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# Bounded Category of an Exact Category

## Categoría acotada de una categoría exacta

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Abstract. Pedersen-Weibel introduced the notion of bounded category of an additive category, which gives the non-connective delooping of the additive category under consideration. In this work, a possible candidate for the bounded category of an exact category is constructed which shares many properties of the bounded categories of Pedersen-Weibel.

*Key words and phrases.* Delooping, Exact Category, Pedersen-Weibel, Bounded Category, Negative K-theory.

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Resumen. Pedersen-Weibel introducen la noción de categoría limitada de una categoría aditiva, la cual da el "*non-connective delooping*" de la categoría aditiva en consideración. En este trabajo, se construye un posible candidato para la categoría limitada de una categoría exacta el cual posee muchas de las propiedades de las categorías limitadas de Pedersen-Weibel.

*Palabras y frases clave.* Delooping, Categoría exacta, Pedersen-Weibel, Categoría limitada, K-teoría negativa.

## 1. Introduction

Quillen's definition of higher K-groups of a ring $R$ led to the idea of viewing algebraic K-theory of the ring $R$ as a connective spectrum in the sense of stable homotopy theory. A connective spectrum is one which has no homotopy groups in negative dimensions. Along the ideas of Quillen, the algebraic K-theory could be defined for an arbitrary exact category. But the connective spectra does not capture the negative K-groups of a ring $R$ or of an exact category. To address this we explain the problem of non-connective delooping.

Given an exact category $\mathscr{E}$, find an exact category $\mathscr{C}(\mathscr{E})$ such that $K_{i+1}\mathscr{C}(\mathscr{E}) = K_i\widehat{\mathscr{E}}$ where $\widehat{\mathscr{E}}$ is the idempotent completion of $\mathscr{E}$ and $i \geq 0$.

Once this is done, we can define $K_{-1}\mathscr{E} = K_0\widehat{\mathscr{C}(\mathscr{E})}$ and all the other negative K-groups inductively.

In the case of an additive category i.e., an exact category whose exact sequences are split, Pedersen and Weibel (see [6]) solved the problem of non-connective delooping using bounded category methods which proved to be useful in geometric topology to understand assembly maps. The negative K-groups of the category of finitely generated projective $R$-modules agreed with the definition of negative algebraic K-theory groups of a ring $R$ given by Bass (see [1]).

For general exact categories, where the exact sequences are not necessarily split, Schlichting solved the problem of non-connective delooping by algebraic methods (see [9]).

Let us recall the main results of Pedersen-Weibel in [6].

**Definition 1.** Let $\mathscr{A}$ be an additive category and $X$ a proper metric space. Then the Bounded Category $\mathscr{C}_X(\mathscr{A})$ of $\mathscr{A}$ is defined as follows:

**Objects:** A $\mathscr{C}_X(\mathscr{A})$ object $A$ is a collection of $\mathscr{A}$ objects indexed by $X$, $\{A_x\}_{x\in X}$ satisfying the locally finite condition: $\{x : A_x \neq 0\} \cap S$ is finite for every bounded set $S \subset X$.

**Morphisms:** A $\mathscr{C}_X(\mathscr{A})$ morphism $f : A \to B$ is a collection of $\mathscr{A}$-morphisms $f_{xy} : A_x \to B_y$ for which $\exists D(f) \geq 0$ such that $f_{xy} = 0$ if $d_X(x, y) > D(f)$.

**Composition of $f : A \to B$ and $g : B \to C$:** is defined by $(g \circ f)_{xy} = \sum_z g_{zy} \circ f_{xz}$. Note that the sum makes sense because of the locally finite condition.

***Remark* 2.** The category $\mathscr{C}_X(\mathscr{A})$ inherits an additive structure from $\mathscr{A}$.

The main theorem of Pedersen-Weibel as given below (see Theorems A and B in [6]) uses the machinery of connective $\Omega$-spectra associated to a symmetric monoidal category and the notion of the bounded category to obtain a delooping of the K-theory spectrum. For the sake of notational simplicity, all the categories involved are assumed to be idempotent complete. Note that only the $K_0$ of an additive category and its idempotent completion are different.

**Theorem 3.** *Let $\mathbb{K}(\mathscr{A})$ denote the connective K-theory spectrum of an additive category. Then $\mathbb{K}(\mathscr{A})$ is homotopy equivalent to $\Omega\mathbb{K}(\mathscr{C}_{\mathbb{Z}}\mathscr{A})$.*

In the hope of obtaining a non-connective delooping for exact categories along the lines of Pedersen-Weibel, a possible candidate for the bounded category of an exact category $\mathscr{E}$ denoted by $\mathscr{E}_{(X,\mathscr{U})}$ is described in this paper and it will be shown that it agrees with the Pedersen-Weibel bounded category in the case of split exact structure. Note that $\mathscr{C}_X(\mathscr{A})$ does not inherit an exact structure in an obvious manner, if $\mathscr{A}$ were given a non-split exact structure.

**Theorem 4** (Main Theorem)**.** *The category $\mathscr{E}_{(X,\mathscr{U})}$ inherits an exact structure from $\mathscr{E}$. Moreover, $\mathscr{E}_{(\mathrm{pt},\mathscr{U})}$ and $\mathscr{E}$ are naturally isomorphic and hence have the same $\mathbb{K}$-theory. The connective spectrum of $\mathscr{E}_{(\mathbb{Z}_{\leq 0},\mathscr{U})}$ is contractible.*

## 2. Definition of Bounded Category of an Exact Category

Let us recall the definition of an Exact category for the sake of completeness. For the equivalence of this definition with the one given by Quillen in [8] (see [5]).

**Definition 5.** Let $\mathscr{E}$ be an additive category. A short sequence in $\mathscr{E}$ is a pair of composable morphisms $L \to M \to N$ such that $L \to M$ is a kernel for $M \to N$ and $M \to N$ is a cokernel for $L \to M$.

Homomorphisms of short sequences are defined in the obvious way as commutative diagrams.

An Exact Category is an additive category $\mathscr{E}$ together with a choice of a class of short sequences, called short exact sequences, closed under isomorphisms and satisfying the axioms below. A short exact sequence is displayed as $L \rightarrowtail M \twoheadrightarrow N$. $L \rightarrowtail M$ is called an admissible monomorphism and $M \twoheadrightarrow N$ is called an admissible epimorphism. The axioms are as follows:

1) The identity morphism of the zero object is an admissible monomorphism and an admissible epimorphism.

2) The class of admissible monomorphisms is closed under composition and cobase changes by pushouts along arbitrary morphisms, i.e., given any admissible monomorphism $L \rightarrowtail M$ and any arbitrary $L \to L'$, their pushout $M'$ exists and the induced morphism $L' \to M'$ is again an admissible monomorphism.

$$\begin{array}{ccc} L & \rightarrowtail & M \\ \downarrow & & \downarrow \\ L' & \rightarrowtail & M' \end{array}$$

3) Dually, the class of admissible epimorphisms is closed under composition and base changes by pullbacks along arbitrary morphisms, i.e., given any admissible epimorphism $M \twoheadrightarrow N$ and any arbitrary $N' \to N$, their pullback $M'$ exists and the induced morphism $M' \to N'$ is again an admissible epimorphism.

$$\begin{array}{ccc} M' & \dashrightarrow & M \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ N' & \longrightarrow & N \end{array}$$

***Remark* 6.** Any extension closed full subcategory of an abelian category is exact. In fact any exact category is an extension closed full subcategory of an abelian category. See Appendix A in [10].

Let $\mathscr{E} \subset \mathscr{U}$ be a pair, where $\mathscr{U}$ is a co-complete abelian category and $\mathscr{E}$ is an extension closed, full subcategory. Since the composition of two admissible monomorphisms is again an admissible monomorphism, if we restrict to the admissible monomorphisms on the objects of $\mathscr{E}$, we obtain a subcategory $co\mathscr{E}$.

We consider a category of functors from the bounded subsets of a proper metric space $X$ to the exact category $\mathscr{E}$ and show that this inherits an exact structure from $\mathscr{E}$. A very similar construction of an exact category parametrized over a metric space is explained in [3]. The category $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ is defined as follows:

**Definition 7.** Let $\mathscr{P}(X)$ be the poset of all subsets of $X$, $\mathscr{P}_{bd}(X)$ be the poset of all bounded subsets of $X$, $co\mathscr{E} \subset \mathscr{E}$ be the subcategory of admissible monomorphisms (also called cofibrations in Waldhausen's language) of $\mathscr{E}$ and $co\mathscr{U} \subset \mathscr{U}$ be the subcategory of monomorphisms of $\mathscr{U}$. Let $N_\rho : \mathscr{P}_{bd}(X) \to \mathscr{P}_{bd}(X)$ be the functor which maps a bounded set $S$ to $N_\rho(S) = \{x \in X : d_X(x, S) \leq \rho\}$. Then $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ has the following:

**Objects:** A $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$-object is a functor of pairs $A : \big(\mathscr{P}(X), \mathscr{P}_{bd}(X)\big) \to (co\mathscr{U}, co\mathscr{E})$ that satisfies:

- $A$ maps the empty set to the zero object.
- In $\mathscr{U}$, the cokernel of $A(S \subset X)$, is non-trivial, for any proper bounded subset inclusion.
- $\exists D \geq 0$ such that $\bigoplus\limits_{S|diam(S)\leq D} A(S) \twoheadrightarrow A(X)$ in $\mathscr{U}$.

**Morphisms:** A $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$-morphism $f : A \to B$ is a $\mathscr{U}$-morphism $f : A(X) \to B(X)$ for which $\exists \rho \geq 0$ which depends only on $f$, such that $\forall S \in \mathscr{P}_{bd}(X)$, $f|_{A(S)}$ factors through $BN_\rho(S) \rightarrowtail B(X)$.

$$\begin{array}{ccc} A(S) & \longrightarrow & A(X) \\ {\scriptstyle f|_{A(S)}} \downarrow & & \downarrow {\scriptstyle f} \\ BN_\rho(S) & \xrightarrow[i]{} & B(X) \end{array} .$$

Such a $\mathscr{U}$-morphism is called a *controlled* morphism and $\rho$ is called a control parameter of $f$.

Now let us consider the full subcategory $\mathscr{E}_{(X,\mathscr{U})}$ of $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ on the objects satisfying $A(S) = \bigoplus\limits_{s\in S} A(s)$ for all subsets $S$ of $X$.

## 3. More about $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$

Before describing the exact structure on $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ and proving the Main Theorem (Theorem 4) we need to understand the morphisms in general and isomorphisms in particular. The following subsections provide various details in this direction.

### 3.1. Additiveness of $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$

**Proposition 8.** *$\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ is an additive category.*

***Proof.*** Firstly, let us verify that the composition of two controlled $\mathscr{U}$-morphisms is controlled. Let $f : A \to B$ and $g : B \to C$ be controlled $\mathscr{U}$-morphisms with parameters $\rho(f)$ and $\sigma(g)$ respectively. Then $\rho(f)+\sigma(g)$ is a control parameter for $g \circ f$ as the following diagram shows:

$$\begin{array}{ccccc}
 & & A(S) & \longrightarrow & A(X) \\
 & & \downarrow & & \downarrow f \\
 & & BN_\rho(S) & \longrightarrow & B(X) \\
 & \swarrow & \downarrow & & \downarrow g \\
CN_\sigma\big(N_\rho(S)\big) & \longrightarrow & CN_{\sigma+\rho}(S) & \longrightarrow & C(X) \; .
\end{array}$$

Notice that in a metric space $X$ the containment $N_\rho(N_\sigma) \subsetneq N_{\rho+\sigma}$ could be strict.

Associativity in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ follows from the associativity in $\mathscr{U}$ and the fact that composition of controlled morphisms is controlled. $1_A : A \longrightarrow A$, the identity morphism exists because we can choose a control parameter $\rho = 0$ and the identity morphism in $\mathscr{U}$. This shows that $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ is a category.

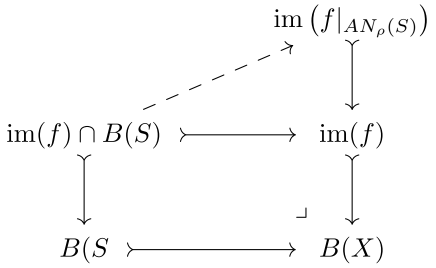
Given $A, B \in Ob\, \mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$, we can define $A \oplus B : \big(\mathscr{P}(X), \mathscr{P}_{bd}(X)\big) \to (co\mathscr{U}, co\mathscr{E})$ as $A \oplus B(Y) := A(Y) \oplus B(Y)$, $\forall Y \in \mathscr{P}(X)$.

Given two morphisms $f, g : A \to B$ in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$, with control parameters $\rho$ and $\sigma$ respectively, then the $\mathscr{U}$-morphism, $f + g$ is in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$, because we can choose a control parameter $\max(\rho, \sigma)$. This shows that $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ is an additive category. ☑

### 3.2. Isomorphisms in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$

Isomorphisms in $\mathscr{U}$ that are controlled need not be isomorphisms in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$, i.e., their inverses need not be controlled. To distinguish this difference we have the following definition.

**Definition 9.** A $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ morphism $f : A \to B$ is said to be *bicontrolled* if $\exists \rho \geq 0$ such that for any $S \in \mathscr{P}_{bd}(X)$, $\mathrm{im}(f) \cap B(S) \rightarrowtail \mathrm{im}(f)$ factors through $\mathrm{im}(f|_{AN_\rho(S)}) \rightarrowtail \mathrm{im}(f)$, where $\mathrm{im}(f)$ is the image of $f : A(X) \to B(X)$ in the Abelian Category $\mathscr{U}$ and $\cap$ is the pullback in $\mathscr{U}$.




The following are two examples of $\mathscr{U}$-isomorphisms that are not bicontrolled.

**Example 10.** Let $\mathscr{U}$ = Category of Abelian groups, $\mathscr{E}$ = Category of finitely generated abelian groups and $X = \mathbb{Z}$. $A, B : \big(\mathscr{P}(\mathbb{Z}), \mathscr{P}_{bd}(\mathbb{Z})\big) \longrightarrow (co\mathscr{U}, co\mathscr{E})$, objects of $\mathscr{C}_\mathbb{Z}(\mathscr{E} \subset \mathscr{U})$, and a morphism $f : A \to B$ with control parameter 0 are defined as shown below.

Define a sequence of integers by $a_n = n(n+1)$. Let $T_i$ denote any proper subset of $N_{2(i+1)}(\{a_i\})$ where $\{a_i\} \in \mathscr{P}(\mathbb{Z})$. Any element of $\mathscr{P}(\mathbb{Z})$ that is not a $T_i$ is termed as $REST$.

The object $A$ is defined as the functor $A(T_i) = \mathbb{Z}, \forall S \in \mathscr{P}(\mathbb{Z})$ and $A$ maps every inclusion in $\mathscr{P}(\mathbb{Z})$ of the type $T_i \subset REST$ to the multiplication by 2 map and all the other inclusions to the identity map.

The object $B$ is defined as the constant functor $B(S) = \mathbb{Z} \forall S \in \mathscr{P}(\mathbb{Z})$ and $B$ maps every inclusion in $\mathscr{P}(\mathbb{Z})$ to the identity morphism.

Let $f : A \to B$, be the natural transformation, when restricted to the subsets $REST$ is the identity map. $f$ is controlled as it is a natural transformation. $f$ is not bicontrolled because $A(X) \cap B(\{a_n\}) \to A(X)$ factors only through $AN_\rho(\{a_n\}) \to A(X)$ where $\rho \geq 2(n+1)$. So there is no uniform bound $\rho$ that works all $S \in \mathscr{P}(\mathbb{Z})$.

On the other hand $A(X) \cap B(S) \cong A(S) \cap B(S)$, $\forall S \in \mathscr{P}(\mathbb{Z})$.

The following is an example from Pedersen-Weibel's paper [6].

**Example 11.** Let $\mathscr{U}$ = Category of Abelian groups, $\mathscr{E}$ = Category of finitely generated abelian groups and $X = \mathbb{Z}$. Let $W : \big(\mathscr{P}(\mathbb{Z}), \mathscr{P}_{bd}(\mathbb{Z})\big) \longrightarrow (co\mathscr{U}, co\mathscr{E})$, an object of $\mathscr{C}_\mathbb{Z}(\mathscr{E} \subset \mathscr{U})$, be defined as follows:

$$\forall\, Y \in \mathscr{P}(\mathbb{Z}), \qquad W(Y) := \bigoplus_{Y_{\leq 0}} \mathbb{Z},$$

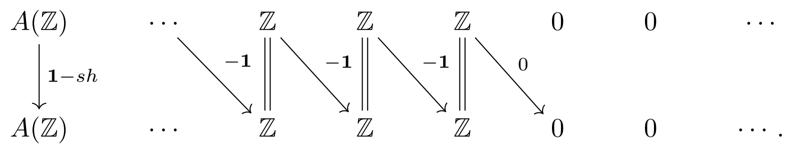
where $Y_{\leq 0} = \{y \in Y : y \leq 0\}$. Note that if $Y$ is bounded then $W(Y) \in Ob\mathscr{E}$. Let $sh$ be the shift operator on $W(\mathbb{Z})$, which moves the coordinates of an element of $W(\mathbb{Z})$ one place to the right

$$sh(\dots, 0, a_{-n}, a_{-n+1}, \dots, a_{-1}, a_0) = (\dots, 0, a_{-n}, a_{-n+1}, \dots, a_{-2}, a_{-1})$$

and $\mathbf{1}$ be the identity operator on $W$. Consider the map $\mathbf{1} - sh : W \to W$. Explicitly

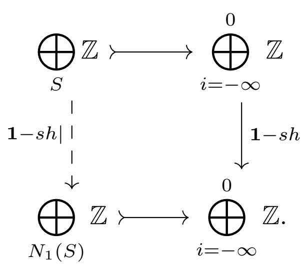
$$\mathbf{1} - sh(\dots, 0, a_{-n}, a_{-n+1}, \dots, a_{-1}, a_0) = (\dots, 0, a_{-n}, a_{-n+1} - a_{-n}, \dots, a_{-1} - a_{-2}, a_0 - a_{-1}).$$

Pictorially $\mathbf{1} - sh$ can be seen as follows




**Claim 12.** $(\mathbf{1} - sh)$ is a $\mathscr{C}_{\mathbb{Z}}(\mathscr{E} \subset \mathscr{U})$ morphism.

***Proof.*** If we choose $\rho = 1$ then $\forall S \in \mathscr{P}_{bd}(\mathbb{Z})$ we have the following diagram.




The restriction of $\mathbf{1} - sh$ to $\bigoplus_S \mathbb{Z}$ gives the required natural transformation.

☑

**Claim 13.** $(\mathbf{1} - sh)$ is not bicontrolled.

***Proof.*** First notice that $(\mathbf{1} - sh)$ is both monic and epic in $\mathscr{U}$. Let $S = \{-n\}$ where $n > 1$. Notice that for $1 \in \mathbb{Z} = W(S)$,

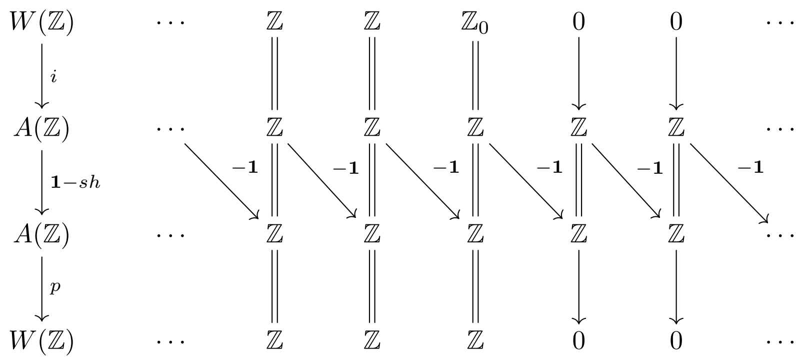
$$(\mathbf{1} - sh)^{-1}(1) = \sum_{i=0}^{\infty} sh^i(1) = (\dots, 0, \underbrace{1, 1, \dots, 1}_{n+1}) \in \bigoplus_{i=-n}^{0} \mathbb{Z} = N_n(S).$$

As we vary $n$ we see that no uniform $\rho$ works as a bicontrol parameter and hence, the factorization $\operatorname{im}(\mathbf{1}-sh)\cap W(S) = W(S) \to (\mathbf{1}-sh)|_{W(N_\rho(S))}$ is not possible. The point is that there is no uniform bound for the above factorization. Observe that $\rho$ depends on $n$. ☑

It is important to note that composition does not respect bicontrolledness.

**Example 14.** Let $\mathscr{U}$ = Category of Abelian groups, $\mathscr{E}$ = Category of finitely generated abelian groups and $X = \mathbb{Z}$ as in Example 11.

Consider the composition of the maps as shown in the picture below. The subscript 0 is used to make clear the definitions of $W$ and $A$.




$\mathbf{1}-sh$ as shown above is just a monomorhpism and $\operatorname{im}(\mathbf{1}-sh)$ is all the tuples whose sum is 0. Explicitly

$$(\mathbf{1}-sh)^{-1}(\dots,0,a_{-m},a_{-m+1},\dots,a_{-1},a_0,a_1,\dots,a_n,0,\dots) = \left(\dots,0,a_{-m},a_{-m}+a_{-m+1},\dots,\sum_{i=-m}^{0} a_i,\dots,\sum_{i=-m}^{n-1} a_i,0,\dots\right).$$

Notice that all the three maps $i$, $\mathbf{1}-sh$ and $p$ are bicontrolled because choosing the bicontrol parameter $\rho = 0$ works for all of them. But the composition $p(\mathbf{1}-sh)i = (\mathbf{1}-sh) : W(\mathbb{Z}) \to W(\mathbb{Z})$, is precisely the map discussed in Example 11 which is not bicontrolled.

## 4. Exact structure on $\mathscr{C}_{\mathbf{X}}(\mathscr{E}\subset\mathscr{U})$ and $\mathscr{E}_{(\mathbf{X},\mathscr{U})}$

The idea behind defining an exact structure is as follows. The exact structure on $\mathscr{C}_X(\mathscr{E}\subset\mathscr{U})$ should inherit the exact sequences from $\mathscr{E}$ and should also contain split exact sequences that come from the additive structure on $\mathscr{C}_X(\mathscr{E}\subset\mathscr{U})$. And finally we prove that $\mathscr{E}_{(X,\mathscr{U})}$ is an extension closed subcategory of $\mathscr{C}_X(\mathscr{E}\subset\mathscr{U})$. Here comes the definition.

**Definition 15.** A *strict short sequence* in the additive category $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ is a sequence

$$A \overset{m}{\rightarrowtail} B \overset{q}{\twoheadrightarrow} C$$

satisfying the following:

1) The corresponding sequence in $\mathscr{E}$ $\; A(S) \overset{m}{\rightarrowtail} B(S) \overset{q}{\twoheadrightarrow} C(S) \;$ is a short exact sequence for all $S \in \mathscr{P}_{bd}(X)$.
2) $m$ and $q$ have control and bicontrol parameters 0.

We call $m$ a *strict admissible monomorphism* and $q$ a *strict admissible epimorphism* and we denote them by arrows as shown above.

Let $\mathscr{J}$ denote the class of all strict short sequences and $\widehat{\mathscr{J}}$ the class of all short sequences that are obtained by including all the short sequences that are isomorphic to strict short sequences in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$.

## 5. Proof of the Main Theorem

**Lemma 16.** *Let*

$$\begin{array}{ccc} A & \overset{m}{\rightarrowtail} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \underset{n}{\rightarrowtail} & Q \end{array}$$

*be a cocartesian square of $R-$modules where $m$ is a monic. Then the square is bicartesian.*

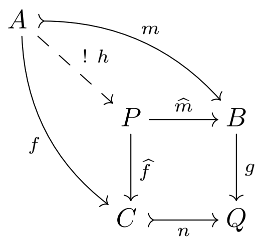
***Proof.*** $Q := (B \oplus C)/M$, where

$$M = \big\{\big(m(a), -f(a)\big) : a \in A\big\} \subseteq B \oplus C.$$

Here $g(b) = (b, 0) + M$ and $n(c) = (0, c) + M$. The pullback of $n$ along $g$ is:

$$P := \big\{(b, c) \in B \oplus C : n(c) = g(b)\big\}$$

We have the following diagram.

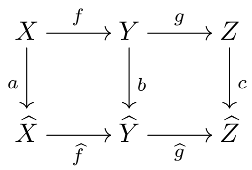


where $\widehat{m}(b,c) = b$ and $\widehat{f}(b,c) = c$.

By the universal property of $P$, $h$ is the unique map such that $m = \widehat{m}h$ and $f = \widehat{f}h$. But the map $\widehat{h} : A \to P$, given by $\widehat{h}(a) = \big(m(a), f(a)\big)$, satisfies the conditions for the universal properties. So by uniqueness, $h = \widehat{h}$.

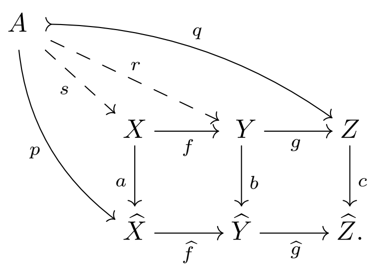
$h$ is monic because $m$ is so. Let $(b,c) \in P$. Then $g\widehat{m}(b,c) = \widehat{f}n(b,c)$, i.e., $(b,0) + M = (0,c) + M \implies (b,-c) \in M \implies \exists a \in A$ such that $\big(m(a), -f(a)\big) = h(a) = (b,c)$. Hence $h$ is onto. ☑

**Lemma 17.** *Let*



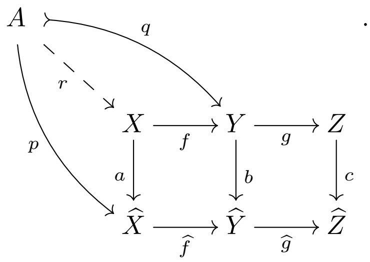

*be a commutative diagram. Assume the square on the right side is a pullback. Then the square on the left side is a pullback if and only if the whole diagram is a pullback. i.e., $X$ is a pullback of $\widehat{g}\widehat{f}$ along $c$.*

***Proof.*** First we will show that $X$ has the desired universal property for the whole diagram, assuming that the left hand square is a pullback. Assume the following diagram commutes (do not consider the dotted arrows):




Since the right hand square is a pullback $\exists$ a unique $r$ such that $\widehat{f}p = br$ and $q = gr$. Now since the left hand square is a pullback, $\exists$ a unique $s$ such that $r = fs$ and $p = as$. So $\exists$ a unique $s$ such that $q = gr = gfs$ and $p = as$. So $X$ is the pullback of the whole diagram.

Now let us assume that the whole diagram is a pullback and show that the left hand square is so. Assume the following diagram is commutative:



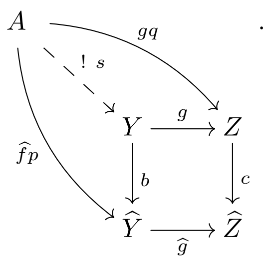

Since

$$bq = \widehat{f}p \tag{1}$$

we have $\widehat{g}bq = \widehat{g}\widehat{f}p \implies c(gq) = \widehat{g}\widehat{f}(p)$. Since the whole diagram is a pullback $\exists$ a unique $r$ such that

$$(a)r = p; \qquad (gf)r = gq. \tag{2}$$

We need to show that $fr = q$. We have the following diagram:




Now, since the right hand square is a pull back and $\widehat{g}(\widehat{f}p) = c(gq)$, $\exists$ a unique $s$ such that

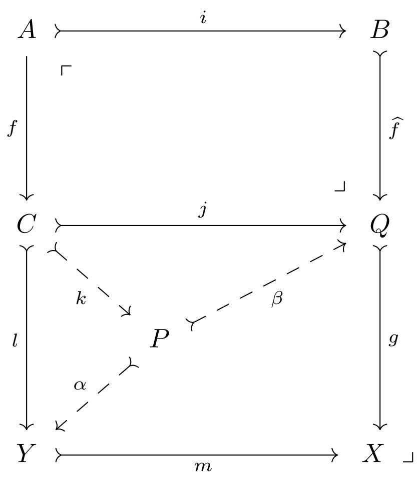
$$bs = \widehat{f}p; \qquad gs = gq. \tag{3}$$

But $q$ satisfies (3). So by uniqueness $s = q$.

Also $bfr = \widehat{f}ar = \widehat{f}p = bq$ by (1) and (2) and $gfr = gq$ by (2). So $fr$ also satisfies (3). So, by uniqueness of $s$, $fr = q = s$. ☑

***Remark* 18.** The dual versions of the above two lemmas are also true and the proofs are very similar.

**Lemma 19.** *Given the following diagram*




*where the $Q$ is the pushout of $i$ along $f$ and $A$ is the pullback of $m$ along $g\widehat{f}$, tailed arrows are monic and the rest arbitrary and $P$ is the pullback of lower square, then $k$ is an isomorphism.*

***Proof.*** The proof consists of analysing $A$ and $P$ in the light of the hypotheses.

1) Let $M = \big\langle \big(i(a), -f(a)\big)\big\rangle$ be the submodule of $B \oplus C$. $Q$ is the pushout of $i$ along $f$ means $Q = (B \oplus Q)/M$. $\widehat{f}(b) = (b, 0) + M$ and $j(c) = (0, c) + M$.

2) $A$ is the pullback of the upper square by Lemma 16. This means

$$\begin{aligned} A &\cong \big\{(b, c) \in B \oplus C : (b, 0) + M = (0, c) + M\big\} \\ &\cong \big\{(b, c) \in B \oplus C : (b, -c) \in M\big\} \\ &\cong \big\{(b, c) \in B \oplus C : (b, -c) = \big(i(a), -f(a)\big)\big\} \\ &\cong \big\{\big(i(a), f(a)\big)\big\} \cong \big\{\big(i(a), lf(a)\big)\big\}. \end{aligned}$$

The last line is true since $l$ is monic.

3) $A$ is the pullback of $m$ along $g\widehat{f} \implies$

$$\begin{aligned} A &\cong \big\{(b, y) \in B \oplus Y : m(y) = g\widehat{f}(b)\big\} \\ &\cong \big\{(b, y) \in B \oplus Y : m(y) = g\big((b, 0) + M\big)\big\}. \end{aligned}$$

4) Let $P$ be the pullback of the lower square. So

$$P = \big\{\big((b, c) + M, y\big) \in Q \oplus Y : g\big((b, c) + M, y\big) = m(y)\big\}.$$

Since the lower square commutes $k$ is the unique universal map. Using that $\alpha k = l$ and $\beta k = j$, we have a formula for $k$: $k(c) = \big((0,c)+M, l(c)\big)$.

5) $k$ is monic because $l$ is so. Now let us prove that $k$ is epic.

$$
\begin{aligned}
&\big((b,c)+M, y\big) \in P \\
&\qquad \Longrightarrow\ g\big((b,c)+M, y\big) = m(y) \\
&\qquad \Longrightarrow\ g\big((b,0)+M\big) + g\big((0,c)+M\big) = m(y) \\
&\qquad \Longrightarrow\ g\big((b,0)+M\big) + gj(c) = m(y) \\
&\qquad \Longrightarrow\ g\big((b,0)+M\big) + ml(c) = m(y) \\
&\qquad \Longrightarrow\ g\widehat{f}(b) = m\big(y - l(c)\big) \qquad \text{[See 3)]} \\
&\qquad \Longrightarrow\ \big(b, y - l(c)\big) \in A \\
&\qquad \Longrightarrow\ b = i(a) \quad \text{and} \quad y - l(c) = lf(a) \qquad \text{[See the last line of 2)]} \\
&\qquad \Longrightarrow\ b = i(a) \quad \text{and} \quad y = l\big(c + f(a)\big).
\end{aligned}
$$

Note that $c+f(a) \in C$. Now $k\big(c+f(a)\big) = \big((0,c+f(a))+M, l(c+f(a))\big) = \big((0,c+f(a))+M, y\big) = \big((b,c)+M, y\big)$ because $(b,c) - \big(0, c+f(a)\big) = \big(b, -f(a)\big) = \big(i(a), -f(a)\big) \in M$. Therefore $k\big(c+f(a)\big) = \big((b,c)+M, y\big)$ and $k$ is an isomorphism. ☑

**Lemma 20.** *Let $\mathscr{C}$ be an additive category and $\mathscr{J}$ be a class of short sequences satisfying all the axioms of Definition 5, but being closed under ismorphisms. Then the class $\widehat{\mathscr{J}}$ obtained by including all the short sequences in $\mathscr{C}$ that are isomorphic to those in $\mathscr{J}$ is an exact structure.*

***Proof*.** Let $A' \to B' \to C'$ be in $\widehat{\mathscr{J}}$. This means we have the following commutative diagram:

$$
\begin{array}{ccccc}
A & \xrightarrow{\ m\ } & B & \xrightarrow{\ q\ } & C \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\
A' & \xrightarrow[\ m'\ ]{} & B' & \xrightarrow[\ q'\ ]{} & C'
\end{array}
$$

where $A \to B \to C$ is in $\mathscr{J}$ and $f$, $g$ and $h$ are $\mathscr{C}$-isomorphisms. Notice that both the left and right squares above are bicartesian i.e., the universal property for a pushout is satisfied by $B'$, $m'$, $g$ for the left square and by $C'$, $q'$, $h$ for the right square. Similarly the universal property for a pullback is satisfied by $A$, $m$, $f$ for the left square and by $B$, $q$, $g$ for the right square.

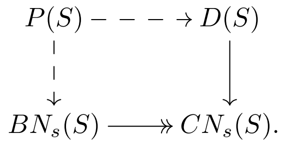
Using Lemma 17 and its dual version we can conclude:

1) Cobase change of $m'$ along $t : A' \to X$ is the same as cobase change of $m$ along $tf$.

2) Base change of $q'$ along $u : Y \to C'$ is the same as base change of $q$ along $h^{-1}u$. ☑

Given a strict admissible epimorphism $q : B \twoheadrightarrow C$ in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ and $f : D \to C$ with a control parameter $s$ in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$, let us define as an object $P$ and maps $p : P \to D$ and $g : P \to B$ with the following description:

For each $S \in \mathscr{P}(X)$, $P(S)$ is the pullback of the following diagram.




***Remarks* 21.**

1) Note that if $S = X$ then $P(X)$ is the pullback of $q$ along $f$ in the Abelian Category $\mathscr{U}$. If $S$ is a bounded subset then $P(S) \to D(S)$ is an admissible epimorphism in $\mathscr{E}$.

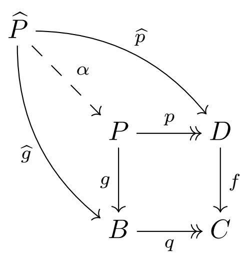
2) The map $p : P \to D$ is the map $p : P(X) \to D(X)$. Note that this is a map in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ with a control parameter 0. The map $g : P \to B$ is the map $g : P(X) \to B(X)$. Note that this is a map in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ with a control parameter $s$.

**Lemma 22.** *$p : P \to D$ is bicontrolled.*

***Proof*.** This is obvious because the way $P(S)$ is defined is such that $\operatorname{im}\big(p : P(S)\big) = D(S)$; so, a bicontrol parameter 0 works for $p$. ☑

**Lemma 23.** *$P$, $p$ and $g$ defined as above satisfy the universal property of a pullback in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$.*

***Proof*.** We will show that $P$ defined as base change has the universal property of a pullback in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$. Consider the following commutative diagram in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$. We need the existence of a unique $\alpha$ as shown below.



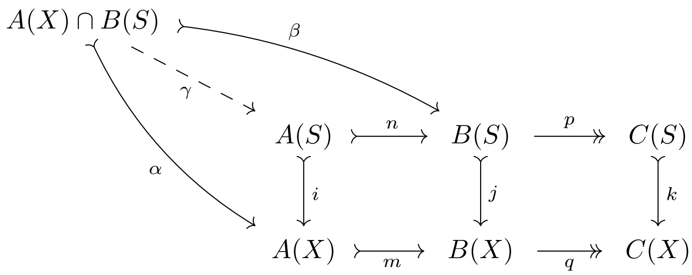
By the manner we defined $P$ we already have a unique $\alpha : \widehat{P}(X) \to P(X)$ in $\mathscr{U}$. We need to show that $\alpha$ is controlled.

Let $\rho = \max\{\text{control parameter of } f\widehat{p}, \text{control parameter of } \widehat{g}\}$. This works as a control parameter for $\alpha$ by the manner we defined $\alpha$. ☑

**Lemma 24.** *Let $q : B \to C$ be a natural transformation that has a bicontrol parameter 0 and $B(S) \twoheadrightarrow C(S)$ is an admissible epimorphism in $\mathscr{U}$ or $\mathscr{E}$ depending on $S$. Then $q$ can be extended to a strict short sequence.*

***Proof.*** Define $A(S)$ so that $A(S) \rightarrowtail B(S)$ is the kernel of $B(S) \twoheadrightarrow C(S)$. We have a natural transformation $m : A \to B$. Need to check $m$ is bicontrol of parameter 0.

Now we have the following diagram

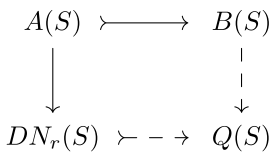



1) $kp\beta = qj\beta = qm\alpha = 0 = k0$. Since $k$ is monic, $p\beta = 0$.

2) Since $n : A(S) \rightarrowtail B(S)$ is the kernel of $p : B(S) \twoheadrightarrow C(S)$, $\exists$ a unique $\gamma$ such that $n\gamma = \beta$ and $\gamma$ is a monic. This is equivalent to saying that $m$ has a bicontrol parameter 0. ☑

Given a strict admissible monomorphism $m : A \rightarrowtail B$ in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$, let us define an object $Q$ and the and maps $n : D \to Q$ and $g : B \to Q$ with the following description:

For each $S \in \mathscr{P}(X)$, $Q(S)$ is defined as the pushout of the following diagram:



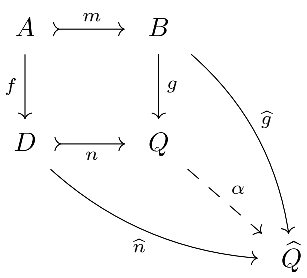

***Remarks* 25.**

1) Note that if $S = X$ then $Q(X)$ is the pushout of $m$ along $f$ in $\mathscr{U}$.

2) The map $n : D \to Q$ is the map $n : D(X) \to Q(X)$ which is controlled with a control parameter 0. The map $g : B \to Q$ is the map $n : B(X) \to Q(X)$ which is controlled with a control parameter 0.

**Lemma 26.** *$Q$, $n$ and $g$ defined as above satisfy the universal property of a pushout in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$.*

***Proof.*** We will show that $Q$ defined as cobase change has the universal property of a pushout in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$. Consider the following commutative diagram in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$. We need the existence of a unique $\alpha$ as shown below.

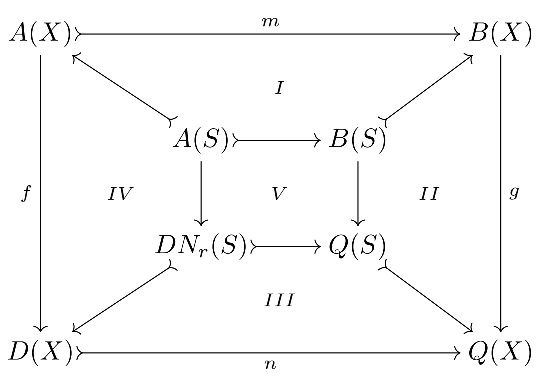



By the manner we defined $Q$ we already have a unique $\alpha : Q(X) \to \widehat{Q}(X)$ in $\mathscr{U}$. We need to show that $\alpha$ is controlled.

Let $\rho = \max\{\text{control parameter of } \widehat{n}f, \text{control parameter of } \widehat{g}\}$. This works as a control parameter for $\alpha$ by the manner we defined $\alpha$. ☑

**Lemma 27.** *$n : D \to Q$ is bicontrolled.*

***Proof.*** We have the following commutative diagram in $\mathscr{U}$.




The squares are numbered for convenience and the back square with $m$, $n$, $f$ and $g$ is numbered $VI$.

1) Square $I$ is a pullback because $m$ has bicontrol parameter 0.

2) Squares $V$ and $VI$ are pullbacks by Lemma 16 and Freyd's embedding theorem which states that if $\mathscr{U}$ is a small abelian category, then there exists a ring $R$ and a full, faithful and exact functor $F : \mathscr{U} \to R$-Mod, where $R$-Mod is the abelian category of left $R$-modules (see [4]).

3) By Lemma 17, the gluing of squares $I$ and $VI$ is a pullback.

4) By commutativity the gluing of squares $V$ and $III$ is the same as the gluing of squares $I$ and $VI$ and hence is a pullback.

5) By Lemma 19, square $III$ is a pullback which means that $n$ is bicontrolled with a parameter $r$. ☑

**Lemma 28.** *Let $m : A \to B$ be a natural transformation that has a bicontrol parameter 0 and $A(S) \rightarrowtail B(S)$ is an admissible monomorphism in $\mathscr{U}$ or $\mathscr{E}$ depending on $S$. Then $m$ can be extended to a strict short sequence.*

***Proof.*** Define $C(S)$ so that $B(S) \twoheadrightarrow C(S)$ is the cokernel of $A(S) \rightarrowtail B(S)$. We have a natural transformation $q : B \to C$. Need to check $q$ has a bicontrol parameter 0.

Now we have the following diagram

$$\begin{array}{ccc} & & \mathrm{im}(q : B(S)) \\ & = & \downarrow \\ C(S) & \rightarrowtail & C(X) \end{array}$$

which shows that $q$ has a bicontrol parameter 0. ☑

Now combining all the previous Lemmas we can finish the proof of the exactness part of the Main theorem as follows.

**Lemma 29.** *The class $\mathscr{J}$ of strict short sequences in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ satisfies all the axioms in Definition 5, other than being closed under isomorphisms.*

***Proof.*** Let the following be a strict short sequence in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$:

$$A \overset{m}{\rightarrowtail} B \overset{q}{\twoheadrightarrow} C\ .$$

1) $m$ is a kernel of $q$ and $q$ is a cokernel of $m$ by definition of a strict short sequence.

2) The class of strict admissible epimorphisms is closed under base changes by Lemmas 22, 23 and 24. and Remarks 21.

3) The class of strict admissible monomorphisms is closed under cobase changes by Lemmas 26, 27, 28 and Remarks 25.

4) The fact that $0 \to 0 \to 0$ is a strict short sequence is obvious. ☑

**Theorem 30.** *$\mathscr{E}_{(X,\mathscr{U})}$ is extension closed in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ and hence inherits the exact structure from $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$.*

***Proof.*** Let $0 \to A \rightarrowtail B \twoheadrightarrow C \to 0$ be an exact sequence in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ with $A$ and $B$ in $\mathscr{E}_{(X,\mathscr{U})}$. Without loss of generality we can assume that the above exact sequence is strict (maps are bicontrolled with parameter $\rho = 0$). Also it suffices to prove that for a subset $S \subset X$ and $x \notin S$, $B(x, S) \cong B(x) \oplus B(S)$. Here $(x, S)$ is to be understood as $\{x\} \cup S$. The above exact sequence and the definition of objects in $\mathscr{C}_X(\mathscr{E} \subset \mathscr{U})$ gives us the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(x) \oplus A(S) & \rightarrowtail & B(x) \oplus B(S) & \twoheadrightarrow & C(x) \oplus C(S) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & A(x, S) & \rightarrowtail & B(x, S) & \twoheadrightarrow & C(x, S) & \longrightarrow & 0.
\end{array}$$

Applying five lemma, we can conclude $B(x, S) \cong B(x) \oplus B(S)$. ☑

The fact that $\mathscr{E}_{(X,\mathscr{U})}$ is naturally isomorphic to $\mathscr{E}$ is straight forward. To prove that the connective spectrum $\mathbb{K}(\mathscr{E}_{(X,\mathscr{U})})$ is contractible, let us recall the following Proposition of Pedersen-Weibel [6].

**Proposition 31.** *$\mathbb{K}(\mathscr{C}_{\mathbb{Z}_{\geq 0}}(\mathscr{A}))$ is contractible.*

***Proof.*** We restrict our attention to the metric space of non-negative integers. There is an endo functor $T : \mathscr{C}_{\mathbb{Z}_{\geq 0}}(\mathscr{A}) \to \mathscr{C}_{\mathbb{Z}_{\geq 0}}(\mathscr{A})$ which is a right shift functor and a natural transformation $t : 1 \to T$. More explicitly, for any object $A \in \mathscr{C}_{\mathbb{Z}_{\geq 0}}(\mathscr{A})$, $T(A_n) = A_{n-1}$ and $t_A : A \to T(A)$ is an isomorphism, the inverse given by the left shift. Now the following diagram helps us understand that $\sum_{i \geq 0} T^n$ is an endo functor of $\mathscr{C}_{\mathbb{Z}_{\geq 0}}(\mathscr{A})$.

$$\begin{array}{cccccc}
A_0 & A_1 & A_2 & A_3 & A_4 & \cdots \\
 & A_0 & A_1 & A_2 & A_3 & \cdots \\
 & & A_0 & A_1 & A_2 & \cdots \\
 & & & A_0 & A_1 & \cdots \\
 & & & & A_0 & \cdots
\end{array}$$

$\sum_{n\geq 0} T^n$ picks up the direct sum of $A_i's$ vertically and although it is a sum of infinitely many shifts, at each stage we only get finitely many summands. Notice that $t$ induces a natural isomorphism $t : \sum_{n\geq 0} T^n \to \sum_{n\geq 1} T^n$.

As endo functors of $\mathscr{C}_{\mathbb{Z}_{\geq 0}}(\mathscr{A})$ we have a natural equivalence $1 + \sum_{n\geq 1} T^n \sum_{n\geq 1} T^n$. Hence, as self-maps of the loop space $\mathbb{K}\big(\mathscr{C}_{\mathbb{Z}_{\geq 0}}(\mathscr{A})\big)$, $1 \sim 0$. ☑

**Proposition 32.** $\mathbb{K}(\mathscr{E}_{(X,\mathscr{U})})$ *is contractible.*

***Proof***. Let $\mathscr{E} \subset \mathscr{U}$ be a small exact category embedded fully in an abelian category. Then there is a functor from the category of proper metric spaces and proper and eventually Lipschitz maps(see [7] for more details) to the category of small exact categories defined in the natural way: $X \mapsto \mathscr{E}_{(X,\mathscr{U})}$.

Given a proper eventually Lipschitz map $f : X \to Y$, we have a functor $f_* : \mathscr{E}_{(X,\mathscr{U})} \to \mathscr{E}_{(Y,\mathscr{U})}$ by defining $f_*(A)(S) := A\big(f^{-1}(S)\big)$, where $S$ is a subset of $Y$. The proper and eventually Lipschitz conditions ensure that we do get controlled maps in $\mathscr{C}_Y(\mathscr{E} \subset \mathscr{U})$.

In our case, let $X = Y = \mathbb{Z}_{\geq 0}$ and $sh$ given by $\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ where the map is addition by 1. This induces an endo functor $sh$ (a little abuse of notation) on $\mathscr{E}_{(\mathbb{Z}_{\geq 0},\mathscr{U})}$. Notice that for $A \in \mathscr{E}_{(\mathbb{Z}_{\geq 0},\mathscr{U})}$, $sh\ A(S) = A(S-1)$, where $S \subset \mathbb{Z}_{\geq 0}$ and $S-1 := \{s-1 : s \in S\}$. Hence, if we apply $sh$ to a bounded set $[a,b] \cap \mathbb{Z}_{\geq 0}$, for $b+1$ times, we get that $0 \in \mathscr{E}_{(\mathbb{Z}_{\geq 0},\mathscr{U})}$. This gives us that $\sum_{n=0}^{\infty} sh^n$ is a well defined endo functor on $\mathscr{E}_{(\mathbb{Z}_{\geq 0},\mathscr{U})}$.

Let us define the natural transformation $t : sh \to id$ as shown in the diagram below.

$$\begin{array}{ccc} sh\ A(S) = A(S-1) & \longrightarrow & sh\ A(\mathbb{Z}_{\geq 0}) = A(\mathbb{Z}_{\geq 0}) \\ \downarrow & & \| \\ AN_1(S) & \longrightarrow & A(\mathbb{Z}_{\geq 0}). \end{array}$$

This natural transformation induces a natural transformation between the endo functors $1 + \sum_{n\geq 1} sh^n = \sum_{n \geq 0} sh^n$ and $\sum_{n \geq 1} sh^n$. Hence, as self-maps of the loop space $\mathbb{K}(\mathscr{E}_{(\mathbb{Z}_{\geq 0}, \mathscr{U})})$, $1 \sim 0$. ☑

This concludes the proof of the Main Theorem.

## 6. Comparison with Pedersen-Weibel Bounded Category and Further Questions

Let $\mathscr{A}$ be an additive category, i.e., an exact category with split short exact sequences. There is an obvious functor $F : \mathscr{C}_{\mathbb{Z}}(\mathscr{A}) \to \mathscr{C}_{\mathbb{Z}}(\mathscr{A} \subset \mathscr{U})$ where $\mathscr{C}_{\mathbb{Z}}(\mathscr{A})$ is the Pedersen-Weibel definition of the bounded category. The functor $F : \mathscr{C}_{\mathbb{Z}}(\mathscr{A}) \to \mathscr{A}_{(\mathbb{Z}, \mathscr{U})}$ is defined as follows.

An object of $\mathscr{C}_{\mathbb{Z}}(\mathscr{A})$ is a set $A = \{A_n :\ A_n \in \mathscr{A}, n \in \mathbb{Z}\}$. Now the object $F(A) : \mathscr{P}(\mathbb{Z}) \to co\mathscr{U}$ in the category $\mathscr{A}_{(\mathbb{Z}, \mathscr{U})}$ is defined as follows. Let $S$ be a subset of $\mathbb{Z}$. Then $F(A)(S) := \bigoplus_{i \in S} A_i$.

Let $f : A \to B$ be a morphism in $\mathscr{C}_{\mathbb{Z}}(\mathscr{A})$. Note that $\mathscr{U}$ contains countable sums of objects of $\mathscr{A}$. Hence $F(f)$ is a map in $\mathscr{U}$.

$f \in Mor\mathscr{C}_{\mathbb{Z}}(\mathscr{A})$ means that there exists a number $\rho$ such that $f_{xy} = 0$ for all $x$ and $y$ such that $|x - y| > \rho$. $F(f)$ Now we have the following diagram

$$\begin{array}{ccc} F(A)(S) & \rightarrowtail & F(A) \\ \downarrow & & \downarrow {\scriptstyle F(f)} \\ F(B)N_\rho(S) & \rightarrowtail & F(B), \end{array}$$

showing that $F(f)$ is actually controlled and hence a map in $\mathscr{A}_{(\mathbb{Z}, \mathscr{U})}$.

Notice that in the case of an additive category $\mathscr{A}$, Pedersen-Weibel construction $\mathscr{C}_{\mathbb{Z}}(\mathscr{A})$ coincides with $\mathscr{A}_{(\mathbb{Z}, \mathscr{U})}$ and hence they have the same $\mathbb{K}$-theory.

**Question.** Is the category $\mathscr{E}_{(X, \mathscr{U})}$ a the deloop of $\mathscr{E}$?

**Problem.** Let a group $\Gamma$ act on a matrix space $X$. This action can be extended to $\mathscr{E}_{(X, \mathscr{U})}$. In this context, define an equivariant version of K-theory as carried out in [2] and analyse the fixed spectra.

**Question.** Can one deloop the $\mathbb{K}$-theory of Waldhausen Categories [11] using the bounded category methods?

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