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Fourier-Mukai Transform for Twisted Derived Categories of Surfaces

La transformada de Fourier-Mukai para categorías derivadas torcidas de superficies

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Abstract. In this paper we study the classification of surfaces under twisted derived categories.

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Resumen. En este artículo estudiamos la clasificación de superficies bajo las categorías derivadas torcidas.

Palabras y frases clave. Categorías derivadas torcidas, grupos de Brauer, espacios moduli.

1. Introduction

Bridgeland proved in [3] a classification of surfaces under derived categories. In this paper we study partial results in the classification of surfaces under twisted derived categories. First of all we check that an equivalence of categories $\Phi : D^b(X, \alpha) \to D^b(Y, \beta)$ preserves the Kodaira dimension, which follows from an isomorphism between the canonical rings $R(X)$ and $R(Y)$ induced by the equivalence (Proposition 22). This was proved by Orlov in [15] for the untwisted case. We show that the Brauer group of a surface X of Kodaira dimension $k(X) = -\infty$ is zero (Proposition 36), which implies that there is nothing new in the classification of surfaces of this kind. For surfaces of general type, that means of Kodaira dimension 2, we have that an equivalence between twisted derived categories as above implies an isomorphism between the surfaces (Proposition 35) just as in the untwisted case. Now if we are in the case of surfaces of Kodaira dimension 1, we have that from an equivalence as above with $\beta = 0$ we get that X can be described as a moduli space defined over Y (Proposition 46). We remark that we do not treat the case of surfaces of Kodaira dimension 0 in this paper. Finally we show that for an elliptic surface X with a section and Kodaira dimension 1 there are no nonisomorphic surfaces Y derived equivalent to X that belongs to the Shafarevich group $\mathrm{Sh}(X)$ (Proposition 52).

2. Basic Facts

Let X be a smooth projective variety. We define the *cohomological Brauer group* of X to be the torsion part of the cohomology group $H^2\big(X, \mathcal{O}_X^*\big)$ in the analytic topology (or, in $H^2_{et}\big(X, \mathcal{O}_X^*\big)$ for the étale topology). We denote it by $\mathrm{Br}'(X)$.

Definition 1. A *twisted variety* (X, α) consists of a variety X together with a Brauer class $\alpha \in \mathrm{Br}'(X)$.

If (X, α) is a twisted variety, $\alpha \in \mathrm{Br}'(X)$ can be represented as a Čech 2-cocycle on an open analytic cover $\{U_i\}_{i\in I}$ of X by sections

$$\alpha_{ijk} \in \Gamma\big(U_i \cap U_j \cap U_k, \mathcal{O}_X^*\big).$$

We say that $\mathcal{F}$ is an *α-twisted quasi-coherent (coherent) sheaf* if this consists of a pair $\big(\mathcal{F}_i, \{\varphi_{ij}\}_{i,j\in I}\big)$ where $\mathcal{F}_i$ is a quasi-coherent (coherent) sheaf on U_i and

$$\varphi_{ij} : \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$$

is an isomorphism satisfying the following conditions (i.e. the α-twisted cocycle conditions):

i) $\varphi_{ii} = \mathrm{id}$,

ii) $\varphi_{ij} = \varphi_{ji}^{-1}$,

iii) $\varphi_{jk} \circ \varphi_{ij} \circ \varphi_{ki} = \alpha_{ijk}.\,\mathrm{id}$.

If for every $i \in I$, $\mathcal{F}_i$ is only a sheaf of $\mathcal{O}_X$-modules on U_i, we say that $\mathcal{F}$ is an α-twisted sheaf and we denote by $\mathrm{Mod}(X, \alpha)$ the abelian category of α-twisted sheaves.

***Remark* 2.** If X is a smooth projective variety (defined over an arbitrary field) and $\alpha \in H^2_{et}\big(X, \mathcal{O}_X^*\big)$, the abelian category $\mathrm{Coh}(X, \alpha)$ contains a locally free α-twisted coherent sheaf.

Theorem 3. ([13, App. C]) *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ be an additive, left exact functor. Assume that $\mathcal{A}$ has enough injectives, so that the derived functor*

$$RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

exists. Let $X^\bullet$ be a complex in $D^+(\mathcal{A})$. Then there exists a spectral sequence $E_k^{i,j}$ such that

$$E_2^{i,j} = R^i F\big(H^j(X^\bullet)\big) \Rightarrow H^{i+j}\big(RF(X^\bullet)\big).$$

We recall some spectral sequences defined in the derived category $\mathrm{D}^b(X)$ on a smooth variety X (cf. [11, Ch. II and III]):

$$E_2^{p,q} = \mathcal{E}xt^p\big(\mathcal{F}^\bullet, \mathcal{H}^q(\mathcal{E}^\bullet)\big) \Rightarrow \mathcal{E}xt^{p+q}(\mathcal{F}^\bullet, \mathcal{E}^\bullet), \tag{1}$$

$$E_2^{p,q} = \mathcal{E}xt^p\big(\mathcal{H}^{-q}(\mathcal{F}^\bullet), \mathcal{E}^\bullet\big) \Rightarrow \mathcal{E}xt^{p+q}(\mathcal{F}^\bullet, \mathcal{E}^\bullet), \tag{2}$$

$$E_2^{p,q} = \mathcal{T}or^{-p}\big(\mathcal{H}^q(\mathcal{F}^\bullet), \mathcal{E}^\bullet\big) \Rightarrow \mathcal{T}or^{-(p+q)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet), \tag{3}$$

for any $\mathcal{E}^\bullet, \mathcal{F}^\bullet$ in $\mathrm{D}^b(X)$.

We see now some applications of this spectral sequences in twisted derived categories. Let (X, α) be a smooth variety and $\mathcal{P} \in \mathrm{D}^b(X, \alpha)$. We use the spectral sequence (2) to show that the support of the object $\mathcal{P}$ remains the same under taking its dual. Take a locally free α^{-1}-twisted sheaf L on X and consider the spectral sequence

$$\mathcal{E}xt^p\big(\mathcal{H}^{-q}(\mathcal{P}\otimes L), \mathcal{O}_X\big) \Rightarrow \mathcal{E}xt^{p+q}(\mathcal{P}\otimes L, \mathcal{O}_X) = \mathcal{H}^{p+q}\big(\mathcal{P}^\vee \otimes L^\vee\big).$$

Hence

$$\begin{aligned}\operatorname{supp}\big(\mathcal{P}^\vee\otimes L^\vee\big) &= \\ &\bigcup \operatorname{supp}\big(\mathcal{H}^i\big(\mathcal{P}^\vee\otimes L^\vee\big)\big) \subseteq \bigcup \operatorname{supp}\big(\mathcal{H}^i(\mathcal{P}\otimes L)\big) = \operatorname{supp}(\mathcal{P}\otimes L).\end{aligned}$$

Since L is a locally free α^{-1}-twisted sheaf,

$$\operatorname{supp}\big(\mathcal{P}^\vee\big) = \operatorname{supp}\big(\mathcal{P}^\vee\otimes L^\vee\big) \subseteq \operatorname{supp}(\mathcal{P}\otimes L) = \operatorname{supp}(\mathcal{P})$$

and from $\big(\mathcal{P}^\vee\big)^\vee \cong \mathcal{P}$, we get the other inclusion. Thus

$$\operatorname{supp}(\mathcal{P}) = \operatorname{supp}\big(\mathcal{P}^\vee\big).$$

Let $\mathcal{A}$ be a k-linear category. A *Serre functor* is a k-linear equivalence $S : \mathcal{A} \to \mathcal{A}$ such that for any two objects $A, B \in \mathcal{A}$ there exists an isomorphism

$$\eta_{A,B} : \operatorname{Hom}(A, B) \xrightarrow{\sim} \operatorname{Hom}\big(B, S(A)\big)^\vee$$

of k-vector spaces which is functional in A and B.

Example 4. Let X be a smooth projective variety. The functor

$$\begin{aligned} S : \mathrm{D}^b(X) &\to \mathrm{D}^b(X) \\ \mathcal{E} &\mapsto \mathcal{E}\otimes\omega_X\big[\dim(X)\big],\end{aligned}$$

where ω_X is the dualizing sheaf of X, is a Serre functor.

Example 5. If (X, α) is a twisted smooth projective variety, the functor

$$S_{(X,\alpha)} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(X,\alpha)$$
$$\mathcal{E} \mapsto \mathcal{E} \otimes \omega_X \big[\dim(X)\big],$$

is a Serre functor.

Definition 6. A collection of objects Ω in the category $\mathrm{D}^b(X,\alpha)$ is a spanning class of (or spans) $\mathrm{D}^b(X,\alpha)$ if for all $G \in \mathrm{D}^b(X,\alpha)$ the following equivalent conditions hold:

i) If $\mathrm{Hom}(F, G[i]) = 0$ for all $F \in \Omega$ and all $i \in \mathbb{Z}$ then $G \cong 0$,

ii) If $\mathrm{Hom}(G[i], F) = 0$ for all $F \in \Omega$ and all $i \in \mathbb{Z}$ then $G \cong 0$.

The equivalence in the last definition follows immediately by using the Serre functor $S_{(X,\alpha)}$. The proof of the following proposition is identical to that of the untwisted case (cf. [11, Prop. 3.16]).

Proposition 7. *Let (X, α) be a twisted smooth projective variety. The objects of the form $k(x)$ with $x \in X$ a closed point span the derived category $\mathrm{D}^b(X,\alpha)$.*

Proof. We need to show that for a given $\mathcal{E}^\bullet \in \mathrm{D}^b(X,\alpha)$ there exists a point $x \in X$ and an integer n such that $\mathrm{Hom}\big(\mathcal{E}^\bullet, k(x)[n]\big) \neq 0$. By the untwisted version of the proposition we have that $\mathrm{Hom}\big(\mathcal{E}^\bullet \otimes L, k(x)[n]\big) \neq 0$ where L is a α^{-1}-locally free sheaf and this implies that $Hom\big(\mathcal{E}^\bullet, k(x)[n]\big) \neq 0$. ☑

The following lemma follows as in ([11, Lemma 3.31]). We only need to tensor any element of $\mathrm{D}^b(S,\alpha)$ by a α^{-1}-locally free sheaf to reduce the statement to the untwisted case.

Lemma 8. *Let $\pi : S \to T$ be a morphism of schemes, and for each point $t \in T$, let $i_t : S_t \to S$ denote the inclusion of the fibre $\pi^{-1}(t)$ in S. Let E be an object of $\mathrm{D}^b(S,\alpha)$ such that for all $t \in T$, $Li_t^*(E)$ is a sheaf on S_t. Then E is a twisted sheaf on S, flat over T.*

The last lemma has a useful application. Suppose $\Phi_{\mathcal{P}} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ is a FM equivalence (see Definition 13) such that for all $x \in X$ there exists $f(x) \in Y$ with $\Phi_{\mathcal{P}}\big(k(x)\big) = k\big(f(x)\big)$. Hence

$$\mathcal{P}|_{\{x\}\times Y} \cong k\big(f(x)\big) \tag{4}$$

for all $x \in X$ and then by the previous lemma, $\mathcal{P}$ is a twisted sheaf (which is X-flat). By taking local sections of $\mathcal{P}$ we define a morphism $X \to Y$ and by the isomorphism (4), we get that this induces f on closed points. We call this morphism again f. By following the same argument given in [11, Cor. 5.23], we obtain

$$\Phi_{\mathcal{P}}(-) = \big(L \otimes (-)\big) \circ f_* \tag{5}$$

where L is a line bundle and that f is an isomorphism because $\Phi_{\mathcal{P}}$ is an equivalence.

3. Ample (Antiample) Canonical Bundle

Throughout this section we consider all the varieties to be smooth and projective.

Definition 9. An object $P \in \mathrm{D}^b(X, \alpha)$ is called a point of codimension d if

i) $S_{(X,\alpha)}(P) \cong P[d]$, (where $S_{(X,\alpha)}$ is the Serre functor),

ii) $\mathrm{Hom}(P, P[i]) = 0$ for $i < 0$,

iii) The object P is simple, i.e. $k := \mathrm{Hom}(P, P)$.

We follow the untwisted proofs of the next two lemmas in order to get a twisted version of them (cf. [11, Lemma 4.5 and Prop. 4.6], and the original proof in [2]).

Lemma 10. *Let $\mathcal{F}^\bullet \in \mathrm{D}^b(X, \alpha)$ be a simple complex concentrated in dimension 0 such that $\mathrm{Hom}\big(\mathcal{F}^\bullet, \mathcal{F}^\bullet[i]\big) = 0$ for $i < 0$. Then $\mathcal{F}^\bullet \cong k(x)[m]$ for some closed point $x \in X$ and integer m.*

Lemma 11. *Let X be a smooth projective variety of dimension n. If ω_X is ample or antiample, then the point like objects in $\mathrm{D}^b(X, \alpha)$ are the objects P isomorphic to $k(x)[m]$, where $x \in X$ is a closed point and $m \in \mathbb{Z}$.*

Proof. It can be easily seen that the objects of the form $k(x)[m]$ are point like objects in $\mathrm{D}^b(X, \alpha)$. Now, we show that all point like objects are of this form. Take $P \in \mathrm{D}^b(X, \alpha)$ a point like object. By $i)$ in Definition 9

$$\mathcal{H}^i\big(P \otimes \omega_X[n-d]\big) \cong \mathcal{H}^i(P).$$

Thus

$$\mathcal{H}^{i+n-d}(P \otimes \omega_X) \cong \mathcal{H}^i(P),$$

i.e.

$$\mathcal{H}^{i+n-d}(P) \otimes \omega_X \cong \mathcal{H}^i(P). \tag{6}$$

If $n > d$, then we take the maximum integer i among the indices of the non-vanishing cohomologies $\mathcal{H}^i$. This yields to a contradiction by using (6). On the other hand, if $n < d$, we take i to be minimal, and (6) also yields to a contradiction. Thus, $n = d$ and hence

$$\mathcal{H}^i(P) \otimes \omega_X \cong \mathcal{H}^i(P) \tag{7}$$

Now, we show that this isomorphism implies that $\mathcal{H}^i(P)$ is supported in dimension 0. Recall that the Hilbert polynomial

$$P_{\mathcal{F}}(k) = \chi\big(\mathcal{F} \otimes \omega_X^k\big)$$

has degree

$$\deg(P_{\mathcal{F}}) = \dim(\operatorname{supp} \mathcal{F})$$

when ω_X $\big($or $\omega_X^\vee\big)$ is ample and $\mathcal{F}$ is any coherent sheaf. Let $\mathcal{E} \in \operatorname{Coh}\big(X, \alpha^{-1}\big)$ be a locally free α^{-1}-twisted sheaf and denote by $\mathcal{F}^i := \mathcal{H}^i(\mathcal{P}) \otimes \mathcal{E}$. Hence

$$\mathcal{F}^i \otimes \omega_X \cong \mathcal{F}^i \tag{8}$$

and $\mathcal{F}^i$ is a coherent sheaf on X. If $n = \dim(\operatorname{supp}\big(\mathcal{F}^i\big)) > 0$, we deduce from the isomorphism (8) that for all k, $P_{\mathcal{F}^i}(k)$ is a fixed number, i.e. the polynomial $P_{\mathcal{F}^i}$ is a constant polynomial, a contradiction. Then $\mathcal{F}^i$ is supported in dimension 0, and since $\mathcal{E}$ is locally free, $\mathcal{H}^i(P)$ has also support of dimension 0. Thus, P is a complex concentrated in dimension 0 and by Lemma 10, $P \cong k(x)[m]$ for some closed point x and integer m. ☑

Definition 12. Let $\mathcal{D}$ be a triangulated category with a Serre functor S. An object $L \in \mathcal{D}$ is invertible if for any point like object $P \in \mathcal{D}$ there exists $n_P \in \mathbb{Z}$ such that

$$\operatorname{Hom}(L, P[i]) = \begin{cases} k(P), & \text{if} \quad i = n_P; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 13. Let (X, α) and (Y, β) be two twisted varieties. A functor $F : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ is of Fourier–Mukai type (or a Fourier–Mukai functor) if there exists $\mathcal{P} \in \mathrm{D}^b\big(X \times Y, \alpha^{-1} \boxtimes \beta\big)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{P}}$, where we denote by $p : X \times Y \to Y$ and $q : X \times Y \to X$ the natural projections, $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ is the exact functor defined by

$$\Phi_{\mathcal{P}} := Rp_*\Big(\mathcal{P} \overset{L}{\otimes} q^*(-)\Big).$$

If the Fourier–Mukai functor is an equivalence we will call it a Fourier–Mukai transform.

From now, we will often write a functor and its derived functor in the same way.

In the category of twisted coherent sheaves Canonaco and Stellari proved in [6] that every equivalence can be seen as a Fourier–Mukai transform. In fact, they showed the following more general statement

Theorem 14. *Let (X, α) and (Y, β) be twisted varieties and let $F : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ be an exact functor such that, for any $\mathcal{F}, \mathcal{G} \in \operatorname{Coh}(X, \alpha)$,*

$$\operatorname{Hom}_{\mathrm{D}^b(Y,\beta)}\big(F(\mathcal{F}), F(\mathcal{G})[j]\big) = 0 \quad \textit{if} \quad j < 0.$$

Then there exist $\mathcal{P} \in \mathrm{D}^b\left(X \times Y, \alpha^{-1} \boxtimes \beta\right)$ and an isomorphism of functors $F \cong \Phi_{\mathcal{P}}$. Moreover, $\mathcal{P}$ is uniquely determined up to isomorphism.

By this theorem, we focus only on Fourier–Mukai transforms. If we take any exact functor

$$\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta),$$

then by an application of the Grothendieck–Verdier duality (cf. [5, Theorem 2.4.1]) as was given by Mukai (a good exposition by Orlov is found in [15]) we can prove that the functor $\Phi_{\mathcal{P}}$ has a left and a right adjoint functor with kernels

$$\mathcal{P}_L := \mathcal{P}^{\vee} \otimes p^{*}\omega_Y\big[\dim(Y)\big]$$

and

$$\mathcal{P}_R := \mathcal{P}^{\vee} \otimes q^{*}\omega_X\big[\dim(X)\big]$$

respectively. In particular, if $\Phi_{\mathcal{P}}$ is an equivalence, these adjoints must be quasi-inverses to $\Phi_{\mathcal{P}}$. However, from the uniqueness of the kernel of a twisted Fourier–Mukai transform we conclude that $\mathcal{P}_L$ is isomorphic to $\mathcal{P}_R$ and then

$$\mathcal{P}^{\vee} \cong \mathcal{P}^{\vee} \otimes \left(p^{*}\omega_Y \otimes q^{*}\omega_X^{\vee}[\dim(X) - \dim(Y)]\right).$$

This isomorphism implies: $\dim(X) = \dim(Y)$.

***Remark* 15.** If $\Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y, \beta)$ is an equivalence, the isomorphism $\mathcal{P}_L \cong \mathcal{P}_R$ and projection formula imply that for any point $x \in X$,

$$\Phi_{\mathcal{P}}\big(k(x)\big) = \omega_Y \otimes \Phi_{\mathcal{P}}\big(k(x)\big).$$

Let X, Y and Z be three smooth varieties. Define the projections π_{XZ}, π_{XY} and π_{YZ} from $X \times Y \times Z$ to $X \times Z$, $X \times Y$ and $Y \times Z$ respectively. Let $\mathcal{P} \in \mathrm{D}^b\left(X \times Y, q^{*}(\alpha)^{-1}.p^{*}(\beta)\right)$ and $\mathcal{Q} \in \mathrm{D}^b\left(Y \times Z, u^{*}(\beta)^{-1}.t^{*}(\gamma)\right)$ where q, p and u, t are the natural projections:

$$\begin{array}{ccc} X \times Y & \xrightarrow{\;p\;} & Y \\ {\scriptstyle q}\downarrow & & \\ X & & \end{array}, \qquad \begin{array}{ccc} Y \times Z & \xrightarrow{\;t\;} & Z \\ {\scriptstyle u}\downarrow & & \\ Y & & \end{array}.$$

We define the object

$$\mathcal{R} := \pi_{XZ*}(\pi_{XY}^{*}\mathcal{P} \otimes \pi_{YZ}^{*}\mathcal{Q}),$$

and let us show that this element is in $\mathrm{D}^b\left(X \times Z, s^{*}(\alpha)^{-1}.r^{*}(\gamma)\right)$ where r and s denote the projections from $-X \times Z$ to Z and X respectively. Let π_X, π_Y

and π_Z denote the projections from $X \times Y \times Z$ to X, Y and Z respectively. The object $\pi_{XY}^*(\mathcal{P}) \otimes \pi_{YZ}^*(\mathcal{Q})$ is in

$$\begin{aligned}\mathrm{D}^b\big(X \times Y \times Z, &\pi_{XY}^*\big(q^*(\alpha)^{-1}.p^*(\beta)\big).\pi_{YZ}^*\big(u^*(\beta)^{-1}.t^*(\gamma)\big)\big)\\ &\cong \mathrm{D}^b\big(X \times Y \times Z, \pi_X^*(\alpha)^{-1}.\pi_Y^*(\beta).\pi_Y^*(\beta)^{-1}.\pi_Z^*(\gamma)\big)\\ &\cong \mathrm{D}^b\big(X \times Y \times Z, \pi_X^*(\alpha)^{-1}.\pi_Z^*(\gamma)\big)\\ &\cong \mathrm{D}^b\big(X \times Y \times Z, \pi_{XZ}^*\big(s^*(\alpha)^{-1}\big).\pi_{XZ}^*\big(r^*(\gamma)\big)\big)\\ &\cong \mathrm{D}^b\big(X \times Y \times Z, \pi_{XZ}^*\big(s^*(\alpha)^{-1}.r^*(\gamma)\big)\big).\end{aligned}$$

Hence

$$\mathcal{R} = \pi_{XZ*}\big(\pi_{XY}^*(\mathcal{P}) \otimes \pi_{YZ}^*(\mathcal{Q})\big) \in \mathrm{D}^b\big(X \times Z, s^*(\alpha)^{-1}.r^*(\gamma)\big).$$

We note that the following twisted version of a result of Mukai holds by just following his proof.

Proposition 16 (Mukai, [14])**.** *The composition of two Fourier–Mukai transforms*

$$\mathrm{D}^b(X,\alpha) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b(Y,\beta) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b(Z,\gamma)$$

is isomorphic to the Fourier–Mukai transform

$$\Phi_{\mathcal{R}} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Z,\gamma).$$

We follow only a part of the proof given in [11, Prop. 4.11] of the untwisted version of the next proposition originally proved by Bondal and Orlov in [2].

Proposition 17. *Let X be a smooth projective variety with ample (or anti-ample) canonical bundle. If there exists an exact equivalence $F : \mathrm{D}^b(X,\alpha) \xrightarrow{\sim} \mathrm{D}^b(Y,\beta)$ with Y a smooth projective variety, then there exists an isomorphism $f : X \xrightarrow{\sim} Y$ with $f^*(\beta) = \alpha$.*

Proof. First, note that from the definition of point like objects there exists a bijection between the set of point like objects in $\mathrm{D}^b(X,\alpha)$ and the point like objects in $\mathrm{D}^b(Y,\beta)$. Since we have

$$\big\{\text{points like objects in } \mathrm{D}^b(X,\alpha)\big\} = \big\{k(x)[m] : x \in X \text{ closed and } m \in \mathbb{Z}\big\}$$

and

$$\big\{k(y)[m] : y \in Y \text{ closed and } m \in \mathbb{Z}\big\} \hookrightarrow \big\{\text{point like objects in } \mathrm{D}^b(Y,\beta)\big\}$$

we conclude that $F\big(k(x)[n]\big)$ is a point like object but we still do not know whether it is of the form $k(y)[m]$ for some closed point $y \in Y$ and $m \in \mathbb{Z}$.

Claim. Every point like object in $\mathrm{D}^b(Y,\beta)$ is of the form $k(y)[m]$ for some closed point $y \in Y$ and $m \in \mathbb{Z}$.

Proof of the claim. Suppose not and let P be a point like object not isomorphic to any $k(y)[m]$. We know that for every $y \in Y$ there exists $x_y \in X$ and $m_y \in \mathbb{Z}$ such that

$$F\big(k(x_y)[m_y]\big) = k(y).$$

From the bijection between point like objects in $\mathrm{D}^b(X,\alpha)$ and in $\mathrm{D}^b(Y,\beta)$, we find $x_P \in X, m_P \in \mathbb{Z}$ such that $x_P \neq x_y$ for all $y \in Y$ and

$$F\big(k(x_P)[m_P]\big) = P.$$

Then,

$$\begin{aligned}\operatorname{Hom}\big(k(y)[n],P\big) &= \operatorname{Hom}\big(F\big(k(x_y)[m_y]\big)[n], F\big(k(x_P)[m_P]\big)\big)\\ &= \operatorname{Hom}\big(k(x_y)[m_y+n], k(x_P)[m_P]\big)\\ &= \operatorname{Hom}\big(k(x_y), k(x_P)[m_P-m_y-n]\big)\\ &= 0\end{aligned}$$

for all y. Hence, since by Proposition 7 the set

$$\big\{k(y)[n] : y \in Y \text{ closed}, n \in \mathbb{Z}\big\}$$

span the category $\mathrm{D}^b(Y,\beta)$, we conclude $P = 0$. This completes the proof of the claim. ☑

Thus, for every $x \in X$ there exists $y_x \in Y$ and $m_x \in \mathbb{Z}$ such that $F\big(k(x)\big) = k(y_x)[m_x]$. Besides, for every $x \in X$ there exists V_x a neighborhood of x such that for every $z \in V_x$, $F\big(k(z)\big) = k(y_z)[m_x]$ ([11, Cor. 6.14]) and we can conclude that $m_x = m_z$ for all $z \in X$. Therefore we can assume that $F\big(k(x)\big) = k(y_x)$ for all x in X and so F defines a bijection $f : X \to Y$ by $x \mapsto y_x$. Since $F \cong \Phi_{\mathcal{P}}$, we have $\mathcal{P}|_{\{x\}\times Y} \cong k(y_x)$ and from this we can assume that f is a morphism (cf. commentary after Lemma 8). Since F is an equivalence, we conclude that f is injective. The surjectivity of the map was shown above. By using F^{-1} we also show that f^{-1} is a morphism. On the other hand, $\mathcal{P}$ is a sheaf supported on the graph of f and the second projection gives an isomorphism $\mathrm{supp}(\mathcal{P}) \cong Y$. Then if we consider $\mathcal{P}$ as a sheaf over its support, we can consider it as a twisted sheaf over Y. Besides, we also know that it is a twisted sheaf of constant fibre dimension 1, i.e. an untwisted line bundle L over Y. Then $F \cong L \otimes f_*(-)$ (up to shift) Therefore, f is an isomorphism with $f^*(\beta) = \alpha$. ☑

Let (X,α) be a twisted variety and let $\mathcal{F}$ be an α-twisted sheaf. We proceed to define the exterior algebra $\bigwedge \mathcal{F}$. By definition $\mathcal{F} = (\mathcal{F}_i, \varphi_{ij})_{i,j\in I}$ where $\mathcal{F}_i$ is a coherent sheaf on an element U_i of an open covering $\{U_i\}_{i\in I}$ of X and

$$\varphi_{ij} : \mathcal{F}_i|_{U_i\cap U_j} \to \mathcal{F}_j|_{U_i\cap U_j}$$

are morphisms that satisfies the α-twisted cocycle conditions. We define the exterior algebras as usual for any coherent sheaf $\mathcal{F}_i$ and we need only to check that the resulting transition maps satisfies the cocycle conditions. But this follows inmediately and it shows that for any $r \in \mathbb{N}$, $\bigwedge^r \mathcal{F}$ is a α^r-twisted sheaf. In particular, if $\mathcal{F}$ is a locally free α-twisted sheaf of rank r, the maximal exterior power of $\mathcal{F}$, $\bigwedge^r \mathcal{F}$, is a line bundle called the *determinant bundle of* $\mathcal{F}$ and we denote it by $\det(\mathcal{F})$. Now, we follow the proofs of the untwisted version of the following three lemmas and the corresponding corollary to get a twisted version of them (cf. [11]).

Lemma 18. *Let Z be a normal variety and $\mathcal{F} \in \mathrm{Coh}(Z,\alpha)$. If L_1 and L_2 are two line bundles with $\mathcal{F}\otimes L_1 \cong \mathcal{F}\otimes L_2$, then $L_1^r \cong L_2^r$ where r is the generic rank of $\mathcal{F}$.*

Proof. By definition $\mathcal{F} = (\mathcal{F}_i, \varphi_{ij})_{i,j\in I}$, where $\mathcal{F}_i$ is coherent sheaf on an open set U_i of an open covering $\{U_i\}_{i\in I}$ of X. Let $f = \{f_i\}_{i\in I}$ be the isomorphism $f : \mathcal{F}\otimes L_1 \cong \mathcal{F}\otimes L_2$ given in the statement, i.e.

$$f_i : \mathcal{F}_i \otimes L_1 \cong \mathcal{F}_i \otimes L_2$$

is an isomorphism for every $i \in I$ such that the following diagram commutes

$$\begin{array}{ccc}
(\mathcal{F}_j\otimes L_1)|_{U_i\cap U_j} & \xrightarrow{\;f_j|_{U_i\cap U_j}\;} & (\mathcal{F}_j\otimes L_2)|_{U_i\cap U_j} \\
{\scriptstyle\varphi^1_{ij}}\Big\uparrow & & \Big\uparrow{\scriptstyle\varphi^2_{ij}} \\
(\mathcal{F}_i\otimes L_1)|_{U_i\cap U_j} & \xrightarrow[\;f_i|_{U_i\cap U_j}\;]{} & (\mathcal{F}_i\otimes L_2)|_{U_i\cap U_j}
\end{array}$$

where φ^k_{ij} are defined by $\varphi_{ij}\otimes \mathrm{id}$, $k = 1,2$. First, let us suppose that $\mathcal{F}$ is a locally free α-twisted sheaf of rank r. The last diagram induces the commutative diagram

$$\begin{array}{ccc}
\big(\det(\mathcal{F}_j)\otimes L_1^r\big)|_{U_i\cap U_j} & \xrightarrow{\;\widetilde{f_j}\;} & \big(\det(\mathcal{F}_j)\otimes L_2^r\big)|_{U_i\cap U_j} \\
{\scriptstyle\widetilde{\varphi}^1_{ij}}\Big\uparrow & & \Big\uparrow{\scriptstyle\widetilde{\varphi}^2_{ij}} \\
\big(\det(\mathcal{F}_i)\otimes L_1^r\big)|_{U_i\cap U_j} & \xrightarrow[\;\widetilde{f_i}\;]{} & \big(\det(\mathcal{F}_i)\otimes L_2^r\big)|_{U_i\cap U_j}
\end{array}.$$

Hence $\det(\mathcal{F})\otimes L_1^r \cong \det(\mathcal{F})\otimes L_2^r$ and so $L_1^r \cong L_2^r$.

In general, let $\mathcal{F}$ be an α-twisted coherent sheaf. Dividing by the torsion part, we can assume that $\mathcal{F}$ is torsion free. Since Z is normal, $\mathcal{F}$ is a locally free α-twisted sheaf on an open set U with $\operatorname{codim}(Z-U)\geq 2$. Therefore by the argument given above we have that $L_1^r|_U \cong L_2^r|_U$. Then it defines a trivializing section $s \in H^0\big(U, L_1^r \otimes L_2^{-r}\big)$ which can be extended to another trivializing section $\widetilde{s} \in H^0\big(Z, L_1^r \otimes L_2^{-r}\big)$ and it defines an isomorphism $L_1^r \cong L_2^r$. ☑

The following lemma is as in [11, Lemma 6.4]. We reduce the proof to the untwisted case by tensoring with a locally free $q^*(\alpha)\cdot p^*(\beta)^{-1}$-twisted sheaf.

Lemma 19. *If $\Phi_{\mathcal{P}} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ is an equivalence, then the projection $q : \operatorname{supp}(\mathcal{P}) \twoheadrightarrow X$ is surjective.*

***Remark* 20.** Since the support of a complex does not change when we take tensor product with a line bundle, one has

$$\operatorname{supp}(\mathcal{P}) = \operatorname{supp}\left(\mathcal{P}^\vee\right) = \operatorname{supp}(\mathcal{P}_R) = \operatorname{supp}(\mathcal{P}_L).$$

Thus, we also deduce from the equivalence that $p : \operatorname{supp}(P) \twoheadrightarrow Y$ is surjective. Hence, there exist two irreducible components $Z_1 \subset \operatorname{supp}\left(\mathcal{H}^i(\mathcal{P})\right)$ and $Z_2 \subset \operatorname{supp}\left(\mathcal{H}^j(\mathcal{P})\right)$ that project onto X and Y respectively. Note that the components could be different.

Lemma 21. *Let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ be an equivalence and let $Z \subseteq \operatorname{supp}(\mathcal{P})$ be a closed irreducible subvariety with normalization $\nu : \widetilde{Z} \to Z$. Then there exists an integer $r > 0$ such that*

$$\pi_X^*\omega_X^r \cong \pi_Y^*\omega_Y^r$$

where $\pi_X := q\circ\nu$ and $\pi_Y := p\circ\nu$.

***Proof*.** We apply Lemma 18 as in the untwisted case ([11, Lemma 6.9]) ☑

The following result is the twisted version of a result of Orlov (cf. [15]). We follow the proof given in ([11, Prop. 6.1]).

Theorem 22. *Let X and Y be two projective varieties with $\alpha \in \mathrm{Br}'(X)$ and $\beta \in \mathrm{Br}'(Y)$. Any equivalence of categories $F : \mathrm{D}^b(X,\alpha) \xrightarrow{\sim} \mathrm{D}^b(Y,\beta)$ implies an isomorphism of the canonical rings $R(X) \cong R(Y)$.*

***Proof*.** Let d be the diagonal morphism $d : X \hookrightarrow X \times X$. Then $d_*\mathcal{O}_X$ can be regarded as a $\alpha \boxtimes \alpha^{-1}$-twisted sheaf. Denote $\mathcal{O}_\Delta := d_*\mathcal{O}_X$, which viewed as a Fourier–Mukai kernel induces the identity $\mathrm{id} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(X,\alpha)$.

The equivalence F is given by a Fourier–Mukai transform

$$\Phi_{\mathcal{P}} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$$

with $\mathcal{P} \in \mathrm{D}^b(X \times Y, \alpha^{-1} \boxtimes \beta)$. Then the Fourier–Mukai transform

$$\Phi_{\mathcal{Q}} : \mathrm{D}^b\left(X, \alpha^{-1}\right) \to \mathrm{D}^b\left(Y, \beta^{-1}\right)$$

with

$$\mathcal{Q} := \mathcal{P}^{\vee} \otimes q^*\omega_X[n] \cong \mathcal{P}^{\vee} \otimes p^*\omega_Y[n] \in \mathrm{D}^b\left(Y \times X, \beta^{-1} \boxtimes \alpha\right)$$

is also an equivalence. Indeed, since the composition

$$\mathrm{D}^b(X, \alpha) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b(Y, \beta) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b(X, \alpha)$$

is isomorphic to the identity, and the kernel of this composition is given by $\mathcal{R} = \pi_{13*}\big(\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{Q}\big)$, one has $\mathcal{R} \cong \mathcal{O}_\Delta \in \mathrm{D}^b\left(X \times X, \alpha^{-1} \boxtimes \alpha\right)$. Consider the automorphism $\tau_{12} : X \times X \to X \times X$ that interchanges the two factors,

$$\mathcal{O}_\Delta \cong \tau_{12}^*\mathcal{O}_\Delta \cong \tau_{12}^*\mathcal{R} \cong \pi_{13*}\tau_{13}^*\big(\pi_{12}^*\mathcal{P} \otimes \pi_{23}^*\mathcal{Q}\big) \cong \pi_{13*}\big(\pi_{12}^*\mathcal{Q} \otimes \pi_{23}^*\mathcal{P}\big).$$

Thus the composition of

$$\mathrm{D}^b\left(X, \alpha^{-1}\right) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b\left(Y, \beta^{-1}\right) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b\left(X, \alpha^{-1}\right)$$

is isomorphic to the identity.

In the same way we can prove that

$$\mathrm{D}^b\left(Y, \beta^{-1}\right) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b\left(X, \alpha^{-1}\right) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b\left(X, \beta^{-1}\right)$$

is isomorphic to the identity.

Moreover $\mathcal{P} \boxtimes \mathcal{Q} \in \mathrm{D}^b\left((X \times X) \times (Y \times Y), \alpha^{-1} \boxtimes \alpha \boxtimes \beta \boxtimes \beta^{-1}\right)$ defines the Fourier–Mukai equivalence

$$\Phi_{\mathcal{P}\boxtimes\mathcal{Q}} : \mathrm{D}^b\left(X \times X, \alpha^{-1} \boxtimes \alpha\right) \longrightarrow \mathrm{D}^b\left(Y \times Y, \beta^{-1} \boxtimes \beta\right).$$

Now, we show that this equivalence implies an isomorphism between the canonical rings. Since $d_*\big(\omega_X^m\big)$ can be considered as an element in $\mathrm{D}^b\big(X \times X, \alpha^{-1} \boxtimes \alpha\big)$, by defining $S := \Phi_{\mathcal{P}\boxtimes\mathcal{Q}}\big(d_*\omega_X^m\big)$ we have that

$$\Phi_S : \mathrm{D}^b(Y, \beta) \to \mathrm{D}^b(Y, \beta)$$

is an equivalence that can be obtained as the composition

$$\mathrm{D}^b(Y, \beta) \xrightarrow{\Phi_{\mathcal{Q}}} \mathrm{D}^b(X, \alpha) \xrightarrow{\Phi_{d_*\omega_X^m}} \mathrm{D}^b(X, \alpha) \xrightarrow{\Phi_{\mathcal{P}}} \mathrm{D}^b(Y, \beta).$$

That is,

$$\Phi_S \cong \Phi_{\mathcal{P}} \circ \Phi_{d_*\omega_X^m} \circ \Phi_{\mathcal{Q}}.$$

Note that $\Phi_{d_*\omega_X^m} = S^m_{(X,\alpha)}[-mn]$ where $S_{(X,\alpha)}$ denotes the Serre functor defined on the category $\mathrm{D}^b(X,\alpha)$. From the fact that equivalences commutes with Serre functor, we conclude that

$$\Phi_S \cong S^m_{(Y,\beta)}[-mn].$$

Then, the uniqueness of the kernel of a Fourier–Mukai transform yields

$$S \cong d_*\omega_Y^m,$$

i.e. $\Phi_{\mathcal{Q}\boxtimes\mathcal{P}}\big(d_*\omega_X^m\big) \cong d_*\big(\omega_Y^m\big)$. Thus

$$\begin{aligned} H^0\big(X,\omega_X^m\big) &= Hom_{D^b(X\times X,\alpha\boxtimes\alpha^{-1})}\big(d_*\mathcal{O}_X, d_*\omega_X^m\big) \\ &\cong Hom_{\mathrm{D}^b(Y\times Y,\beta\boxtimes\beta^{-1})}\big(d_*\mathcal{O}_Y, d_*\omega_Y^m\big) \\ &= H^0\big(Y,\omega_Y^m\big). \end{aligned}$$

Since the algebra structure is given by composition of Ext's just by using

$$\mathrm{Ext}^i_{\mathrm{D}^b(X\times X,\alpha^{-1}\boxtimes\alpha)}\big(d_*\mathcal{O}_X, d_*\big(\omega_X^k\big)\big) \cong \mathrm{Ext}^i_{\mathrm{D}^b(X\times X,\alpha^{-1}\boxtimes\alpha)}\big(d_*\omega_X^m, d_*\big(\omega_X^{m+k}\big)\big)$$

then $R(X) \cong R(Y)$. ☑

The following result in the untwisted case is due to Kawamata (cf. [12], [11, Prop. 6.18]) but copying his proof yields a proof in the twisted case.

Theorem 23 (Kawamata)**.** *Let X and Y be smooth projective varieties and let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X,\alpha) \longrightarrow \mathrm{D}^b(Y,\beta)$ be an equivalence such that the canonical bundle ω_X is big or anti-big (i.e. $\omega_X^\vee$ is big). Then there exists a birational morphism $f : X \dashrightarrow Y$ with $f^*(\beta) = \alpha$.*

***Remark* 24.** If X and Y are two smooth projective varieties with a birational correspondence

$$\begin{array}{ccc} Z & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle\pi_X}\big\downarrow & & \\ X & & \end{array}$$

where Z is a normal smooth variety. If $\pi_X^*\omega_X^r \cong \pi_Y^*\omega_Y^r$, then $\pi_X^*\omega_X \cong \pi_Y^*\omega_Y$.

***Remark* 25.** Let X and Y be two K-equivalent surfaces, i.e. there exists a birational correspondence

$$\begin{array}{ccc} Z & \xrightarrow{\pi_Y} & Y \\ {\scriptstyle\pi_X}\big\downarrow & & \\ X & & \end{array}$$

such that $\pi_X^*\omega_X \cong \pi_Y^*\omega_Y$. Then $X \cong Y$.

4. Classification of Surfaces Under Twisted Derived Categories

In this section we show that a theorem of Kawamata remains true when we consider twisted derived categories.

Definition 26. If L is a line bundle on a projective scheme X, we define the numerical Kodaira dimension $\nu(X,L)$ to be the maximal integer m such that there exists a proper morphism $\phi : W \to X$ with W of dimension m and $\big([\phi^*(L)]^m.W\big) \neq 0$. In particular, if $L = \omega_X$, we denote $\nu(X) := \nu(X,\omega_X)$.

Lemma 27. ([11, Lemma 6.26]) *Let $\pi : Z \to X$ be a projective morphism of proper schemes and $L \in \mathrm{Pic}(X)$.*

i) If L is a nef line bundle on X then $\pi^*(L)$ is nef.

ii) If π is surjective, then L is nef if and only if $\pi^*(L)$ is nef.

Lemma 28. ([11, Lemma 6.28]) *Let $\pi : Z \to X$ be a projective morphism of projective schemes and $L \in \mathrm{Pic}(X)$.*

i) Then $\nu(X,L) \geq \nu(Z,\pi^*L)$.

ii) If $\pi : Z \to X$ is surjective, then $\nu(X,L) = \nu(Z,\pi^*L)$.

Proposition 29. ([11, Prop. 6.17]) *Let X and Y be smooth projective varieties and let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ be an equivalence. Then $\nu(X) = \nu(Y)$.*

***Proof*.** Since $\Phi_{\mathcal{P}}$ is an equivalence, there exists a component Z of $\mathrm{supp}(\mathcal{P})$ such that $p : Z \to Y$ is surjective. If $\nu : \widetilde{Z} \to Z$ is the normalization, then by Lemma 21, there exists an integer r such that $\pi_X^*\omega_X^r \cong \pi_Y^*\omega_Y^r$ where $\pi_X = q \circ \nu$ and $\pi_Y = p \circ \nu$. Hence

$$\nu\big(\widetilde{Z},\pi_X^*\omega_X^r\big) = \nu\big(\widetilde{Z},\pi_Y^*\omega_Y^r\big)$$

and then

$$\nu(X,\omega_X) \geq \nu\big(\widetilde{Z},\pi_X^*\omega_X\big) = \nu\big(\widetilde{Z},\pi_X^*\omega_X^r\big) = \nu\big(\widetilde{Z},\pi_Y^*\omega_Y^r\big) = \nu(Y,\omega_Y).$$

The other inequality holds by considering $\Phi_{\mathcal{P}_R}$ instead of $\Phi_{\mathcal{P}}$. ☑

Definition 30. A rational surface is a surface that is birationally equivalent to $\mathbb{P}^2$.

Definition 31. A ruled surface is a smooth projective surface X, together with a surjective morphism $\pi : X \to C$ to a nonsingular curve C, such that the fibre X_y is isomorphic to $\mathbb{P}^1$ for every point $y \in C$.

Theorem 32 (Castelnuovo)**.** *A surface is rational if and only if the irregularity and second geometric genus are trivial, i.e. $h^1\big(X,\mathcal{O}_X\big) = h^0\big(X,\omega_X^2\big) = 0$.*

Definition 33. A smooth surface X is an elliptic surface if there exists a curve C and a morphism $\pi : X \to C$ whose general fibre is an elliptic curve.

The proof of the following result is identical to the proof of its untwisted version given in [11, Prop. 12.15], which was originally proved by Kawamata in [12].

Theorem 34 (Kawamata)**.** *Let X be a smooth projective surface containing a $(-1)-$curve and Y a smooth projective variety and let $\Phi_{\mathcal{P}} : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ be an equivalence. Then one of the following holds:*

i) $X \cong Y$*;*

ii) X *is a relatively minimal elliptic rational surface.*

4.1. Surfaces with kod $= -\infty, 2$

We also have the following twisted version of a proposition due to Bridgeland and Maciocia. The proof is identical to ([11, Prop. 12.16]).

Proposition 35. *Let X be a surface of general type and Y a smooth projective variety. If $\mathrm{D}^b(X,\alpha) \cong \mathrm{D}^b(Y,\beta)$, then $X \cong Y$.*

Proof. Since X is of general type, Y is also of general type by Theorem 22. Moreover, by Theorem 23, X and Y are birational. If X is not minimal, by Theorem 34, $X \cong Y$. Thus we can assume that X and Y are minimal surfaces. Since the minimal model of a surface of general type is unique, the birational morphism between X and Y yields an isomorphism $X \cong Y$. ☑

Let X be a rational surface. Thus, $H^i\big(X, \mathcal{O}_X\big) = 0$ for any $i > 0$. From the exponential short exact sequence we obtain the isomorphism

$$\mathrm{Br}'(X) \cong H^3(X,\mathbb{Z}).$$

Since the cohomological Brauer group is a birational invariant,

$$\mathrm{Br}'(X) \cong \mathrm{Br}'(\mathbb{P}^2) = H^3(\mathbb{P}^2,\mathbb{Z}) = 0.$$

Now, let $\pi : X \to C$ be a ruled surface. Consider the Leray spectral sequence associated to π:

$$E_2^{p,q} = H^p(C, R^q\pi_*\mathcal{O}_X^*) \Rightarrow H^{p+q}(X,\mathcal{O}_X^*).$$

Since $H^q\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}\big) = 0$ for $q > 1$, we obtain

$$R^q\pi_*\mathcal{O}_X = 0, \quad \text{for} \quad q \geq 1. \tag{9}$$

The exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ yields a long exact sequence

$$\cdots \to R^q\pi_*\mathcal{O}_X \to R^q\pi_*\mathcal{O}_X^* \to R^{q+1}\pi_*\mathbb{Z} \to R^{q+1}\pi_*\mathcal{O}_X \to \cdots$$

so that by Equation (9),

$$R^q\pi_*\mathcal{O}_X^* \cong R^{q+1}\pi_*\mathbb{Z}, \quad \text{for any} \quad q \geq 1. \tag{10}$$

Clearly $R^q\pi_*\mathcal{O}_X^* = 0$ for $q \geq 2$ and $R^0\pi_*\mathcal{O}_X^* = O_C^*$. On the other hand, the sheaf $R^2\pi_*\mathbb{Z}$ is a local system of coefficients with stalk $\mathbb{Z}$ and the complex structure of the morphism π gives a canonical generator for each stalk on this local system. Thus $R^2\pi_*\mathbb{Z}$ is trivial, i.e. $R^2\pi_*\mathbb{Z} = \mathbb{Z}$. Hence by the isomorphism (10)

$$R^1\pi_*\mathcal{O}_X^* = \mathbb{Z}. \tag{11}$$

The Leray spectral sequence yields a long exact sequence

$$H^0\big(C, R^1\pi_*\mathcal{O}_X^*\big) \to H^2\big(C, \mathcal{O}_C^*\big) \to H^2\big(X, \mathcal{O}_X^*\big) \to H^1\big(C, R^1\pi_*\mathcal{O}_X^*\big). \tag{12}$$

By (11), $H^1\big(C, R^1\pi_*\mathcal{O}_X^*\big) = H^1(C,\mathbb{Z}) = \mathbb{Z}^{2g(C)}$. Since X is smooth, $H^2\big(X, \mathcal{O}_X^*\big)$ is a torsion group. Thus the last map in the sequence (12) is trivial and since $H^2\big(C, \mathcal{O}_C^*\big) = 0$, we obtain $\mathrm{Br}'(X) = H^2\big(X, \mathcal{O}_X^*\big) = 0$ (if X is not smooth we also obtain that $\mathrm{Br}'(X) = H^2\big(X, \mathcal{O}_X^*\big)_{tors} = 0$). Therefore, we have shown the following proposition:

Proposition 36. *If X is a smooth projective surface of* $\mathrm{kod}(X) = -\infty$*, then* $\mathrm{Br}'(X) = 0$.

Proposition 37. *Let X be a smooth projective surface containing a $(-1)-$curve and Y a smooth projective variety. If $\mathrm{Br}'(X) \neq 0$ and $\Phi : \mathrm{D}^b(X,\alpha) \to \mathrm{D}^b(Y,\beta)$ is an equivalence. Then $X \cong Y$.*

Proof. By Theorem 34, either $X \cong Y$ or X is a rational surface that is elliptically fibred. Thus, if X is rational, Proposition 36 implies $\mathrm{Br}'(X) = 0$, a contradiction. ☑

4.2. Surfaces with kod = 1

Definition 38. A vector bundle $\mathcal{F}$ on a curve C is decomposable if it is isomorphic to a direct sum $\mathcal{F}_1 \oplus \mathcal{F}_2$ of two non-zero vector bundles. Otherwise, we say that $\mathcal{F}$ is indecomposable.

Lemma 39. ([16, Cor 14.8]) *Let $\mathcal{F}$ be an indecomposable vector bundle of rank r and degree d on an elliptic curve E. The following conditions are equivalent.*

i) $\mathcal{F}$ is stable;

ii) $\mathcal{F}$ *is simple;*

iii) d *and* r *are relatively prime.*

Theorem 40. ([9, Prop. I. 3.24]) *Let* X *be a minimal projective surface of Kodaira dimension 1. Then there is a unique curve* C *and a unique morphism* $\pi : X \to C$ *making* X *an elliptic surface.*

Definition 41. Let $\pi : X \to C$ be an elliptic surface and $c \in C$. The fibre $\pi^{-1}(c)$ is called a *multiple* fibre if there is a divisor D on X with $\pi^{-1}(c) = mD$ for some integer $m > 1$.

Let $\pi : X \to C$ be a relatively minimal elliptic surface with $\mathrm{kod}(X) = 1$. The cohomology class of the fibre $F_x := \pi^{-1}(x)$ is denoted by $f \in H^2(X, \mathbb{Z})$. Note that F_x is a smooth elliptic curve for generic $x \in C$. The canonical bundle formula (cf. [1, V.12]) states that

$$\omega_X \cong \pi^*\mathcal{L} \otimes \mathcal{O}\Big(\sum (m_i - 1)F_i\Big) \tag{13}$$

where $\mathcal{L} \in \mathrm{Pic}(C)$ and F_i are the multiple fibres. Hence $c_1(X) = \lambda f$ (in $H^2\big(X, \mathbb{Q}\big)$) for some $\lambda \neq 0$ (because $\mathrm{kod}(X) = 1$). We also define the moduli space $M_H(v)$ similarly as for K3 surfaces to be the moduli space of semi-stable (with respect to H) sheaves E with $v(E) = v$.

***Remark* 42.** Suppose $v = (0, rf, d)$ and E a stable sheaf of rank r and degree d on a smooth fibre. By the Hirzebruch-Riemann-Roch formula and $f.c_1(X) = 0$ one has $\chi(E) = d$. On the other hand, if $[E] \in M_H(v)$ corresponds to a stable sheaf E, $\mathrm{supp}(E)$ is connected, so that $\mathrm{supp}(E) \subseteq F_x$ for some fibre F_x because $v(E) = (0, rf, d)$ (if $\mathrm{supp}(E)$ has an horizontal component it would intersect non-trivially the fibre class f).

Definition 43. Let $\pi : X \to C$ be an elliptic surface with $\mathrm{kod}(X) = 1$ and let $\lambda_{X/C}$ denote the smallest positive number such that there exists a divisor σ on X with $\sigma.f = \lambda_{X/C}$. We also denote it sometimes by only λ_X (recall that from Theorem 40 there is only one C and morphism making X an elliptic fibration).

Theorem 44. ([5, Theorem 3.2.1]) *The functor* $F = \Phi_{\mathcal{P}} : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ *is fully faithful, if and only if, for each* $x \in X$,

$$\mathrm{Hom}_{\mathrm{D}^b(Y)}\big(F\big(k(x)\big), F\big(k(x)\big)\big) = \mathbb{C},$$

and for each pair of points $x_1, x_2 \in X$, *and each integer* i,

$$\mathrm{Ext}^i_{\mathrm{D}^b(Y)}\big(F\big(k(x_1)\big), F\big(k(x_2)\big)\big) = 0$$

unless $x_1 = x_2$ *and* $0 \leq i \leq \dim X$. *Assuming the above conditions satisfied,* F *is an equivalence if and only if, for every point* $x \in X$,

$$F\big(k(x)\big) \overset{L}{\otimes} \omega_Y \cong F\big(k(x)\big).$$

Căldăraru proved in [5] a version of the following proposition in the case of K3 surfaces. In that case the proof followed immediately from the last theorem because of the triviality of the canonical bundle for K3 surfaces. This is not the case for properly elliptic surfaces.

Proposition 45. *Let X be a properly elliptic surface, i.e.* $\operatorname{kod}(X) = 1$ *that is relatively minimal, and let $v = (0, rf, d)$ be a Mukai vector with $gcd(r, d) = 1$. Let M be a connected component of the moduli space of stable sheaves with Mukai vector v and let $\alpha = \operatorname{Obs}(X, v)$ (see Definition 3.3.3 in [5]). Then we have*

$$\mathrm{D}^b(X) \cong \mathrm{D}^b\left(M, \alpha^{-1}\right).$$

Proof. The $\pi_M^* \alpha$-universal sheaf $\mathcal{E}$ on $X \times M$ defines a functor

$$\Phi_{\mathcal{E}} : \mathrm{D}^b\left(M, \alpha^{-1}\right) \to \mathrm{D}^b(X).$$

Let $[\mathcal{F}] \in M$ be a point corresponding to a stable sheaf $\mathcal{F}$ on X and Mukai vector $v = (0, rf, d)$. Then, by definition of the universal sheaf, $\Phi_{\mathcal{E}}\big(k([\mathcal{F}])\big) = \mathcal{F}$. We check the conditions of Theorem 44. Let $[\mathcal{F}]$ and $[\mathcal{G}]$ be two distinct points in M corresponding to two nonisomorphic stable sheaves $\mathcal{F}$ and $\mathcal{G}$ on X respectively. Since $\mathcal{F}$ is a stable sheaf,

$$\operatorname{Hom}\left(\Phi_{\mathcal{E}}\big(k([\mathcal{F}])\big), \Phi_{\mathcal{E}}\big(k([\mathcal{F}])\big)\right) = \operatorname{Hom}(\mathcal{F}, \mathcal{F}) = \mathbb{C}.$$

If $i < 0$ or $i > 2$, trivially $\operatorname{Ext}^i\left(\Phi_{\mathcal{E}} k([\mathcal{F}]), \Phi_{\mathcal{E}} k([\mathcal{G}])\right) = 0$. Since $\mathcal{F}$ and $\mathcal{G}$ are stables,

$$\operatorname{Hom}\left(\Phi_{\mathcal{E}}\big(k([\mathcal{F}])\big), \Phi_{\mathcal{E}}\big(k([\mathcal{G}])\big)\right) = \operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0.$$

By Serre duality,

$$\operatorname{Ext}^2(\mathcal{F}, \mathcal{G}) = \operatorname{Hom}(\mathcal{G}, \mathcal{F} \otimes \omega_X)^{\vee}. \tag{14}$$

Let us show that $\mathcal{F} \cong \mathcal{F} \otimes \omega_X$. If $\mathcal{F}$ is supported on a non-singular fibre, by the canonical bundle formula (cf. (13)), the restriction of ω_X to the non-singular fibre is trivial. Hence $\mathcal{F} \cong \mathcal{F} \otimes \omega_X$. Since the dimension of $\operatorname{Hom}\left(\mathcal{E}_{[\mathcal{F}]}, \mathcal{E}_{[\mathcal{F}]} \otimes \omega_X\right)$ is upper semi-continuous on M (cf. [10, III. 7.7.8]), for all $[\mathcal{F}] \in M$ there is a non-zero morphism $\mathcal{E}_{[\mathcal{F}]} \to \mathcal{E}_{[\mathcal{F}]} \otimes \omega_X$ (i.e. $\mathcal{F} \to \mathcal{F} \otimes \omega_X$ is non-zero). Since $\operatorname{rk}(\mathcal{F}) = \operatorname{rk}(\mathcal{F} \otimes \omega_X)$ and

$$c_1(\mathcal{F}).f = c_1(\mathcal{F}).f + c_1(X).f = c_1(\mathcal{F} \otimes \omega_X)$$

and both sheaves $\mathcal{F}$ and $\mathcal{F} \otimes \omega_X$ are stable, we obtain an isomorphism

$$\mathcal{F} \cong \mathcal{F} \otimes \omega_X$$

for all $\mathcal{F}$ stable. Thus, by isomorphism (14)

$$\begin{aligned} \operatorname{Ext}^2\left(\Phi_{\mathcal{E}}\big(k([\mathcal{F}])\big), \Phi_{\mathcal{E}}\big(k([\mathcal{G}])\big)\right) &= \operatorname{Ext}^2(\mathcal{F}, \mathcal{G}) \\ &= \operatorname{Hom}(\mathcal{G}, \mathcal{F} \otimes \omega_X)^{\vee} \\ &= \operatorname{Hom}(\mathcal{G}, \mathcal{F})^{\vee} \\ &= 0 \end{aligned}$$

for any two points $[\mathcal{F}] \neq [\mathcal{G}]$ in M (corresponding to two stable sheaves on X). Since

$$\chi(\mathcal{F}, \mathcal{G}) = -\big\langle v(\mathcal{F}), v(\mathcal{G}) \big\rangle = -\langle v, v \rangle = 0,$$

we obtain $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$. Thus, since we have verified all the conditions of Theorem 44, $\Phi_{\mathcal{E}}$ is an equivalence of categories. ☑

The following result is a generalization of a result of Bridgeland and Maciocia (cf. [4]). We follow the proof given in [11] with some little modifications.

Proposition 46. *Let $\pi : Y \to C$ be a relatively minimal elliptic surface with $\mathrm{kod}(Y) = 1$ and let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ be an equivalence. Then there exists a Mukai vector $v = (0, rf, d)$ such that $gcd(r, d) = 1$ and $X \cong M(v)$.*

Proof. If either X or Y is not minimal, then they are isomorphic (see Theorem 34) and we pick $v = (0, f, 1)$. Hence, we may assume that X and Y are minimal surfaces. For any closed point x in X, $E := \Phi\big(k(x)\big)$ satisfies

$$E \otimes \omega_Y \cong E,$$

because of Remark 15. Since $\mathrm{Hom}\,\big(k(x), k(x)\big) = \mathrm{Hom}(E, E)$, E is simple and thus $\mathrm{supp}(E)$ is connected. Since $E \cong E \otimes \omega_Y$, $\mathrm{supp}(E) \subset F_y$ for some fibre $F_y \subset Y$ because $\mathrm{kod}(Y) = 1$ and the isomorphism (13). For general x, we may assume that F_y is a smooth fibre. Thus, since $\mathrm{supp}(E)$ is connected, either $\mathrm{supp}(E) = F_y$ or $\mathrm{supp}(E)$ consists of only a closed point in F_y.

Claim. We can assume that E is a shifted sheaf, i.e. $\mathcal{H}^i(E) = 0$ for all but one $i \in \mathbb{Z}$.

Proof. Consider the spectral sequence

$$E_2^{p,q} = \bigoplus_i \mathrm{Ext}^p\left(\mathcal{H}^i(E), \mathcal{H}^{i+q}(E)\right) \Rightarrow Ext^{p+q}(E, E).$$

Since Y is a surface, $E_2^{p,q}$ are trivial for $p \notin [0, 2]$. In particular

$$\bigoplus_i \mathrm{Ext}^1\left(\mathcal{H}^i(E), \mathcal{H}^i(E)\right) \subset \mathrm{Ext}^1(E, E). \tag{15}$$

Since E is supported on a smooth elliptic curve F_x, all its cohomologies are. This means that if $\mathcal{H}^i(E) \neq 0$, $\mathrm{Ext}^1\big(\mathcal{H}^i(E), \mathcal{H}^i(E)\big) \neq 0$ $\big($because $\mathrm{Ext}^1_{F_x}\left(\mathcal{H}^i(E), \mathcal{H}^i(E)\right) \hookrightarrow \mathrm{Ext}^1_Y\left(\mathcal{H}^i(E), \mathcal{H}^i(E)\right)\big)$. Moreover, since $\mathcal{H}^i(E)$ is supported on a smooth elliptic curve, $\chi\big(\mathcal{H}^i(E), \mathcal{H}^i(E)\big) = -\big\langle v\big(\mathcal{H}^i(E)\big), v\big(\mathcal{H}^i(E)\big)\big\rangle = 0$ and $\mathcal{H}^i(E) = \mathcal{H}^i(E) \otimes \omega_X$ (cf. the proof of Proposition 45). Thus, by Serre duality $\dim \mathrm{Ext}^1_Y\left(\mathcal{H}^i(E), \mathcal{H}^i(E)\right)$ is even (≥ 2) for any $\mathcal{H}^i(E) \neq 0$. Hence by (15), $2n \leq \dim \mathrm{Ext}^1_Y(E, E) = 2$, where n is the number of non-trivial cohomologies $\mathcal{H}^i$. Thus, E is a shifted sheaf. ☑

By composing the original equivalence with a shift, we can assume that E is a sheaf. If E is concentrated in one point y, from $\Phi(k(x)) = k(y)$ we get that X and Y are birational. Hence they are isomorphic because they are minimal surfaces (the minimal model for surfaces of Kodaira dimension 1 is unique).

Thus, we can assume that E is a vector bundle on F_y. Since E is simple, by Lemma 39, E is stable (with respect to some polarization H) and $(\mathrm{rk}(E), \deg(E)) = 1$. Set $v = (0, rf, d)$ where $r := \mathrm{rk}(E), d := \deg(E)$. Then v is isotropic, i.e. $\langle v, v\rangle = 0$. Hence the moduli space $M = M_H(v)$ of stable sheaves with Mukai vector v is 2-dimensional $\big(\dim \mathrm{Ext}(E,E) = 2 + \langle v, v\rangle = 2\big)$.

By Proposition 45, for $\gamma = \mathrm{Obs}(Y, v)$, the $\pi_M^*\gamma$-universal sheaf yields an equivalence

$$\Phi_{\mathcal{E}} : \mathrm{D}^b(M, \gamma^{-1}) \to \mathrm{D}^b(Y).$$

Thus, the composition

$$\Psi := \Phi_{\mathcal{E}}^{-1} \circ \Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(M, \gamma^{-1}).$$

satisfies $\Psi\big(k(x)\big) = k(e)$ where $e \in M$ is the point that corresponds to E. Hence, M is birational to X. Since X is minimal and $\mathrm{kod}(X) = 1$, $M \cong X$. Moreover Ψ defines an isomorphism $f : X \to M$ such that $\Psi \cong L \otimes f_*(-)$, hence $f^*\gamma^{-1} = \alpha$. ☑

Corollary 47. *Let X and Y be relatively minimal elliptic surfaces with $\mathrm{kod}(X) = \mathrm{kod}(Y) = 1$ and let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ be an equivalence. Then one of the following holds:*

i) $X \cong Y$ *and* $\alpha = 1$ *in* $\mathrm{Br}(X)$,

ii) There exists a Mukai vector $v = (0, rf, d)$ such that $gcd(r, d) = 1$ and an isomorphism $f : X \cong M(v)$ with $f^*(\gamma^{-1}) = \alpha$, where $\gamma = \mathrm{Obs}(Y, v)$.

***Remark* 48.** In general, the moduli space $M(v)$ obtained in the previous Proposition is coarse.

Corollary 49. *Let X and Y be relatively minimal elliptic surfaces with $\mathrm{kod}(X) = \mathrm{kod}(Y) = 1$ and let $\Phi : \mathrm{D}^b(X, \alpha) \to \mathrm{D}^b(Y)$ be an equivalence. If Y is elliptically fibred with a section, then $\alpha = 1$ in $\mathrm{Br}(X)$.*

***Proof*.** By the last corollary there exists a Mukai vector $v = (0, rf, d)$ such that $gcd(r, d) = 1$ and an isomorphism $f : X \cong M(v)$ with $f^*(\gamma^{-1}) = \alpha$, where $\gamma = \mathrm{Obs}(Y, v)$. Since $\lambda_Y = 1$, there exists H such that $gcd(d, r(f.H)) = 1$ with H ample. Thus $M(v)$ is a fine moduli space, i.e. $\gamma = 1$ in $\mathrm{Br}'(Y)$ and hence $\alpha = 1$ in $\mathrm{Br}'(X)$. ☑

The previous corollary provides a very interesting application. First we introduce the notion of the Tate–Shafarevich group. For an elliptic surface

$\pi : X \to C$ with a section σ and integral fibres, we define the Tate–Shafarevich group by

$$\mathrm{Sh}(X) := H^1(C, X^{\#})$$

where $X^{\#}$ is the sheaf of abelian groups on C such that

$$X^{\#}(U) = \text{the group of sections of } X_U \to U$$

and the natural group structure on $X^{\#}$ is the one given by the section $\sigma : C \to X$. This group is in 1-1 correspondence with the set of elliptic fibrations $Y \to C$ whose Jacobian is $\pi : X \to C$ (Note that we are in the analytic or étale setup).

***Note* 1.** Let $\pi : X \to C$ be an elliptic surface with a section. For any $\alpha \in \mathrm{Sh}(X)$, let $\pi_\alpha : X_\alpha \to C$ denote the elliptic fibration corresponding to the element α.

Let $\pi : X \to C$ be an elliptic fibration with a section and integral fibres and let $\pi_\alpha : X_\alpha \to C$ be an elliptic fibration in $\mathrm{Sh}(X)$. We proceed to define a morphism $T_\alpha : \mathrm{Sh}(X) \to \mathrm{Br}'(X_\alpha)$. First, for a given $\alpha \in \mathrm{Sh}(X)$ we can define a homomorphism

$$T_\alpha : H^1\big(C, X^{\#}\big) \to H^1\big(C, \mathcal{P}ic(X_\alpha/C)\big) \tag{16}$$

by considering the long exact sequence obtained from the exact sequence

$$0 \longrightarrow X^{\#} \longrightarrow \mathcal{P}ic(X_\alpha/C) \xrightarrow{\deg_\alpha} \mathbb{Z} \longrightarrow 0$$

where $\mathcal{P}ic(X_\alpha/C)$ is the relative Picard sheaf of π_α (note that the relative Picard functor for an elliptic fibration with integral fibres is representable. If the elliptic fibration allows non-integral fibres the functor is non-representable, but it has a maximal representable quotient (cf. [7]) and $\deg_\alpha$ is the map that sends any $L \in \mathcal{P}ic(\pi_\alpha^{-1}(U))/\pi_\alpha^*\mathcal{P}ic(U)$ to its degree along a smooth fibre. From the Leray spectral sequence associated to $\pi_\alpha : X_\alpha \to C$ and $\mathcal{O}^*_{X_\alpha}$, we get the exact sequence

$$H^2\big(C, \mathcal{O}^*_C\big) \to H^2\big(X_\alpha, \mathcal{O}^*_{X_\alpha}\big) \to H^1\big(C, \mathcal{P}ic(X_\alpha/C)\big) \to H^3\big(C, \mathcal{O}^*_C\big),$$

where all cohomologies are taken either in the analytic topology or in the étale topology $\big($note that $R^1\pi_{\alpha,*}\mathcal{O}^*_{X_\alpha} = \mathcal{P}ic(X_\alpha/C)\big)$. Hence, since $H^3\big(C, \mathcal{O}^*_C\big) = H^2\big(C, \mathcal{O}^*_C\big) = 0$,

$$H^1\big(C, \mathcal{P}ic(X_\alpha/C)\big) \cong \mathrm{Br}'(X_\alpha). \tag{17}$$

Since $\mathrm{Sh}(X) = H^1\big(C, X^{\#}\big)$, from (16) and (17) we get the morphism

$$T_\alpha : \mathrm{Sh}(X) \to \mathrm{Br}'(X_\alpha).$$

In particular, for the elliptic fibration $\pi : X \to C$ we get the exact sequence

$$0 \longrightarrow \mathrm{Sh}(X) \xrightarrow{T_0} H^2\big(X, \mathcal{O}_X^*\big) \longrightarrow H^1(C, \mathbb{Z}).$$

Thus T_0 is an isomorphism because $\mathrm{Br}'(X) = H^2\big(X, \mathcal{O}_X^*\big)$ (X is smooth) is a torsion group and $H^1(C, \mathbb{Z})$ is torsion free.

Theorem 50 (Donagi–Pantev, [8]). *Let $\pi : X \to C$ be an elliptic fibration with a section. Fix a positive integer m and let $\alpha, \beta \in \mathrm{Sh}(X)$ be two elements such that α is m-divisible and β is m-torsion. Then there is an equivalence*

$$\Phi : \mathrm{D}^b\left(X_\alpha, T_\alpha(\beta)\right) \cong \mathrm{D}^b\left(X_\beta, T_\beta(\alpha)^{-1}\right).$$

***Remark* 51.** Let X be a relatively elliptic surface with a section and $\alpha \in \mathrm{Sh}(X)$. Due to Theorem 50, there exists an equivalence

$$\mathrm{D}^b(X_\alpha) = \mathrm{D}^b\left(X_\alpha, T_\alpha(0)\right) \cong \mathrm{D}^b\left(X, T_0(\alpha)^{-1}\right).$$

Since T_0 is an isomorphism, we denote the element α and $T_0(\alpha)$ by the same letter α when there is no confusion. For example, if α is of order 2 we get an equivalence $\mathrm{D}^b(X_\alpha) \cong \mathrm{D}^b\left(X, \alpha^{-1}\right) \cong \mathrm{D}^b(X, \alpha)$.

Proposition 52. *Let X be a relatively minimal elliptic surface with a section and* $\mathrm{kod}(X) = 1$. *If $Y \in \mathrm{Sh}(X)$ and $\Phi : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ is an equivalence. Then $X \cong Y$ as elliptic surfaces.*

Proof. Since $Y \in \mathrm{Sh}(X)$, there exists $\alpha \in \mathrm{Sh}(X)$ such that $X_\alpha \cong Y$. By Theorem 50

$$\mathrm{D}^b\left(X, T_0(\alpha)^{-1}\right) \cong \mathrm{D}^b(X_\alpha) \cong \mathrm{D}^b(Y) \cong \mathrm{D}^b(X)$$

and by Corollary 49, $T_0(\alpha)^{-1} = 1$ in $\mathrm{Br}'(X)$. Thus X and Y are isomorphic as elliptic surfaces. ☑

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