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Powers of Two in Generalized Fibonacci Sequences

Potencias de dos en sucesiones generalizadas de Fibonacci

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Abstract. The $k-$generalized Fibonacci sequence $\left(F_n^{(k)}\right)_n$ resembles the Fibonacci sequence in that it starts with $0, \ldots, 0, 1$ (k terms) and each term afterwards is the sum of the k preceding terms. In this paper, we are interested in finding powers of two that appear in $k-$generalized Fibonacci sequences; i.e., we study the Diophantine equation $F_n^{(k)} = 2^m$ in positive integers n, k, m with $k \geq 2$.

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Resumen. La sucesión $k-$generalizada de Fibonacci $\left(F_n^{(k)}\right)_n$ se asemeja a la sucesión de Fibonacci, pues comienza con $0, \ldots, 0, 1$ (k términos) y a partir de ahí, cada término de la sucesión es la suma de los k precedentes. El interés en este artículo es encontrar potencias de dos que aparecen en sucesiones $k-$generalizadas de Fibonacci; es decir, se estudia la ecuación Diofántica $F_n^{(k)} = 2^m$ en enteros positivos n, k, m con $k \geq 2$.

Palabras y frases clave. Números de Fibonacci, cotas inferiores para formas lineales en logaritmos de números algebraicos.

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1. Introduction

Let $k \geq 2$ be an integer. We consider a generalization of Fibonacci sequence called the $k-$*generalized Fibonacci sequence* $F_n^{(k)}$ defined as

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \tag{1}$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We call $F_n^{(k)}$ the n^{th} $k-$*generalized Fibonacci number*. For example, if $k = 2$, we obtain the classical Fibonacci sequence

$$F_0 = 0, \quad F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2.$$

$$(F_n)_{n\geq 0} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots\}.$$

If $k = 3$, the Tribonacci sequence appears

$$(T_n)_{n\geq 0} = \{0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, \ldots\}.$$

If $k = 4$, we get the Tetranacci sequence

$$\left(F_n^{(4)}\right)_{n\geq 0} = \{0, 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, \ldots\}.$$

There are many papers in the literature which address Diophantine equations involving Fibonacci numbers. For example, it is known that $1, 2, 8$ are the only powers of two that appear in our familiar Fibonacci sequence. One proof of this fact follows from Carmichael's Primitive Divisor theorem [3], which states that for n greater than 12, the n^{th} Fibonacci number F_n has at least one prime factor that is not a factor of any previous Fibonacci number.

We extend the above problem to the k–generalized Fibonacci sequences, that is, we are interested in finding out which powers of two are k–generalized Fibonacci numbers; i.e., we determine all the solutions of the Diophantine equation

$$F_n^{(k)} = 2^m, \tag{2}$$

in positive integers n, k, m with $k \geq 2$.

We begin by noting that the first $k+1$ non–zero terms in the $k-$generalized Fibonacci sequence are powers of two, namely

$$F_1^{(k)} = 1, \quad F_2^{(k)} = 1, \quad F_3^{(k)} = 2, \quad F_4^{(k)} = 4, \quad \ldots, \quad F_{k+1}^{(k)} = 2^{k-1}, \tag{3}$$

while the next term in the above sequence is $F_{k+2}^{(k)} = 2^k - 1$. Hence, the triples

$$(n, k, m) = (1, k, 0) \qquad \text{and} \qquad (n, k, m) = (t, k, t-2), \tag{4}$$

are solutions of equation (2) for all $2 \leq t \leq k+1$. Solutions given by (4) will be called *trivial solutions*.

2. Main Result

In this paper, we prove the following theorem.

Theorem 1. *The only nontrivial solution of the Diophantine equation* (2) *in positive integers n, k, m with $k \geq 2$, is $(n, k, m) = (6, 2, 3)$, namely $F_6 = 8$.*

Our method is roughly as follows. We use lower bounds for linear forms in logarithms of algebraic numbers to bound n polynomially in terms of k. When k is small, the theory of continued fractions suffices to lower such bounds and complete the calculations. When k is large, we use the fact that the dominant root of the $k-$generalized Fibonacci sequence is exponentially close to 2, so we can replace this root by 2 in our calculations and finish the job.

3. Preliminary Inequalities

It is known that the characteristic polynomial of the $k-$generalized Fibonacci numbers $\left(F_n^{(k)}\right)_n$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible over $\mathbb{Q}[x]$ and has just one root outside the unit circle. Throughout this paper, $\alpha := \alpha(k)$ denotes that single root, which is located between $2(1 - 2^{-k})$ and 2 (see [7]). To simplify notation, in general we omit the dependence on k of α.

The following "Binet–like" formula for $F_n^{(k)}$ appears in Dresden [4]:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}, \tag{5}$$

where $\alpha = \alpha_1, \ldots, \alpha_k$ are the roots of $\Psi_k(x)$. It was also proved in [4] that the contribution of the roots which are inside the unit circle to the formula (5) is very small, namely that the approximation

$$\left| F_n^{(k)} - \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1} \right| < \frac{1}{2} \qquad \text{holds for all} \qquad n \geq 2 - k. \tag{6}$$

We will use the estimate (6) later. Furthermore, in [1], we proved that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \qquad \text{for all} \qquad n \geq 1. \tag{7}$$

The following lemma is a simple result, which is a small variation of the right–hand side of inequality (7) and will be useful to bound m in terms of n.

Lemma 2. *For every positive integer $n \geq 2$, we have*

$$F_n^{(k)} \leq 2^{n-2}. \tag{8}$$

Moreover, if $n \geq k + 2$, then the above inequality is strict.

Proof. We prove the Lemma 2 by induction on n. Indeed, by recalling (3), we have that $F_t^{(k)} = 2^{t-2}$ for all $2 \leq t \leq k+1$, so it is clear that inequality (8) is true for the first k terms of n. Now, suppose that (8) holds for all terms $F_m^{(k)}$ with $m \leq n-1$ for some $n \geq k+2$. It then follows from (1) that

$$\begin{aligned} F_n^{(k)} &\leq 2^{n-3} + 2^{n-4} + \cdots + 2^{n-k-2} = 2^{n-k-2}\big(2^{k-1} + 2^{k-2} + \cdots + 1\big) \\ &= 2^{n-k-2}\big(2^k - 1\big) < 2^{n-2}. \end{aligned}$$

Thus, inequality (8) holds for all positive integers $n \geq 2$. ☑

Now assume that we have a nontrivial solution (n, k, m) of equation (2). By inequality (7) and Lemma 2, we have

$$\alpha^{n-2} \leq F_n^{(k)} = 2^m < 2^{n-2}.$$

So, we get

$$n \leq m\left(\frac{\log 2}{\log \alpha}\right) + 2 \qquad \text{and} \qquad m < n-2. \tag{9}$$

If $k \geq 3$, then it is a straightforward exercise to check that $1/\log\alpha < 2$ by using the fact that $2(1-2^{-k}) < \alpha$. If $k = 2$, then α is the golden section so $1/\log\alpha = 2.078\ldots < 2.1$. In any case, the inequality $1/\log\alpha < 2.1$ holds for all $k \geq 2$. Thus, taking into account that $\log 2/\log\alpha < 2.1\log 2 = 1.45\ldots < 3/2$, it follows immediately from (9) that

$$m + 2 < n < \frac{3}{2}m + 2. \tag{10}$$

We record this estimate for future referencing.

To conclude this section of preliminaries, we consider for an integer $s \geq 2$, the function

$$f_s(x) = \frac{x-1}{2+(s+1)(x-2)} \qquad \text{for} \qquad x > 2(1-2^{-s}). \tag{11}$$

We can easily see that

$$f_s'(x) = \frac{1-s}{(2+(s+1)(x-2))^2} \qquad \text{for all} \qquad x > 2(1-2^{-s}), \tag{12}$$

and $2 + (s+1)(x-2) \geq 1$ for all $x > 2(1-2^{-s})$ and $s \geq 3$. We shall use this fact later.

4. An Inequality for n and m in Terms of k

Since the solution to equation (2) is nontrivial, in the remainder of the article, we may suppose that $n \geq k+2$. So, we get easily that $n \geq 4$ and $m \geq 3$.

By using (2) and (6), we obtain that

$$\left|2^m - f_k(\alpha)\alpha^{n-1}\right| < \frac{1}{2}. \tag{13}$$

Dividing both sides of the above inequality by $f_k(\alpha)\alpha^{n-1}$, which is positive because $\alpha > 1$ and $2^k > k+1$, so $2 > (k+1)\bigl(2-(2-2^{-k+1})\bigr) > (k+1)(2-\alpha)$, we obtain the inequality

$$\left|2^m \cdot \alpha^{-(n-1)} \cdot (f_k(\alpha))^{-1} - 1\right| < \frac{2}{\alpha^{n-1}}, \tag{14}$$

where we used the facts $2 + (k+1)(\alpha - 2) < 2$ and $1/(\alpha - 1) \leq 2$, which are easily seen.

Recall that for an algebraic number η we write $h(\eta)$ for its logarithmic height whose formula is

$$h(\eta) := \frac{1}{d}\Biggl(\log a_0 + \sum_{i=1}^{d} \log\Bigl(\max\bigl\{|\eta^{(i)}|, 1\bigr\}\Bigr)\Biggr),$$

with d being the degree of η over $\mathbb{Q}$ and

$$f(X) := a_0 \prod_{i=1}^{d}\bigl(X - \eta^{(i)}\bigr) \in \mathbb{Z}[X] \tag{15}$$

being the minimal primitive polynomial over the integers having positive leading coefficient a_0 and η as a root.

With this notation, Matveev (see [6] or Theorem 9.4 in [2]) proved the following deep theorem.

Theorem 3. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$B \geq \max\bigl\{|b_1|, \ldots, |b_t|\bigr\},$$

and

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be real numbers such that

$$A_i \geq \max\bigl\{Dh(\gamma_i), |\log \gamma_i|, 0.16\bigr\}, \qquad i = 1, \ldots, t.$$

Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp\bigl(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t\bigr).$$

In order to apply Theorem 3, we take $t \coloneqq 3$ and

$$\gamma_1 \coloneqq 2, \qquad \gamma_2 \coloneqq \alpha, \qquad \gamma_3 \coloneqq f_k(\alpha).$$

We also take the exponents $b_1 \coloneqq m$, $b_2 \coloneqq -(n-1)$ and $b_3 \coloneqq -1$. Hence,

$$\Lambda \coloneqq \gamma_1^{b_1} \cdot \gamma_2^{b_2} \cdot \gamma_3^{b_3} - 1. \tag{16}$$

Observe that the absolute value of Λ appears in the left–hand side of inequality (14). The algebraic number field containing $\gamma_1, \gamma_2, \gamma_3$ is $\mathbb{K} \coloneqq \mathbb{Q}(\alpha)$. As α is of degree k over $\mathbb{Q}$, it follows that $D = [\mathbb{K} : \mathbb{Q}] = k$. To see that $\Lambda \neq 0$, observe that imposing that $\Lambda = 0$ yields

$$2^m = \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1}.$$

Conjugating the above relation by some automorphism of the Galois group of the splitting field of $\Psi_k(x)$ over $\mathbb{Q}$ and then taking absolute values, we get that for any $i > 1$,

$$2^m = \left| \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1} \right|.$$

But the above relation is not possible since its left–hand side is greater than or equal to 8, while its right–hand side is smaller than $2/(k-1) \leq 2$ because $|\alpha_i| < 1$ and

$$|2 + (k+1)(\alpha_i - 2)| \geq (k+1)|\alpha_i - 2| - 2 > k - 1. \tag{17}$$

Thus, $\Lambda \neq 0$.

Since $h(\gamma_1) = \log 2$, it follows that we can take $A_1 \coloneqq k \log 2$. Furthermore, since $h(\gamma_2) = (\log \alpha)/k < (\log 2)/k = (0.693147\cdots)/k$, it follows that we can take $A_1 \coloneqq 0.7$.

We now need to estimate $h(\gamma_3)$. First, observe that

$$h(\gamma_3) = h(f_k(\alpha)) = h\left(\frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \right). \tag{18}$$

Put

$$g_k(x) = \prod_{i=1}^{k} \left(x - \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \right) \in \mathbb{Q}[x].$$

Then the leading coefficient a_0 of the minimal polynomial of

$$\frac{\alpha - 1}{2 + (k+1)(\alpha - 2)}$$

over the integers (see definition (15)) divides $\prod_{i=1}^{k}(2+(k+1)(\alpha_i-2))$. But,

$$\left|\prod_{i=1}^{k}\big(2+(k+1)(\alpha_i-2)\big)\right| = (k+1)^k\left|\prod_{i=1}^{k}\left(2-\frac{2}{k+1}-\alpha_i\right)\right|$$
$$= (k+1)^k\left|\Psi_k\left(2-\frac{2}{k+1}\right)\right|.$$

Since

$$\big|\Psi_k(y)\big| < \max\left\{y^k, 1+y+\cdots+y^{k-1}\right\} < 2^k \quad \text{for all} \quad 0<y<2,$$

it follows that

$$a_0 \le (k+1)^k\left|\Psi_k\left(2-\frac{2}{k+1}\right)\right| < 2^k\,(k+1)^k.$$

Hence,

$$\begin{aligned} h\left(\frac{\alpha-1}{2+(k+1)(\alpha-2)}\right) &= \frac{1}{k}\left(\log a_0+\sum_{i=1}^{k}\log\max\left\{\left|\frac{\alpha_i-1}{2+(k+1)(\alpha_i-2)}\right|,1\right\}\right)\\ &< \frac{1}{k}\big(k\log 2+k\log(k+1)+k\log 2\big)\\ &= \log(k+1)+\log 4\\ &< 4\log k. \end{aligned} \tag{19}$$

In the above inequalities, we used the facts $\log(k+1)+\log 4 < 4\log k$ for all $k \ge 2$ and

$$\left|\frac{\alpha_i-1}{2+(k+1)(\alpha_i-2)}\right| < 2 \qquad \text{for all} \qquad 1 \le i \le k,$$

which holds because for $i>1$, $\big|2+(k+1)(\alpha_i-2)\big| > k-1 \ge 1$ (see (17)), and

$$2+(k+1)(\alpha-2) > \frac{85}{100} > \frac{1}{2},$$

which is a straightforward exercise to check using the fact that $2(1-2^{-k}) < \alpha < 2$ and $k \ge 2$.

Combining (18) and (19), we obtain that $h(\gamma_3) < 4\log k$, so we can take $A_3 := 4k\log k$. By recalling that $m < n-1$ from (10), we can take $B := n-1$. Applying Theorem 3 to get a lower bound for $|\Lambda|$ and comparing this with inequality (14), we get

$$\exp\big(-C(k)\times\big(1+\log(n-1)\big)\,(k\log 2)\,(0.7)\,(4k\log k)\big) < \frac{2}{\alpha^{n-1}},$$

where $C(k) := 1.4 \times 30^6 \times 3^{4.5} \times k^2 \times (1+\log k) < 1.5 \times 10^{11}\, k^2\, (1+\log k)$.

Taking logarithms in the above inequality, we have that

$$(n-1)\log\alpha - \log 2 < 3 \times 10^{11}\, k^4\, \log k\,(1+\log k)\,\bigl(1+\log(n-1)\bigr),$$

which leads to

$$n-1 < 3.68 \times 10^{12}\, k^4\, \log^2 k\, \log(n-1),$$

where we used the facts $1+\log k \le 3\log k$ for all $k \ge 2$, $1+\log(n-1) \le 2\log(n-1)$ for all $n \ge 4$ and $1/\log\alpha < 2.1$ for all $k \ge 2$.

Thus,

$$\frac{n-1}{\log(n-1)} < 3.68 \times 10^{12}\, k^4\, \log^2 k. \tag{20}$$

Since the function $x \mapsto x/\log x$ is increasing for all $x > e$, it is easy to check that the inequality

$$\frac{x}{\log x} < A \qquad \text{yields} \qquad x < 2A\,\log A,$$

whenever $A \ge 3$. Indeed, for if not, then we would have $x > 2A\log A > e$, therefore

$$\frac{x}{\log x} > \frac{2A\log A}{\log(2A\log A)} > A,$$

where the last inequality follows because $2\log A < A$ holds for all $A \ge 3$. This is a contradiction.

Thus, taking $A := 3.68 \times 10^{12}\, k^4\, \log^2 k$, inequality (20) yields

$$\begin{aligned} n-1 &< 2(3.68 \times 10^{12}\, k^4\, \log^2 k)\, \log(3.68 \times 10^{12}\, k^4\, \log^2 k) \\ &< (7.36 \times 10^{12} k^4 \log^2 k)\,(29 + 4\log k + 2\log\log k) \\ &< 3.32 \times 10^{14} k^4 \log^3 k. \end{aligned}$$

In the last chain of inequalities, we have used that $29+4\log k+2\log\log k < 45\log k$ holds for all $k \ge 2$. We record what we have just proved.

Lemma 4. *If (n,k,m) is a nontrivial solution in integers of equation* (2) *with $k \ge 2$, then $n \ge k+2$ and the inequalities*

$$m+2 < n < 3.4 \times 10^{14}\, k^4\, \log^3 k$$

hold.

5. The Case of Small k

We next treat the cases when $k \in [2, 169]$. After finding an upper bound on n the next step is to reduce it. To do this, we use several times the following lemma from [1], which is an immediate variation of a result due to Dujella and Pethö from [5].

Lemma 5. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\gamma q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-k},$$

in positive integers m, n and k with

$$m \leq M \qquad \text{and} \qquad k \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

In order to apply Lemma 5, we let

$$z := m \log 2 - (n-1)\log\alpha - \log\mu, \tag{21}$$

where $\mu := f_k(\alpha)$. Then $e^z - 1 = \Lambda$, where Λ is given by (16). Therefore, (14) can be rewritten as

$$|e^z - 1| < \frac{2}{\alpha^{n-1}}. \tag{22}$$

Note that $z \neq 0$ since $\Lambda \neq 0$, so we distinguish the following cases. If $z > 0$, then $e^z - 1 > 0$, therefore, from (22), we obtain

$$0 < z < \frac{2}{\alpha^{n-1}},$$

where we used the fact that $x \leq e^x - 1$ for all $x \in \mathbb{R}$. Replacing z in the above inequality by its formula (21) and dividing both sides of the resulting inequality by $\log\alpha$, we get

$$0 < m\left(\frac{\log 2}{\log\alpha}\right) - n + \left(1 - \frac{\log\mu}{\log\alpha}\right) < 5\cdot\alpha^{-(n-1)}, \tag{23}$$

where we have used the fact $1/\log\alpha < 2.1$ once again. With

$$\gamma := \frac{\log 2}{\log\alpha}, \qquad \widehat{\mu} := 1 - \frac{\log\mu}{\log\alpha}, \qquad A := 5, \qquad \text{and} \qquad B := \alpha,$$

the above inequality (23) yields

$$0 < m\gamma - n + \widehat{\mu} < AB^{-(n-1)}. \tag{24}$$

It is clear that γ is an irrational number because $\alpha > 1$ is a unit in $\mathcal{O}_{\mathbb{K}}$, so α and 2 are multiplicatively independent.

In order to reduce our bound on n, we take $M := \lfloor 3.4 \times 10^{14}\, k^4 \log^3 k \rfloor$ (upper bound on m from Lemma 4) and we use Lemma 5 on inequality (24) for each $k \in [2, 169]$. A computer search with Mathematica revealed that the maximum value of $\log(Aq/\epsilon)/\log B$ is $330.42\cdots$, which, according to Lemma 5, is an upper bound on $n-1$. Hence, we deduce that the possible solutions (n, k, m) of the equation (2) for which k is in the range $[2, 169]$ and $z > 0$, all have $n \in [4, 331]$, and therefore $m \in [2, 328]$, since $m < n - 2$.

Next we treat the case $z < 0$. First of all, observe that if $k \geq 3$, then one checks easily that $2/\alpha^{n-1} < 1/2$ for all $n \geq 4$, by using the fact that $2(1 - 2^{-k}) < \alpha$; but the same is true when $k = 2$, since in this case α is the golden section. Thus, from (22), we have that $|e^z - 1| < 1/2$ and therefore $e^{|z|} < 2$. Since $z < 0$, we have

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{4}{\alpha^{n-1}}.$$

In a similar way as in the case when $z > 0$, we obtain

$$0 < (n-1)\gamma - m + \widehat{\mu} < AB^{-(n-1)}, \tag{25}$$

where now

$$\gamma := \frac{\log \alpha}{\log 2}, \qquad \widehat{\mu} := \frac{\log \mu}{\log 2}, \qquad A := 6 \qquad \text{and} \qquad B := \alpha.$$

Here, we also took $M := \lfloor 3.4 \times 10^{14}\, k^4 \log^3 k \rfloor$ which is an upper bound on $n - 1$ by Lemma 4, and we applied Lemma 5 to inequality (25) for each $k \in [2, 169]$. In this case, with the help of Mathematica, we found that the maximum value of $\log(Aq/\epsilon)/\log B$ is $330.68\cdots$. Thus, the possible solutions (n, k, m) of the equation (2) in the range $k \in [2, 169]$ and $z < 0$, all have $n \in [4, 331]$, so $m \in [2, 328]$.

Finally, we used Mathematica to compare $F_n^{(k)}$ and 2^m for the range $4 \leq n \leq 331$ and $2 \leq m \leq 328$, with $m + 2 < n < 3m/2 + 2$ and checked that the only nontrivial solution of the equation (2) in this range is that given by Theorem 1. This completes the analysis in the case $k \in [2, 169]$.

6. The Case of Large k

From now on, we assume that $k > 169$. For such k we have

$$n < 3.4 \times 10^{14}\, k^4 \log^3 k < 2^{k/2}.$$

Let $\lambda > 0$ be such that $\alpha + \lambda = 2$. Since α is located between $2(1-2^{-k})$ and 2, we get that $\lambda < 2 - 2(1-2^{-k}) = 1/2^{k-1}$, i.e., $\lambda \in (0, 1/2^{k-1})$. Besides,

$$\begin{aligned}\alpha^{n-1} &= (2-\lambda)^{n-1}\\ &= 2^{n-1}\left(1-\frac{\lambda}{2}\right)^{n-1}\\ &= 2^{n-1}e^{(n-1)\log(1-\lambda/2)} \geq 2^{n-1}e^{-\lambda(n-1)},\end{aligned}$$

where we used the fact that $\log(1-x) \geq -2x$ for all $x < 1/2$. But we also have that $e^{-x} \geq 1 - x$ for all $x \in \mathbb{R}$, so, $\alpha^{n-1} \geq 2^{n-1}\big(1-\lambda(n-1)\big)$.

Moreover, $\lambda(n-1) < (n-1)/2^{k-1} < 2^{k/2}/2^{k-1} = 2/2^{k/2}$. Hence,

$$\alpha^{n-1} > 2^{n-1}\big(1 - 2/2^{k/2}\big).$$

It then follows that the following inequalities hold:

$$2^{n-1} - \frac{2^n}{2^{k/2}} < \alpha^{n-1} < 2^{n-1} + \frac{2^n}{2^{k/2}},$$

or

$$\left|\alpha^{n-1} - 2^{n-1}\right| < \frac{2^n}{2^{k/2}}. \tag{26}$$

We now consider the function $f_k(x)$ given by (11). Using the Mean–Value Theorem, we get that there exists some $\theta \in (\alpha, 2)$ such that

$$f_k(\alpha) = f_k(2) + (\alpha - 2)f_k'(\theta).$$

Observe that when $k \geq 3$, we obtain $\left|f_k'(\theta)\right| = (k-1)/\big(2+(k+1)(\theta-2)\big)^2 < k$ (see the inequality (12) and the comment following it), and when $k = 2$, we have that α is the golden section and therefore $\left|f_2'(\theta)\right| = 1/(3\theta-4)^2 < 25/16$, since $\theta > \alpha > 8/5$. In any case, we obtain $\left|f_k'(\theta)\right| < k$. Hence,

$$\left|f_k(\alpha) - f_k(2)\right| = |\alpha - 2|\left|f_k'(\theta)\right| = \lambda\left|f_k'(\theta)\right| < \frac{2k}{2^k}. \tag{27}$$

Writing

$$\alpha^{n-1} = 2^{n-1} + \delta \qquad \text{and} \qquad f_k(\alpha) = f_k(2) + \eta,$$

then inequalities (26) and (27) yield

$$|\delta| < \frac{2^n}{2^{k/2}} \qquad \text{and} \qquad |\eta| < \frac{2k}{2^k}. \tag{28}$$

Besides, since $f_k(2) = 1/2$, we have

$$f_k(\alpha)\,\alpha^{n-1} = 2^{n-2} + \frac{\delta}{2} + 2^{n-1}\eta + \eta\delta. \tag{29}$$

So, from (13) and the inequalities (28) and (29) above, we get

$$\begin{aligned}\left|2^m - 2^{n-2}\right| &= \left|\left(2^m - f_k(\alpha)\alpha^{n-1}\right) + \frac{\delta}{2} + 2^{n-1}\eta + \eta\delta\right| \\ &< \frac{1}{2} + \frac{2^{n-1}}{2^{k/2}} + \frac{2^n k}{2^k} + \frac{2^{n+1}k}{2^{3k/2}}.\end{aligned}$$

Factoring 2^{n-2} in the right–hand side of the above inequality and taking into account that $1/2^{n-1} < 1/2^{k/2}$ (because $n \geq k+2$ by Lemma 4), $4k/2^k < 1/2^{k/2}$ and $8k/2^{3k/2} < 1/2^{k/2}$ which are all valid for $k > 169$, we conclude that

$$\left|2^{m-n+2} - 1\right| < \frac{5}{2^{k/2}}. \tag{30}$$

By recalling that $m < n - 2$ (see (9)), we have that $m - n + 2 \leq -1$, then it follows from (30) that

$$\frac{1}{2} \leq 1 - 2^{m-n+2} < \frac{5}{2^{k/2}}.$$

So, $2^{k/2} < 10$, which is impossible since $k > 169$.

Hence, we have shown that there are no solutions (n, k, m) to equation (2) with $k > 169$. Thus, Theorem 1 is proved.

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