

Dual π -Rickart Modules

Módulos π -Rickart duales

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ABSTRACT. Let R be an arbitrary ring with identity and M a right R -module with $S = \text{End}_R(M)$. In this paper we introduce dual π -Rickart modules as a generalization of π -regular rings as well as that of dual Rickart modules. The module M is said to be *dual π -Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $\text{Im } f^n = eM$. We prove that some results of dual Rickart modules can be extended to dual π -Rickart modules for this general settings. We investigate relations between a dual π -Rickart module and its endomorphism ring.

Key words and phrases. π -Rickart modules, Dual π -Rickart modules, Fitting modules, Generalized left principally projective rings, π -regular rings.

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RESUMEN. Sea R un anillo arbitrario con identidad y M un R -módulo derecho con $S = \text{End}_R(M)$. En este artículo introducimos los módulos π -Rickart duales como una generalización de los anillos π -regulares así como también de los módulos Rickart. El módulo M se dice *dual π -Rickart* si para cada $f \in S$, existe $e^2 = e \in S$ y un entero positivo n tales que $\text{Im } f^n = eM$. Demostramos que algunos resultados de los módulos de Rickart pueden ser extendidos a los módulos π -Rickart duales para este marco general. Finalmente, investigamos las relaciones entre un módulo π -Rickart dual y su anillo de endomorfismos.

Palabras y frases clave. Módulos π -Rickart, módulos π -Rickart duales, módulos ajustados, anillos izquierdos principalmente proyectivos generalizados, anillos π -regulares.

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1. Introduction

Throughout this paper R denotes an associative ring with identity, and modules are unitary right R -modules. For a module M , $S = \text{End}_R(M)$ is the ring of all right R -module endomorphisms of M . In this work, for the (S, R) -bimodule M , $l_S(\bullet)$ and $r_M(\bullet)$ are the left annihilator of a subset of M in S and the right annihilator of a subset of S in M , respectively. A ring is *reduced* if it has no nonzero nilpotent elements. *Baer rings* [8] are introduced as rings in which the right (left) annihilator of every nonempty subset is generated by an idempotent. Principally projective rings were introduced by Hattori [3] to study the torsion theory, that is, a ring R is called *left (right) principally projective* if every principal left (right) ideal is projective. The concept of left (right) principally projective rings (or left (right) Rickart rings) has been comprehensively studied in the literature. Regarding a generalization of Baer rings as well as principally projective rings, recall that a ring R is called *generalized left (right) principally projective* if for any $x \in R$, the left (right) annihilator of x^n is generated by an idempotent for some positive integer n . A number of papers have been written on generalized principally projective rings (see [4] and [7]). A ring R is (*von Neumann*) *regular* if for any $a \in R$ there exists $b \in R$ with $a = aba$. The ring R is called *π -regular* if for each $a \in R$ there exist a positive integer n and an element x in R such that $a^n = a^n x a^n$. Similarly, call a ring R *strongly π -regular* if for every element $a \in R$ there exist a positive integer n (depending on a) and an element $x \in R$ such that $a^n = a^{n+1} x$, equivalently, there exists $y \in R$ such that $a^n = y a^{n+1}$. Every regular ring is π -regular and every strongly π -regular ring is π -regular. There are regular or π -regular rings which are not strongly π -regular.

According to Rizvi and Roman, a module M is said to be *Rickart* [13] if for any $f \in S$, $r_M(f) = eM$ for some $e^2 = e \in S$. The class of Rickart modules is studied extensively by different authors (see [1] and [9]). Recently the concept of a Rickart module is generalized in [16] by the present authors. The module M is called *π -Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $r_M(f^n) = eM$. Dual Rickart modules are defined by Lee, Rizvi and Roman in [10]. The module M is called *dual Rickart* if for any $f \in S$, $\text{Im } f = eM$ for some $e^2 = e \in S$.

In the second section, we investigate general properties of dual π -Rickart modules and Section 3 contains the results on the structure of endomorphism ring of a dual π -Rickart module. In what follows, we denote by \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{Z}_n , the ring of integers, rational numbers, real numbers and the ring of integers modulo n , respectively, and $J(R)$ denotes the Jacobson radical of a ring R .

2. Dual π -Rickart Modules

In this section, we introduce the concept of a dual π -Rickart module that generalizes the notion of a dual Rickart module as well as that of a π -regular

ring. We prove that some properties of dual Rickart modules hold for this general setting. Although every direct summand of a dual π -Rickart module is dual π -Rickart, a direct sum of dual π -Rickart modules is not dual π -Rickart. We give an example to show that a direct sum of dual π -Rickart modules may not be dual π -Rickart. It is shown that the class of some abelian dual π -Rickart modules is closed under direct sums.

We start with our main definition.

Definition 1. Let M be an R -module with $S = \text{End}_R(M)$. The module M is called *dual π -Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $\text{Im } f^n = eM$.

For the sake of brevity, in the sequel, S will stand for the endomorphism ring of the module M considered. Dual π -Rickart modules are abundant. Every semisimple module, every injective module over a right hereditary ring and every module of finite length are dual π -Rickart. Also every quasi-projective strongly co-Hopfian module, every quasi-injective strongly Hopfian module, every Artinian and Noetherian module is dual π -Rickart (see Corollary 19). Every finitely generated module over a right Artinian ring is a dual π -Rickart module (see Proposition 20).

Proposition 2. *Let R be a ring. Then the right R -module R is a dual π -Rickart module if and only if R is a π -regular ring.*

Proof. If the right R -module R is a dual π -Rickart module and $f \in R$, then there exist $e^2 = e \in R$ and a positive integer n such that $\text{Im } f^n = eR$. There exist $x, y \in R$ such that $e = f^n x$ and $f^n = ey$. Multiplying the first equation from the right by f^n , we have $f^n x f^n = ey = f^n$. Conversely, assume that R is a π -regular ring. Let $g \in R$. Then there exist a positive integer n and $x \in R$ such that $g^n = g^n x g^n$. Hence $e = g^n x$ is an idempotent of R . Since $e \in g^n R$ and $g^n = g^n x g^n = e g^n \in eR$, we have $\text{Im } g^n = eR$. Therefore the right R -module R is dual π -Rickart. □

It is clear that every dual Rickart module is dual π -Rickart. The following example shows that every dual π -Rickart module need not be dual Rickart.

Example 3. Let R denote the ring $\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ and M the right R -module $\begin{pmatrix} 0 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$ with usual matrix operations. If $f \in S$, then there exist $a, b, c \in \mathbb{Z}_2$ such that

$$f \begin{pmatrix} 0 & x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & ax \\ by & cx + bz \end{pmatrix}$$

By using this image of f , we prove that there exists a positive integer n such that $\text{Im } f^n$ is a direct summand of M . Consider the following cases for $a, b, c \in \mathbb{Z}_2$.

Case 1. If $a = b = c = 1$, then f is an epimorphism.

Case 2. If $a = 0, b = 0, c = 1$, then $f^2 = 0$.

Case 3. If $a = 0, b = 1, c = 1$ or $a = 0, b = 1, c = 0$, then in either case

$$\text{Im } f = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \mid x, y \in \mathbb{Z}_2 \right\} \text{ is a direct summand of } M.$$

Case 4. If $a = 1, b = 0, c = 1$, then $\text{Im } f = \left\{ \begin{pmatrix} 0 & x \\ 0 & x \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$ is a direct summand of M .

Case 5. If $a = 1, b = 0, c = 0$, then $\text{Im } f = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$ is a direct summand of M .

Case 6. If $a = 1, b = 1, c = 0$, then f is an identity map.

Case 7. If $a = 0, b = 0, c = 0$, then f is a zero map.

In all cases there exists a positive integer n such that $\text{Im } f^n$ is a direct summand of M and so M is a dual π -Rickart module. The module M is not dual Rickart, since $\text{Im } f = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \mid x \in \mathbb{Z}_2 \right\}$ in the second case.

Our next aim is to find conditions under which a dual π -Rickart module is dual Rickart.

Proposition 4. *Let M be a dual Rickart module. Then M is dual π -Rickart. The converse holds if S is a reduced ring.*

Proof. The first statement is clear. Suppose that S is a reduced ring and M is a dual π -Rickart module. Let $f \in S$. There exist a positive integer n and an idempotent $e \in S$ such that $\text{Im } f^n = eM$. If $n = 1$, there is nothing to do. Assume that $n > 1$, then $(1 - e)f^n M = 0$ and so $(1 - e)f^n = 0$. Since S is a reduced ring, e is central and $((1 - e)f)^n = 0$. Also it implies $(1 - e)f = 0$ or $f = ef$. Thus $\text{Im } f \leq eM$. The reverse inclusion $eM \leq \text{Im } f$ follows from $eM = f^n M \leq f(f^{n-1})M \leq fM$. Therefore $eM = \text{Im } f$ and M is a dual Rickart module. \square

By using a different condition on an endomorphism ring of a module we show that a dual π -Rickart module is dual Rickart. To do this we need the following lemma.

Lemma 5. *Let M be a module. Then M is dual π -Rickart and S is a domain if and only if every nonzero element of S is an epimorphism.*

Proof. The sufficiency is clear. For the necessity, let M be a dual π -Rickart module and $0 \neq f \in S$. Then there exist a positive integer n and an idempotent $e \in S$ such that $\text{Im } f^n = eM$. Hence $f^n = ef^n$. Since S is a domain and f^n is nonzero, we have $e = 1$ and so $\text{Im } f^n = M$. This implies that $\text{Im } f = M$. Thus f is an epimorphism. \square

Recall that a module M has C_2 condition if any submodule N of M which is isomorphic to a direct summand of M is a direct summand, while a module M is said to have D_2 condition if any submodule N of M with M/N isomorphic to a direct summand of M , then N is a direct summand of M . In the next result we obtain relations between π -Rickart and dual π -Rickart modules by using C_2 and D_2 conditions. An endomorphism f of a module M is called *morphic* [12] if $M/fM \cong \text{Ker } f$. The module M is called *morphic* if every endomorphism of M is morphic.

Theorem 6. *Let M be a module. Then we have the following.*

- (1) *If M is a dual π -Rickart module with D_2 condition, then it is π -Rickart.*
- (2) *If M is a π -Rickart module with C_2 condition, then it is dual π -Rickart.*
- (3) *If M is projective morphic, then it is π -Rickart if and only if it is dual π -Rickart.*

Proof. Since $M/\text{Ker } f^n \cong \text{Im } f^n$ for any positive integer n , D_2 and C_2 conditions complete the proof of (1) and (2). The proof of (3) is clear. \square

The next result is an immediate consequence of Theorem 6.

Corollary 7. *Let M be a module with C_2 and D_2 conditions. Then M is a dual π -Rickart module if and only if it is π -Rickart.*

In [10, Proposition 2.6], it is shown that M is a dual Rickart module if and only if the short exact sequence $0 \rightarrow \text{Im } f \rightarrow M \rightarrow M/\text{Im } f \rightarrow 0$ splits for any $f \in S$. In this direction we can give a similar characterization for dual π -Rickart modules.

Lemma 8. *The following are equivalent for a module M .*

- (1) *M is a dual π -Rickart module.*
- (2) *For every $f \in S$ there exists a positive integer n such that the short exact sequence $0 \rightarrow \text{Im } f^n \rightarrow M \rightarrow M/\text{Im } f^n \rightarrow 0$ splits.*

Proof. For any $f \in S$ and any positive integer n consider the short exact sequence $0 \rightarrow \text{Im } f^n \rightarrow M \rightarrow M/\text{Im } f^n \rightarrow 0$. The short exact sequence splits in M if and only if $\text{Im } f^n$ is a direct summand of M if and only if M is a dual π -Rickart module. \square

One may suspect that every submodule of a dual π -Rickart module is dual π -Rickart. The following example shows that this is not the case.

Example 9. Consider \mathbb{Q} as a \mathbb{Z} -module. Then $S = \text{End}_{\mathbb{Z}}(\mathbb{Q})$ is isomorphic to \mathbb{Q} . Since every element of S is an isomorphism or zero, \mathbb{Q} is dual π -Rickart. Now consider the submodule \mathbb{Z} and $f \in \text{End}_{\mathbb{Z}}(\mathbb{Z})$ defined by $f(x) = 2x$, where $x \in \mathbb{Z}$. Since the image of any power of f can not be a direct summand of \mathbb{Z} , the submodule \mathbb{Z} is not dual π -Rickart.

Although every submodule of a dual π -Rickart module need not be dual π -Rickart by Example 9, we now prove that every direct summand of dual π -Rickart modules is also dual π -Rickart.

Proposition 10. *Let M be a dual π -Rickart module. Then every direct summand of M is also dual π -Rickart.*

Proof. Let $M = N \oplus P$ with $S_N = \text{End}_R(N)$. Define $g = f \oplus 0|_P$, for any $f \in S_N$ and so $g \in S$. By hypothesis, there exist a positive integer n and $e^2 = e \in S$ such that $\text{Im } g^n = eM$ and $g^n = f^n \oplus 0|_P$. Hence $eM = \text{Im } g^n = f^n N \leq N$. Let $M = eM \oplus Q$ for some submodule Q . Thus $N = eM \oplus (N \cap Q) = f^n N \oplus (N \cap Q)$. Therefore N is dual π -Rickart. \square

Corollary 11. *Let R be a π -regular ring with $e = e^2 \in R$. Then eR is a dual π -Rickart R -module.*

Here we give the following result for π -regular rings.

Corollary 12. *Let $R = R_1 \oplus R_2$ be a π -regular ring with direct sum of the rings R_1 and R_2 . Then the rings R_1 and R_2 are also π -regular.*

We now characterize π -regular rings in terms of dual π -Rickart modules.

Theorem 13. *Let R be a ring. Then R is π -regular if and only if every cyclic projective R -module is dual π -Rickart.*

Proof. The sufficiency is clear. For the necessity, let $M = mR$ be a projective module. Then $R = r_R(m) \oplus I$ for some right ideal I of R . Let $I \xrightarrow{\varphi} M$ denote the isomorphism and $f \in S$. By Proposition 2 and Proposition 10, $(\varphi^{-1}f\varphi)^n I = (\varphi^{-1}f^n\varphi)I$ is a direct summand of I for some positive integer n . Hence $I = (\varphi^{-1}f^n\varphi)I \oplus K$ for some right ideal K of I . Thus $\varphi I = (f^n\varphi)I \oplus \varphi K$, and so $M = f^n M \oplus \varphi K$. Therefore M is dual π -Rickart. \square

Theorem 14. *Let R be a ring and consider the following conditions:*

- (1) *Every free R -module is dual π -Rickart,*
- (2) *Every projective R -module is dual π -Rickart,*
- (3) *Every flat R -module is dual π -Rickart.*

Then (3) \Rightarrow (2) \Leftrightarrow (1). Moreover (2) \Rightarrow (3) holds for finitely presented modules.

Proof. (3) \Rightarrow (2) \Rightarrow (1) Clear. (1) \Rightarrow (2) Let M be a projective R -module. Then M is a direct summand of a free R -module F . By (1), F is dual π -Rickart, and so is M due to Proposition 10. (2) \Rightarrow (3) is clear from the fact that finitely presented flat modules are projective. \square

The next example reveals that a direct sum of dual π -Rickart modules need not be dual π -Rickart.

Example 15. Let R denote the ring $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ and M the R -module $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$.

Let $f \in S$. Then there exist $a, c, u, t \in R$ such that

$$f \begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} ax + ur & ay + us \\ cx + tr & cy + ts \end{pmatrix}$$

where $\begin{pmatrix} x & y \\ r & s \end{pmatrix} \in M$.

Consider $f \in S$ defined by $a = c = 0$, $u = 3$ and $t = 2$. This implies that $f \begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} 3r & 3s \\ 2r & 2s \end{pmatrix}$ and for any positive integer n we obtain

$$f^n \begin{pmatrix} x & y \\ r & s \end{pmatrix} = \begin{pmatrix} 3(2^{n-1})r & 3(2^{n-1})s \\ 2^n r & 2^n s \end{pmatrix}.$$

It follows that $f^n M$ can not be a direct summand. On the other hand, consider the submodules $N = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ 0 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ of M . Then $\text{End}_R(N)$ and $\text{End}_R(K)$ are isomorphic to \mathbb{R} . Hence N and K are dual π -Rickart modules but M is not dual π -Rickart.

The following lemma is useful to show that a direct sum of some dual π -Rickart modules is a dual π -Rickart.

Lemma 16. *Let M be a module and $f \in S$. If $\text{Im } f^n = eM$ for some central idempotent $e \in S$ and a positive integer n , then $\text{Im } f^{n+1} = eM$.*

Proof. Let $f \in S$ and $\text{Im } f^n = eM$ for some central idempotent $e \in S$ and a positive integer n . It is clear that $\text{Im } f^{n+1} \subseteq \text{Im } f^n$. Let $f^n(x) \in \text{Im } f^n$, then $f^n(x) = ef^n(x) = f^n e(x)$. Since $e(x) \in \text{Im } f^n$, $e(x) = f^n(y)$ for some $y \in M$. So $f^n(x) = f^n(f^n(y)) = f^{n+1}(f^{n-1}(y)) \in \text{Im } f^{n+1}$. This completes the proof. \checkmark

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $a, e^2 = e \in R$. A module M is called *abelian* [14] if $fem = efm$ for any $f \in S, e^2 = e \in S, m \in M$. Note that M is an abelian module if and only if S is an abelian ring. We now prove that a direct sum of dual π -Rickart modules is dual π -Rickart for some abelian modules.

Proposition 17. *Let M_1 and M_2 be abelian R -modules. If M_1 and M_2 are dual π -Rickart with $\text{Hom}_R(M_i, M_j) = 0$ for $i \neq j$, then $M_1 \oplus M_2$ is a dual π -Rickart module.*

Proof. Let $S_1 = \text{End}_R(M_1), S_2 = \text{End}_R(M_2)$ and $M = M_1 \oplus M_2$. We may describe S as $\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}$. Let $\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \in S$, where $f_1 \in S_1$ and $f_2 \in S_2$. Then there exist positive integers n, m and $e_1^2 = e_1 \in S_1$ and $e_2^2 = e_2 \in S_2$ such that $\text{Im } f_1^n = e_1 M_1$ and $\text{Im } f_2^m = e_2 M_2$. Consider the following cases:

i.) Let $n = m$. Obviously, $\text{Im } \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^n = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$.

ii.) Let $n < m$. By Lemma 16, we have $\text{Im } f_1^n = \text{Im } f_1^m = e_1 M_1$. Clearly, $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M \leq \text{Im } \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m$. Now let $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \text{Im } \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m$. Then $m_1 \in \text{Im } f_1^m = e_1 M_1$ and $m_2 \in \text{Im } f_2^m = e_2 M_2$. Hence $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$. Thus $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$. Therefore $\text{Im } \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}^m \leq \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} M$.

iii.) Let $m < n$. Since M_2 is abelian, the proof is similar to case ii. \checkmark

We close this section with the relations among strongly co-Hopfian modules, Fitting modules and dual π -Rickart modules.

Recall that a module M is called *co-Hopfian* if every injective endomorphism of M is an automorphism, while M is called *strongly co-Hopfian* [5], if for any endomorphism f of M the descending chain

$$\text{Im } f \supseteq \text{Im } f^2 \supseteq \dots \supseteq \text{Im } f^n \supseteq \dots$$

stabilizes.

We now give a relation between abelian and strongly co-Hopfian modules by using dual π -Rickart modules.

Corollary 18. *Let M be a dual π -Rickart module and S an abelian ring. Then M is strongly co-Hopfian.*

Proof. It follows from Lemma 16 and [5, Proposition 2.6]. \square

A module M is said to be a *Fitting module* [5] if for any $f \in S$, there exists an integer $n \geq 1$ such that $M = \text{Ker } f^n \oplus \text{Im } f^n$. Due to Armendariz, Fisher and Snider [2] or [15, Proposition 5.7], the module M is Fitting if and only if S is strongly π -regular.

We now give the following relation between Fitting modules and dual π -Rickart modules.

Corollary 19. *Every Fitting module is a dual π -Rickart module.*

Then we have the following result.

Proposition 20. *Let R be an Artinian ring. Then every finitely generated R -module is dual π -Rickart.*

Proof. Let M be a finitely generated R -module. Then M is an Artinian and Noetherian module. Hence M is a Fitting module and so it is dual π -Rickart. \square

Proposition 21. *Let R be a ring and n a positive integer. If the matrix ring $M_n(R)$ is strongly π -regular, then R^n is a dual π -Rickart R -module.*

Proof. Let $M_n(R)$ be a strongly π -regular ring. Then by [5, Corollary 3.6], R^n is a Fitting R -module and so it is dual π -Rickart. \square

3. The Endomorphism Ring of a Dual π -Rickart Module

In this section we study relations between a dual π -Rickart module and its endomorphism ring. We prove that the endomorphism ring of a dual π -Rickart module is always a generalized left principally projective ring. The converse holds if the module is self-cogenerator. The modules whose endomorphism rings are π -regular are characterized. It is shown that if the module satisfies D_2 condition, then it is dual π -Rickart if and only if the endomorphism ring of the module is a π -regular ring.

Lemma 22. *If M is a dual π -Rickart module, then S is a generalized left principally projective ring.*

Proof. Let $f \in S$. By assumption, there exist $e^2 = e \in S$ and a positive integer n such that $\text{Im } f^n = eM$. Hence $l_S(f^n M) = S(1 - e) = l_S(f^n)$. Thus S is a generalized left principally projective ring. \checkmark

The next result is a consequence of Theorem 10 and Lemma 22.

Corollary 23. *If R is a π -regular ring, then eRe is a generalized left principal projective ring for any $e^2 = e \in R$.*

Corollary 24. *Let M be a dual π -Rickart module and $f \in S$. Then Sf^n is a projective left S -module for some positive integer n .*

Proof. Clear from Lemma 22, since $Sf^n \cong S/l_S(f^n)$. \checkmark

Recall that a module is called *self-cogenerator* if it cogenerates all its factor modules. The following result shows that the converse of Lemma 22 is true for self-cogenerator modules. On the other hand, Theorem 25 generalizes the result [17, 39.11].

Theorem 25. *Let M be a module and $f \in S$.*

- i.) *If Sf^n is a projective left S -module for some positive integer n , then the submodule $N = \bigcap \{\text{Ker } g : g \in S, \text{Im } f^n \leq \text{Ker } g\}$ is a direct summand of M .*
- ii.) *If M is self-cogenerator and S is a generalized left principally projective ring, then M is a dual π -Rickart module.*

Proof. i.) Let Sf^n be a projective left S -module for some positive integer n . Since $Sf^n \cong S/l_S(f^n)$, $l_S(f^n) = Se$ for some $e^2 = e \in S$. We prove $(1 - e)M = N$. Due to $ef^n M = 0$, we have $f^n M \leq (1 - e)M$. By definition of N we have $N \leq (1 - e)M$. Let $g \in S$ with $\text{Im } f^n \leq \text{Ker } g$. Then $gf^n M = 0$ or $gf^n = 0$. Hence $g \in l_S(f^n) = Se$ and $ge = g$. So $g(1 - e)M = 0$ from which we have $(1 - e)M \leq \text{Ker } g$ for all g with $\text{Im } f^n \leq \text{Ker } g$. Thus $(1 - e)M \leq N$. Therefore $(1 - e)M = N$.

ii.) Assume that M is self-cogenerator and S is generalized left principally projective. There exist $e^2 = e \in S$, a positive integer n such that $l_S(f^n) = Se$ and $M/\text{Im } f^n$ is cogenerated by M . By [17, 14.5],

$$\bigcap \{ \text{Ker } g : g \in \text{Hom}(M/\text{Im } f^n, M) \} = 0.$$

Hence

$$\text{Im } f^n = \bigcap \{ \text{Ker } g : g \in S, \text{Im } f^n \leq \text{Ker } g \}.$$

Thus conditions of i.) are satisfied and so $\text{Im } f^n$ is a direct summand. \checkmark

For an R -module M , it is shown that, if S is a von Neumann regular ring, then M is a dual Rickart module (see [10, Proposition 3.8]). We obtain a similar result for dual π -Rickart modules.

Lemma 26. *Let M be a module. If S is a π -regular ring, then M is dual π -Rickart.*

Proof. Let $f \in S$. Since S is π -regular, there exist a positive integer n and $g \in S$ such that $f^n = f^n g f^n$. Then $e = f^n g$ is an idempotent of S . Now we show that $\text{Im } f^n = f^n g M$. It is clear that $f^n M = e f^n M \leq e M$. For the other inclusion, let $m \in M$. Hence $em = f^n g m \in f^n M$. Thus $\text{Im } f^n = e M$. \square

Since every strongly π -regular ring is π -regular, we have the next result.

Corollary 27. *Let M be a module. If S is a strongly π -regular ring, then M is dual π -Rickart.*

The converse statement of Corollary 27 does not hold in general, that is, there exists a dual π -Rickart module having not a strongly π -regular endomorphism ring.

Example 28. Let D be a division ring, M a vector space over D with an infinite basis $\{e_i \in M : i = 1, 2, \dots\}$ and $S = \text{End}_D(M)$. As a semisimple right D -module, M is dual π -Rickart, and by [17, 3.9] S is a regular and so π -regular ring. Assume that S is a strongly π -regular ring and we reach a contradiction. Let $f \in S$ defined by $f(e_i) = e_{i+1}$ for all $i = 1, 2, 3, \dots$. By assumption, there is a positive integer n such that $f^n = f^{n+1} g$ for some $g \in S$. Then $f^n = f^{n+1} g$ implies $f^n S = f^{n+1} S$ and so $f^n M = f^{n+1} M$. Since $f^n(e_i) = e_{i+n}$ for all i , we have $f^n M = \sum_{i>n} e_i D \neq f^{n+1} M$. This is a contradiction. Hence S is not a strongly π -regular ring (see also [15, 5.5]).

The proof of Lemma 29 may be in the context.

Lemma 29. *Let M be a module. Then S is a π -regular ring if and only if there exists a positive integer n such that $\text{Ker } f^n$ and $\text{Im } f^n$ are direct summands of M for any $f \in S$.*

Now we recall some known facts about π -regular rings that will be needed.

Lemma 30. *Let R be a ring. Then*

- i.) If R is π -regular, then $e R e$ is also π -regular for any $e^2 = e \in R$.*
- ii.) If $M_n(R)$ is π -regular for any positive integer n , then so is R .*
- iii.) If R is a commutative ring, then R is π -regular if and only if $M_n(R)$ is π -regular for any positive integer n .*

Proposition 31. *Let R be a commutative π -regular ring. Then every finitely generated projective R -module is dual π -Rickart.*

Proof. Let M be a finitely generated projective R -module. So the endomorphism ring of M is $eM_n(R)e$ with some positive integer n and an idempotent e in $M_n(R)$. Since R is commutative π -regular, $M_n(R)$ is also π -regular, and so is $eM_n(R)e$ by Lemma 30. Hence M is dual π -Rickart by Lemma 26. \square

Theorem 32. *Let M be a module with D_2 condition. Then M is dual π -Rickart if and only if S is π -regular.*

Proof. The necessity holds by Lemma 26. For the sufficiency, let $0 \neq f \in S$. Since M is dual π -Rickart, $\text{Im } f^n$ is a direct summand of M for some positive integer n . Because of $M/\text{Ker } f^n \cong \text{Im } f^n$, D_2 condition implies that $\text{Ker } f^n$ is a direct summand of M . The rest is obvious from Lemma 29. \square

The following is a consequence of Proposition 31 and Theorem 32.

Corollary 33. *Let R be a commutative ring and satisfy D_2 condition. Then the following are equivalent.*

- i.) R is a π -regular ring.
- ii.) Every finitely generated projective R -module is dual π -Rickart.

Recall that a module M is called *quasi-projective* if it is M -projective. Since every quasi-projective module has D_2 condition, we have the following.

Corollary 34. *If M is a quasi-projective dual π -Rickart module, then the endomorphism ring of M is a π -regular ring.*

Theorem 35. *The following are equivalent for a ring R .*

- i.) $M_n(R)$ is π -regular for every positive integer n .
- ii.) Every finitely generated projective R -module is dual π -Rickart.

Proof. i.) \Rightarrow ii.) Let M be a finitely generated projective R -module. Then $M \cong eR^n$ for some positive integer n and $e^2 = e \in M_n(R)$. Hence S is isomorphic to $eM_n(R)e$. By i.), S is π -regular. Thus, due to Lemma 26, M is π -Rickart.

ii.) \Rightarrow i.) $M_n(R)$ can be viewed as the endomorphism ring of a projective R -module R^n for any positive integer n . By ii.), R^n is dual π -Rickart. Then, by Corollary 34, $M_n(R)$ is π -regular. \square

Recall that an R -module M is called *duo* if every submodule of M is fully invariant, i.e., for any submodule N of M , $fN \leq N$ for each $f \in S$. Equivalently, every right R -submodule of M is also left S -submodule. Our next aim is to determine conditions under which any factor module of a dual π -Rickart module is also dual π -Rickart.

Corollary 36. *Let M be a quasi-projective module and N a fully invariant submodule of M . If M is dual π -Rickart, then so is M/N .*

Proof. Let $f \in S$ and π denote the natural epimorphism from M to M/N . Consider the following diagram.

$$\begin{array}{ccc}
 M & \xrightarrow{\pi} & M/N \\
 f \downarrow & & \downarrow f^* \\
 M & \xrightarrow{\pi} & M/N
 \end{array}$$

Since N is fully invariant, we have $\text{Ker } \pi \subseteq \text{Ker } \pi f$. By the Factor Theorem, there exists a unique homomorphism f^* such that $f^* \pi = \pi f$. Hence we define a homomorphism $\varphi : S \rightarrow \text{End}_R(M/N)$ with $\varphi(f) = f^*$ for any $f \in S$. As M is quasi-projective, φ is an epimorphism. Thus $\text{End}_R(M/N) \cong S/\text{Ker } \varphi$. By Corollary 34, S is π -regular, and so is $S/\text{Ker } \varphi$. Therefore, due to Lemma 26 M/N is dual π -Rickart. \square

Corollary 37. *Let M be a quasi-projective duo module. If M is dual π -Rickart, then M/N is also dual π -Rickart for every submodule N of M .*

Corollary 38. *If M be a quasi-projective dual π -Rickart module, then so is $M/\text{Rad}(M)$ and $M/\text{Soc}(M)$.*

Proposition 39. *Let M be a dual π -Rickart module. Then every endomorphism of M with a small image in M is nilpotent.*

Proof. Let $f \in S$ with $\text{Im } f$ small in M . Then $\text{Im } f^n$ is a direct summand of M for some positive integer n . Also $\text{Im } f^n$ is small in M . Hence $f^n = 0$. \square

Corollary 40. *Let M be a dual π -Rickart discrete module. Then $J(S)$ is nil and $S/J(S)$ is von Neumann regular.*

Proof. Since M is discrete, by [11, Theorem 5.4], $J(S)$ consists of endomorphisms with small image. By Proposition 39, $J(S)$ is nil and again by [11, Theorem 5.4], $S/J(S)$ is von Neumann regular. \square

Theorem 41. *The following are equivalent for a module M .*

- i.) M is a dual π -Rickart module.

ii.) S is a generalized left principally projective ring and $f^n M = r_M(l_S(f^n M))$ for all $f \in S$ and a positive integer n .

Proof. i.) \Rightarrow ii.) By Lemma 22, we only need to show that $f^n M = r_M(l_S(f^n M))$ for all $f \in S$. Since M is dual π -Rickart, for any $f \in S$, $f^n M = eM$ for some $e^2 = e \in S$ and a positive integer n . Thus $r_M(l_S(f^n M)) = r_M(l_S(eM)) = eM = f^n M$.

ii.) \Rightarrow i.) Let $f \in S$. Since S is a generalized left principally projective ring, $l_S(f^n M) = Se$ for some $e^2 = e \in S$ and a positive integer n . By hypothesis, $f^n M = r_M(l_S(f^n M)) = r_M(Se) = (1 - e)M$. Thus M is dual π -Rickart. \checkmark

Corollary 42. Let M be a module. Then M is dual π -Rickart if and only if $f^n M = r_M(l_S(f^n M))$ and $r_M(l_S(f^n M))$ is a direct summand of M .

Theorem 43. Let M be a dual π -Rickart module. Then the left singular ideal $Z_l(S)$ of S is nil and $Z_l(S) \subseteq J(S)$.

Proof. Let $f \in Z_l(S)$. Since M is dual π -Rickart, $\text{Im}(f^n) = eM$ for some positive integer n and $e = e^2 \in S$. Then, by Lemma 22, $l_S(f^n) = S(1 - e)$. Since $l_S(f^n)$ is essential in S as a left ideal, we have $l_S(f^n) = S$. This implies that $f^n = 0$ and so $Z_l(S)$ is nil. On the other hand, for any $g \in S$ and $f \in Z_l(S)$, according to previous discussion, $(gf)^n = 0$ for some positive integer n . Hence $1 - gf$ is invertible. Thus $f \in J(S)$. Therefore $Z_l(S) \subseteq J(S)$. \checkmark

Proposition 44. The following are equivalent for a module M .

- i.) M is an indecomposable dual π -Rickart module.
- ii.) Each element of S is either an epimorphism or nilpotent.

Proof. i.) \Rightarrow ii.) Let $f \in S$. Then $f^n M$ is a direct summand of M for some positive integer n . As M is indecomposable, we see that $f^n M = 0$ or $f^n M = M$. This implies that f is an epimorphism or nilpotent.

ii.) \Rightarrow i.) Let $e = e^2 \in S$. If e is nilpotent, then $e = 0$. If e is an epimorphism, then $e = 1$. Hence M is indecomposable. Also for any $f \in S$, $fM = M$ or $f^n M = 0$ for some positive integer n . Therefore M is dual π -Rickart. \checkmark

Theorem 45. Consider the following conditions for a module M .

- i.) S is a local ring with nil Jacobson radical.
- ii.) M is an indecomposable dual π -Rickart module.

Then i.) \Rightarrow ii.). If M is a morphic module, then ii.) \Rightarrow i.).

Proof. i.) \Rightarrow ii.) Clearly, each element of S is either an epimorphism or nilpotent. Then, due to Proposition 44, M is indecomposable dual π -Rickart.

ii.) \Rightarrow i.) Let $f \in S$. Then $f^n M = eM$ for some positive integer n and an idempotent e in S . If $e = 1$, then f is an epimorphism. Since M is morphyic, f is invertible by [12, Corollary 2]. If $e = 0$, then $f^n = 0$. Hence $1 - f$ is invertible. This implies that S is a local ring. Now let $0 \neq f \in J(S)$. Since f is not invertible and M is morphyic, f is nilpotent by Proposition 44. Therefore $J(S)$ is nil. \square

The next result can be obtained from Theorem 45 and [6, Lemma 2.11].

Corollary 46. *Let M be an indecomposable dual π -Rickart module. If M is morphyic, then S is a left and right π -morphyic ring.*

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References

- [1] N. Agayev, S. Halicioglu, and A. Harmanci, *On Rickart Modules*, Bull. Iran. Math. Soc. (2012), *to appears*, available at <http://www.iranjournals.ir/ims/bulletin/>.
- [2] E. P. Armendariz, J. W. Fisher, and R. L. Snider, *On Injective and Surjective Endomorphisms of Finitely Generated Modules*, Comm. Algebra **6** (1978), no. 7, 659–672.
- [3] A. Hattori, *A Foundation of the Torsion Theory over General Rings*, Nagoya Math. J. **17** (1960), 147–158.
- [4] Y. Hirano, *On Generalized $p.p.$ Rings*, Math. J. Okayama Univ. **25** (1983), no. 1, 7–11.
- [5] A. Hmaimou, A. Kaidi, and E. Sanchez Campos, *Generalized Fitting Modules and Rings*, J. Algebra **308** (2007), no. 1, 199–214.
- [6] Q. Huang and J. Chen, *π -Morphyic Rings*, Kyungpook Math. J. (2007), no. 47, 363–372.
- [7] C. Huh, H. K. Kim, and Y. Lee, *$p.p.$ Rings and Generalized $p.p.$ Rings*, Pure Appl. Algebra J. **167** (2002), no. 1, 37–52.
- [8] I. Kaplansky, *Rings of Operators*, Math. Lecture Note Series, New York, United States, 1965.
- [9] G. Lee, S. T. Rizvi, and C. S. Roman, *Rickart Modules*, Comm. Algebra **38** (2010), no. 11, 4005–4027.

- [10] ———, *Dual Rickart Modules*, *Comm. Algebra* **39** (2011), 4036–4058.
- [11] S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, Cambridge University Press, 1990.
- [12] W. K. Nicholson and E. Sanchez Campos, *Morphic Modules*, *Comm. Algebra* **33** (2005), 2629–2647.
- [13] S. T. Rizvi and C. S. Roman, *On Direct Sums of Baer Modules*, *J. Algebra* **321** (2009), 682–696.
- [14] J. E. Roos, *Sur les categories spectrales localement distributives*, *C. R. Acad. Sci. Paris Ser.* **265** (1967), no. A-B, 14–17 (fr).
- [15] A. A. Tuganbaev, *Semiregular, Weakly Regular and π -Regular Rings*, *Journal of Mathematical Sciences* **109** (2002), no. 3, 1509–1588.
- [16] B. Ungor, S. Halicioglu, and A. Harmanci, *A Generalization of Rickart Modules*, *submitted for publication*, available at <http://arxiv.org/abs/1204.2343>, 2012.
- [17] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.

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