4} + 3\binom{k}{3} + 3\binom{k}{2} + \binom{k}{1}.\end{aligned}$$

5. Concluding Remarks

In this work we have shown exact formulas in order to determine the cardinality of some sets in the classes B_3 and B_4 on a finite abelian group G. Choosing those formulas that are maximal, we obtained the best upper bounds known for the extreme functions $f_3(G)$ and $f_4(G)$.

Note that there is still work to be done, some of which is presented as follows.

5.1. The General Case

What can be done to determine the cardinality of the set $s\mathcal{A} - t\mathcal{A}$, where s and t are positive integers and $\mathcal{A}$ is a B_h set, with $h = s + t$?. To answer this question, it is important to perform a similar work to that carried out in this work for B_3 and B_4 sets; that is, writing $s\mathcal{A} - t\mathcal{A}$ as a disjoint union and then studying the cardinality of each one of the sets appearing in that expression. So far, we have that if $\mathcal{A}$ is a B_{s+t} set with $|\mathcal{A}| = k$, then

$$\left|\widehat{s}A \ominus \widehat{t}A\right| = \binom{k}{s}\binom{k-s}{t} \leq |G|$$

so,

$$\frac{k!}{(k-s)!\,s!}\cdot\frac{(k-s)!}{(k-s-t)!\,t!} = \frac{k!}{(k-s-t)!\,s!\,t!} \leq |G|.$$

Hence

$$(k-s-t+1)^{s+t} \leq s!\,t!\,|G|. \tag{5}$$

If $s = t$, from (5) we obtain

$$(k-2s+1)^{2s} \leq (s!)^2\,|G|,$$

which implies that

$$f_{2s}(G) \leq (s!)^{1/s}|G|^{1/2s} + 2s - 1.$$

If $t = s - 1$, we can also use (5) to obtain

$$(k-2s+2)^{2s-1} \leq s!\,(s-1)!\,|G|.$$

In this case we find the upper bound

$$f_{2s-1}(G) \leq (s!(s-1)!)^{1/(2s-1)}|G|^{1/(2s-1)} + 2s - 2.$$

It is important to remark that based on the work of Jia [3], Chen [1] studied the function $f_h(\mathbb{Z}_N) = f_h(N)$ and he obtained the bounds

$$f_{2s}(N) \leq (s!)^2 N^{1/2s} + O(1), \qquad \text{and}$$
$$f_{2s-1}(N) \leq \big(s(s-1)!\big)^{1/(2s-1)} N^{1/(2s-1)} + O(1).$$

Our arguments generalize the results of Chen to finite commutative groups. Additionally to this, we make a proposal for the constants $O(1)$.

5.2. How Can We Improve Results in B_3 and B_4 Sets?

Let $\mathcal{A}$ be a B_3 set on G with cardinality k and $\mathcal{B} = \widehat{2}\mathcal{A} \ominus \mathcal{A}$. By Theorem 3 we know that

$$|\mathcal{B}| = |-\mathcal{B}| = \binom{k}{2}(k-2) = 3\binom{k}{3}.$$

However, the sets $\mathcal{B}$ and $-\mathcal{B}$ are not disjoint sets. In this sense it would be appropriate to obtain good upper bounds for the cardinality of the set $\mathcal{B}\cap(-\mathcal{B})$. Our computational evidence shows us that

$$\big|\mathcal{B}\cap(-\mathcal{B})\big| \leq \frac{|\mathcal{B}|}{2},$$

and if it were true, we would get that

$$\frac{9}{2}\binom{k}{3} \leq |\mathcal{B}| + |-\mathcal{B}| - |\mathcal{B}\cap(-\mathcal{B})| = |\mathcal{B}\cup(-\mathcal{B})| \leq |G|,$$

where we have also used the fact that $\mathcal{B}\cup(-\mathcal{B}) \subseteq G$. Thus

$$(k-2)^3 \leq k\,(k-1)(k-2) \leq \frac{4}{3}\,|G|,$$

and so,

$$f_3(G) \leq \left(\frac{4}{3}\right)^{1/3} |G|^{1/3} + 2.$$

With this argument we would improve the upper bound found in Corollary 4, being this the best upper bound known so far. For this reason, we propose the next conjecture.

Conjecture 11. The function $f_3(G)$ satisfies

$$\limsup_{|G|\to\infty} \frac{f_3(G)}{\sqrt[3]{|G|}} \leq \sqrt[3]{\frac{4}{3}}.$$

Finally, in the case of B_4 sets, we suggest finding upper bounds for the cardinality of the set

$$\big(\widehat{2}\mathcal{A} \ominus 2\mathcal{A}\big) \cap \big(\widehat{3}\mathcal{A} \ominus \mathcal{A}\big).$$

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