A Simple Observation Concerning
Contraction Mappings

Una simple observación acerca de las contracciones

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Abstract. In this short note we show that the results obtained by Walter in [4] remain valid if we change the metric $\sigma$ by another metric. Furthermore, if we use the norm $|\cdot|_{T,\epsilon}$ given in [3], Theorem B in [4] remains valid.

Key words and phrases. Contraction, contraction principle, fixed point.

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Resumen. En esta breve nota se muestra que los resultados obtenidos por Walter en [4] siguen siendo válidos si se cambia la métrica $\sigma$ por otra. Además, si se utiliza la norma $|\cdot|_{T,\epsilon}$ usada en [3], el Teorema B en [4] sigue siendo válido.

Palabras y frases clave. Contracción, principio de la contracción, punto fijo.

1. Introduction

The main motivation of this note was the paper by W. Walter [4]. Thus, we consider $(X, \rho)$ a metric space and $T : X \to X$ a nonlinear map. We say that $T$ is Lipschitz continuous if there exists $\alpha \geq 0$ such that

$$\rho(Tx, Ty) \leq \alpha \rho(x, y), \quad \forall x, y \in X,$$

and if in addition $0 \leq \alpha < 1$, the map $T$ is called a contraction.

The aim of this short note is to prove the following propositions and make some remarks about them.

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Proposition 1. Let \((X, g)\) be a metric space and \(T : X \to X\) a map such that, for a fixed \(n \in \mathbb{N}\), \(T^n\) satisfies
\[
g(T^n x, T^n y) \leq \alpha^n g(x, y) \quad \text{for } x, y \in X. \tag{1}
\]

Then the function \(\zeta\) defined by
\[
\zeta(x, y) := \left[ g^2(x, y) + \frac{1}{\alpha^2} g^2(Tx, Ty) + \cdots + \frac{1}{\alpha^{2(n-1)}} g^2(T^{n-1} x, T^{n-1} y) \right]^{1/2} \tag{2}
\]
is a metric on \(X\), and \(T\) satisfies
\[
\zeta(Tx, Ty) \leq \alpha \zeta(x, y) \quad \text{for } x, y \in X. \tag{3}
\]
Moreover, there exist positive constants \(a, b\) such that
\[
a g(x, y) \leq \zeta(x, y) \leq b g(x, y) \tag{4}
\]
if and only if \(T\) is Lipschitz continuous with respect to \(g\).

**Proof.** It is not difficult to see that \(\zeta\) is a metric on \(X\) and \(g(x, y) \leq \zeta(x, y)\) for all \(x, y \in X\). Now, using the definition of \(\zeta\) we get
\[
\zeta(Tx, Ty) = \left[ g^2(Tx, Ty) + \frac{1}{\alpha^2} g^2(T(Tx), T(Ty)) + \cdots + \frac{1}{\alpha^{2(n-1)}} g^2(T^{n-1} x, T^{n-1} y) \right]^{1/2}
\]
\[
= \left[ g^2(Tx, Ty) + \frac{1}{\alpha^2} g^2(T^2 x, T^2 y) + \cdots + \frac{1}{\alpha^{2(n-2)}} g^2(T^{n-1} x, T^{n-1} y) + \frac{\alpha^2}{\alpha^{2(n-1)}} g^2(T^n x, T^n y) \right]^{1/2}
\]
\[
\leq \left[ g^2(Tx, Ty) + \frac{1}{\alpha^2} g^2(T^2 x, T^2 y) + \cdots + \frac{1}{\alpha^{2(n-2)}} g^2(T^{n-1} x, T^{n-1} y) + \frac{\alpha^2}{\alpha^{2(n-1)}} g^2(x, y) \right]^{1/2}
\]
\[
\leq \left[ \alpha^2 \left( g^2(x, y) + \frac{1}{\alpha^2} g^2(Tx, Ty) + \cdots + \frac{1}{\alpha^{2(n-1)}} g^2(T^{n-1} x, T^{n-1} y) \right) \right]^{1/2}
\]
\[
= \alpha \zeta(x, y), \quad \forall x, y \in X,
\]
where in the last inequality we have used (1). Hence (3) is proved.

Also, if \(g(x, y) \leq bg(x, y)\), it is not difficult to show that \(T\) is Lipschitz continuous with respect to \(g\). In fact,
\[
g(Tx, Ty) \leq \zeta(Tx, Ty) \leq \alpha \zeta(x, y) \leq \alpha b g(x, y), \quad \text{for all } x, y \in X.
\]
Conversely, if $T$ is Lipschitz continuous, then the powers of $T$ are also Lipschitz continuous.

If we assume that

$$\varrho(T^k x, T^k y) \leq a_k \varrho(x, y), \quad x, y \in X, \quad k = 1, 2, \ldots, n-1, \quad (5)$$

then

$$\varrho(x, y) \leq \zeta(x, y) \leq b \varrho(x, y), \quad \text{for} \quad x, y \in X \quad (6)$$

where $b = 1 + a_1 \alpha^{-1} + \cdots + a_{n-1} \alpha^{1-n}$. To get the last inequality we use the right side of (2) and (5).

**Proposition 2.** Let $(X, |\cdot|)$ be a Banach space and $A \in \mathcal{L}(X)$ such that $|A^m| = \alpha^m$. Then the formula

$$\|x\|_\zeta := \left( |x|^2 + \frac{1}{\alpha^2} |Ax|^2 + \cdots + \frac{1}{\alpha^{2(n-1)}} |A^{n-1}x|^2 \right)^{1/2}$$

defines a norm on $X$ equivalent to the original norm, and for the norm of $A$, $\|A\|_\zeta$, we have the inequality $\|A\|_\zeta \leq \alpha$.

**Proof.** It is not difficult to see that $\|\cdot\|_\zeta$ is a norm on $X$ and $|x| \leq \|x\|_\zeta \leq b|x|$ for all $x \in X$, i.e., the norms $|\cdot|$ and $\|\cdot\|_\zeta$ are equivalent. On the other hand,

$$\|Ax\|_\zeta = \left( |Ax|^2 + \frac{1}{\alpha^2} |A^2x|^2 + \cdots + \frac{1}{\alpha^{2(n-1)}} |A^{n-1}x|^2 \right)^{1/2}$$

\[ \leq \left( |Ax|^2 + \frac{1}{\alpha^2} |A^2x|^2 + \cdots + \frac{1}{\alpha^{2(n-2)}} |A^{n-1}x|^2 + \frac{\alpha^{2n}}{\alpha^{2(n-1)}} |x|^2 \right)^{1/2} \]

\[ = \alpha \|x\|_\zeta. \]

This proves that $\|A\|_\zeta \leq \alpha$. \hfill √

2. Some Remarks

**Remark 3.** Proposition 1 is the same as Proposition A in [4], where we change the metric $\sigma$ by the metric $\zeta$. Also, we can see that

$$\zeta(x, y) \leq \sigma(x, y) \quad \text{for all} \quad x, y \in X. \quad (7)$$

The same applications given in [4] such as Contraction principle, Continuous dependence and Approximate iteration can also be obtained changing the metric $\sigma$ by $\zeta$. As an example, it is well known that if $(X, \varrho)$ is a complete metric space and $T : X \to X$ is a contraction then there exists an unique $x \in X$.
such that \( Tx = x \). This is called the contraction principle or the Banach fixed point theorem. For details on contraction principle see [1, p.120]. One way to find the fixed point \( x \) is: given \( x_0 \in X \) arbitrary, the sequence \( \{x_n\} \subset X \) given by

\[
\begin{align*}
  x_0 & \in X, \\
  x_n & = T^n x_0, \quad n = 0, 1, 2, \ldots
\end{align*}
\]

converges to \( x \). The recursion formula given in (8) is known as the successive approximations method to find the fixed point \( x \). Moreover, we have a priori error estimate

\[
\rho(x_n, x) \leq \frac{\alpha^n}{1 - \alpha} \rho(x_0, x_1), \quad n = 0, 1, 2, \ldots,
\]

and a posteriori error estimate

\[
\rho(x_{n+1}, x) \leq \alpha \rho(x_n, x_{n+1}), \quad n = 0, 1, 2, \ldots,
\]

and, we have the rate of convergence

\[
\rho(x_{n+1}, x) \leq \alpha \rho(x_n, x), \quad n = 0, 1, 2, \ldots
\]

Now, if \( T \) is a map such that, for some \( n \in \mathbb{N} \), \( T^n \) is a contraction with constant \( \alpha^n < 1 \) and \( T \) satisfies the hypothesis of Proposition 1 then from (3), we have that \( T \) is a contraction with respect to \( \zeta \) with constant \( \alpha \). Thus, the inequalities (9), (10) and (11) remain valid if we change the metric \( \rho \) by the metric \( \zeta \).

For numerical implementation it is important to know the number of iterations, \( N \), to get a good approximation of the fixed point. Setting \( d = \rho(x, Tx) \) and using the a priori error estimate (9), we have a lower bound for \( N \) given by

\[
N > \frac{\ln(\epsilon) + \ln(1 - \alpha) - \ln d}{\ln K},
\]

thus we have \( \rho(x_n, x) < \epsilon, \epsilon > 0 \). For more details see [2].

**Remark 4.** Proposition 2 is the same as Proposition B in [4] where we change the norm \( \|\cdot\| \) by the norm \( \|\cdot\|_\zeta \). Also, we can easily see that

\[
\|x\|_\zeta \leq \|x\| \quad \text{for all} \quad x \in X.
\]

**Remark 5.** The norm \( \|\cdot\|_\zeta \) is the same norm \( \|\cdot\|_{T, \epsilon} \) given in [3, p. 132]. If we use the norm \( \|\cdot\| \) given in [4] which is equivalent to the norm \( \|\cdot\|_\zeta \), the main result (Theorem 1) in [3] is still valid.
References


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