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Brown Representability and Spaces over a Category

Representabilidad de Brown y espacios sobre una categoría

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Abstract. We prove a Brown Representability Theorem in the context of spaces over a category. We discuss two applications to the representability of equivariant cohomology theories, with emphasis on Bredon cohomology with local coefficients.

Key words and phrases. Brown Representability, Spaces over a category, Bredon Cohomology with local coefficients.

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Resumen. Probamos un teorema de representabilidad de Brown en el contexto de espacios sobre una categoría. Discutimos dos aplicaciones a la representabilidad de teorías de cohomología, con énfasis en cohomología de Bredon con coeficientes locales.

Palabras y frases clave. Representabilidad de Brown, espacios sobre una categoría, cohomología de Bredon con coeficientes locales.

1. Introducción

In this note we present a proof of the Brown Representability Theorem in the context of spaces over a category, Theorem 4.1.

As an application of the representability result we describe an equivariant generalization of Steenrod Square operations for Bredon cohomology with local coefficients, and describe induction structures.

There are several proofs of representability results in the literature. The approach given in the setting of compactly generated triangulated categories [16, Theorem 4.1, p. 206], together with the usual constructions of the homotopy categories of genuine equivariant spectra as triangulated categories yield several particular instances of similar results.

The earliest realization of this program (without the model/triangulated category machinery) has been outlined in [10].

In a slightly different context, L. Gaunce Lewis Jr. sketches the main steps to be done to achieve Brown Representability for $RO(G)$-graded Cohomology theories for a compact Lie group G in chapter XIII of [11]. However, there are no constructions of the G-equivariant stable homotopy category as a triangulated category for an infinite discrete group G. Hence, other approaches to the representability of equivariant cohomology theories need to be studied.

After collecting necessary preliminary results, spaces and spectra over a small category $\mathcal{C}$ are discussed in Section 2. In Section 4, the notion of a $\mathcal{C}$-cohomology theory is introduced and the main result, Theorem 4.1 is proved.

Theorem. 4.1 Let $\mathcal{H}^*_{\mathcal{C}} : \mathcal{C}-\text{Pairs} \to \mathbb{Z}-\text{Mod}$ be a $\mathcal{C}$-cohomology theory defined on contravariant $\mathcal{C}$-spaces. Then, there exists a contravariant $\mathcal{C}$-Ω-spectrum $\mathbb{E}_{\mathcal{H}_{\mathcal{C}}}$ and a natural transformation of $\mathcal{C}$-cohomology theories

$$\mathcal{H}^n_{\mathcal{C}}(X) \longrightarrow [X, E_{\mathcal{H}_{\mathcal{C}}}(n)]_{\mathcal{C}}$$

consisting of group isomorphisms.

Natural transformations between $\mathcal{C}$-cohomology theories are obtained as maps of the representing objects in the homotopy category, Lemma 4.16. Equivariant Cohomology theories out of a functor defined on small groupoids, taking values on Ω-spectra, and sending equivalences to weak homotopy equivalences are constructed in [7, Proposition 6.8, p. 95].

Given a G-equivariant cohomology theory, the orbit Category $\mathrm{Or}(G)$ is constructed, and an $\mathrm{Or}(G)$-cohomology theory is associated in Section 4. Specializing to the orbit category, Theorem 4.1 gives Corollary 5.3, which is a partial converse to the construction of equivariant cohomology theories by functors taking values on the category of Ω-spectra.

Corollary. 5.3 Let $\mathcal{H}^*_{-}$ be an equivariant cohomology theory and let G be a discrete group. Let $\mathcal{H}^*_{\mathrm{Or}(G)}$ be the $\mathrm{Or}(G)$-cohomology theory defined on $\mathrm{Or}(G)$ spaces by applying to a $\mathrm{Or}(G)$ pair (X, A) an $\mathrm{Or}(G)$-cellular approximation $\big(X', A'\big)$ of the G-pair $\widehat{(X, A)}$. In symbols:

$$\mathcal{H}^*_{\mathrm{Or}\,G}(X, A) = \mathcal{H}^*_G\Big(\widehat{(X', A')}\Big).$$

For any $n \in \mathbb{Z}$, the construction $G \mapsto E_{\mathcal{H}_{\mathrm{Or}(G)}}(n)(G/G)$ sends a group isomorphism to a weak homotopy equivalence.

Another consequence of Theorem 4.1 is Corollary 5.8, which describes Steenrod square operations in Bredon Cohomology with local $\mathbb{Z}/2$-coefficients.

Corollary. 5.8 Let M be a local coefficient system with values on $\mathbb{Z}/2$-modules and $H^*_{\mathbb{Z}\,\mathrm{Or}(G)}(\quad, M)$ be Bredon cohomology with coefficients in M. The Steenrod square operations Sq_k correspond to $\mathrm{Or}(G)^{op} \times \mathcal{E}$-homotopy classes of $\mathrm{Or}(G) \times \mathcal{E}$-maps $Sq_k \in \left[Y_{H_{\mathbb{Z}\Delta_G(X)^n, M}}, Y_{H_{\mathbb{Z}\Delta_G(X)^{n+k}, M}}\right]_{\mathrm{Or}(G)^{op}\times\mathcal{E}}$ between the representing objects constructed either in Theorem 4.1 or [2].

Although the more general approach via triangulated categories of [16] does certainly give a proof of the main Theorem 4.1, we keep the arguments elementary and close to the classical exposition of Brown, aiming to address the described applications in the study of equivariant cohomology theories, as understood in [7], as opposed to the more sophisticated $\mathrm{RO}(G)$-Graded setting of [11]. On the other hand, the comparison to recent developments in the study of the representability of Bredon Cohomology with local coefficients [2], as well as the construction of the Steenrod Square operations do profit from the elementary exposition given here.

The description of the natural transformations described in Corollary 5.8 plays a role in the construction of a spectral sequence to compute twisted equivariant K-Theory for proper actions of infinite discrete groups in [1].

2. Spaces and Modules over a Category

We refer the reader to [4] for further reference and for the proof of the results in this section. All spaces have the compactly generated topology, in the sense of [12]. This is necessary for the construction of mapping spaces.

Definition 2.1. Let $\mathcal{C}$ be a small category. A covariant (contravariant) pointed $\mathcal{C}$-space over $\mathcal{C}$ is a covariant (contravariant) functor $\mathcal{C} \longrightarrow$ Spaces to the category of compactly generated, pointed spaces.

A $\mathcal{C}$-map $f : X \to Y$ between $\mathcal{C}$-spaces is a natural transformation consisting of continuous maps. We will denote by $X \wedge Y$ the smash product of pointed spaces.

Let I_+ be the constant $\mathcal{C}$-space assigning to each object the interval $[0,1]$ with an added disjoint base point. A homotopy of pointed $\mathcal{C}$-maps $f_0, f_1 : X \to Y$ is a map of $\mathcal{C}$-spaces $H : X \wedge I_+ \to Y$ which restricted to $X \wedge \{j\}_+$ gives the maps f_j for $j = 0, 1$.

A $\mathcal{C}$-map $i : A \to X$ is said to be a cofibration if it has the homotopy extension property.

The following definition extends the notion of a CW-complex to pointed spaces over a category.

Definition 2.2. Let $\mathcal{C}$ be a small category. A pointed $\mathcal{C}$-CW complex is a contravariant $\mathcal{C}$-space together with a filtration

$$X_0 \subset X_1 \subset \cdots = X_n$$

such that $X = \mathrm{colim}_n X_n$ and each X_n is obtained from the X_{n-1} by a pushout of maps consisting of pointed maps of $\mathcal{C}$-spaces of the form

$$\begin{tikzcd}
\coprod_{i\in I_n} \mathrm{mor}_{\mathcal{C}}(-,c_i)_+ \wedge S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i\in I_n} \mathrm{mor}_{\mathcal{C}}(-,c_i)_+ \wedge D^{n} \arrow[r] & X_n
\end{tikzcd}$$

Here, the space $\mathrm{mor}(-,c_i)$ carries the discrete topology, c_i is an object in $\mathcal{C}$ and i is an element of an indexing set I_n. And $+$ denotes the addition of a disjoint basis point to the space.

Definition 2.3. Let $f : X \to Y$ be a map between $\mathcal{C}$-spaces. f is said to be n-connected (or a weak homotopy equivalence) if for all objects $c \in \mathcal{C}$, the map of spaces $f(c) : X(c) \to Y(c)$ is n-connected (weak homotopy equivalence).

We need the following version of the Whitehead Theorem, easily obtained as a translation to the pointed setting of Theorem 3.4 in [4, p. 222].

Theorem 2.4. *Let $f : Y \to Z$ be a pointed map of $\mathcal{C}$-spaces and X be a pointed $\mathcal{C}$-space. The map on homotopy classes of maps between $\mathcal{C}$-spaces induced by composition with f is denoted by $f_* : [X,Y]_{\mathcal{C}} \to [X,Z]_{\mathcal{C}}$. Then:*

- *f is n-connected if and only if f_* is bijective for any pointed $\mathcal{C}$-CW complex with $dim(X) < n$ and surjective for any free $\mathcal{C}$-CW complex with $dim(X) = n$.*
- *f is a weak homotopy equivalence if and only f_* is bijective for any pointed $\mathcal{C}$-CW complex X.*

There exists a pointed $\mathcal{C}$-CW approximation of every pair of pointed $\mathcal{C}$-spaces, which is easy to obtain by modifying Theorem 3.7 in [4, p. 223] to the pointed setting.

We present two useful constructions for spaces over a category. They are an instance of ends and coends in category theory [9, Chapter IX, 5 and 6]. Well-known constructions like geometric realizations and mapping spaces give examples of coends.

Definition 2.5. Let X be a contravariant, pointed $\mathcal{C}$, and let Y be a covariant pointed $\mathcal{C}$-space over $\mathcal{C}$. Their tensor product $X \otimes_{\mathcal{C}} Y$ is the space defined by

$$\coprod_{C\in Obj(\mathcal{C})} X(C) \wedge Y(C)/\sim$$

where $\sim$ is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for $x \in X(c)$, $y \in Y(d)$ and morphisms $\phi : c \to d$.

Definition 2.6. Let X and Y be pointed $\mathcal{C}$-spaces of the same variance, in the sense that they are both either covariant or contravariant.

Their mapping space $\hom_{\mathcal{C}}(X,Y)$ is the space of natural transformations between the functors X and Y, topologized as subspace of the product of the spaces of pointed maps $\Pi_{C\in Ob(\mathcal{C})}\mathrm{Map}\big(X(C),Y(C)\big)$.

Given a covariant (contravariant) $\mathcal{C}$-space X and a covariant functor $F:\mathcal{C}\to\mathcal{D}$, the induction with respect to F is the $\mathcal{D}$-space given by

$$F_*X = X \underset{\mathcal{C}}{\otimes} \mathrm{mor}_{\mathcal{D}}\big(F(-),--\big)_+$$

respectively

$$F_*X = \mathrm{mor}_{\mathcal{D}}\big(--,F(-)\big)_+ \underset{\mathcal{C}}{\otimes} X$$

Given a contravariant (covariant) $\mathcal{D}$-space the restriction to F, F^*X is the composition $X\circ F$. Both induction and restriction are fucntorial, in the sense that a morphism of $\mathcal{C}$-spaces (i.e., a natural transformation) $f:X\to Y$ induces morphisms $F_*(f):F_*X\to F_*Y$, $F^*(f):F^*X\to F^*Y$ given by $f\otimes\mathrm{id}$, $f\circ F$ in the covariant case and $id\underset{\mathcal{C}}{\otimes} f$, respectively $f\circ F$ in the contravariant case.

Induction and restriction satisfy adjunctions, which are described in [4, Lemma 1.9, p. 208].

Lemma 2.7. *Given a $\mathcal{C}$-space X, a covariant functor $F:\mathcal{C}\to\mathcal{D}$ and a $\mathcal{D}$-space Y, there are natural adjunction homeomorphisms*

- $hom_{\mathcal{D}}(F_*X,Y)\to hom_{\mathcal{C}}(X,F^*Y)$;
- $F_*X\underset{\mathcal{D}}{\otimes} Y\to X\underset{\mathcal{C}}{\otimes} F^*Y$;
- $Y\underset{\mathcal{D}}{\otimes} F_*X\to F^*(Y)\underset{\mathcal{C}}{\otimes} X$;
  for a $\mathcal{C}$-space and a $\mathcal{D}$-space Y of the required variance.

We now present several instances of categories $\mathcal{C}$ which will be relevant for the later development.

Definition 2.8. Let $\mathcal{E}$ be the category with two objects s,t and morphisms $i:s\to t$, $p:t\to s$ such that $p\circ i=\mathrm{id}_s$. An Ex-space, compare [6], is a covariant functor $X:\mathcal{E}\to$ Spaces to the category of compactly generated, Hausdorff spaces. The space $B=X(s)$ is called the base space, $E=X(t)$ is the total space and the maps $X(p):E\to B$, $X(s):B\to E$ are called the projection and the section, respectively.

Definition 2.9. Let G be a group. Consider the category $\mathcal{G}$ consisting of only one object c, and where the morphisms are the elements of G. A contravariant (covariant) functor $X : \mathcal{G} \to$ Spaces is equivalent to giving a compactly generated, pointed space $X := X(c)$ and a right (left) action of G leaving the base point fixed.

Definition 2.10. Let $\mathcal{F}$ be a family of subgroups of the discrete group G closed under intersection and conjugation. The orbit category $\mathrm{Or}(G, \mathcal{F})$ has as objects homogeneous spaces G/H for $H \in \mathcal{F}$, and a morphism is a G-equivariant map $F : G/H \longrightarrow G/K$. If X is a pointed G-space, we define the contravariant $\mathrm{Or}(G, \mathcal{F})$-space associated to X to be the functor $G/H \longmapsto X^H$. The covariant $\mathrm{Or}(G, \mathcal{F})$-space associated to X is the functor $G/H \mapsto G/H_+ \wedge_G X$.

***Remark* 2.11.** The example given in Definition 2.10 has been successfully used in the literature to describe the homotopy theory of spaces with an action of a group G and $\mathrm{Or}(G)$-spaces. The level model structure on $\mathrm{Or}(G)$ spaces, with level wise weak equivalences and cofibrations having the left homotopy extension property is Quillen equivalent to the Homotopy category of compactly generated, Weak Hausdorff G-spaces, compare [4], [11]. Fibrant objects are given by free $\mathrm{Or}(G)$-CW complexes. For more on this, we remit the reader to Section 5.

3. Cohomology Theories Over a Category

The following definition contains a set of axioms for $\mathcal{C}$-cohomology theories.

Definition 3.1. Let $\mathcal{C}$ be a small category. A reduced $\mathcal{C}$-cohomology theory is a sequence of weak $\mathcal{C}$-homotopy invariant, contravariant functors $\mathcal{H}^n_{\mathcal{C}} : \mathcal{C}$-Pairs $\to \mathbb{Z} - \mathrm{Mod}$, together with natural transformations

$$\delta^n_{(X,A)} : \mathcal{H}^n_{\mathcal{C}}(A) \to \mathcal{H}^{n+1}_{\mathcal{C}}(X, A)$$
$$\sigma^n_{(X,A)} : \mathcal{H}^n_{\mathcal{C}}(X) \to \mathcal{H}^{n+1}_{\mathcal{C}}\big(\Sigma(X)\big)$$

(where $\Sigma(X, A)$ denotes the object wise reduced suspension) satisfying

- The boundary homomorphisms fit into a long exact sequence
$$\cdots \longrightarrow \mathcal{H}^n_{\mathcal{C}}(A) \xrightarrow{\delta^n_{(X,A)}} \mathcal{H}^{n+1}_{\mathcal{C}}(X, A) \xrightarrow{p^*} \mathcal{H}^{n+1}_{\mathcal{C}}(X) \xrightarrow{i^*} \mathcal{H}^{n+1}_{\mathcal{C}}(A) \longrightarrow \cdots$$
- For any wedge $\vee X_i$ of pointed $\mathcal{C}$-spaces, the inclusions $X_i \to \vee X_i$ induce an isomorphism
$$\mathcal{H}^*_{\mathcal{C}}(\vee_i X_i) \cong \Pi_i \mathcal{H}^*_{\mathcal{C}}(X_i).$$
- For any pair (X, A), the homomorphisms $\sigma^n_{(X,A)}$ are isomorphisms.

Definition 3.2. An Ω-spectrum is a sequence of pointed spaces $\mathbb{E} = (E_n)_{n\in\mathbb{Z}}$ together with structure maps $\sigma_n : E_n \wedge S^1 \longrightarrow E_{n+1}$, such that the adjoint maps $\Omega E_{n+1} \longrightarrow E_n$ are homotopy equivalences. A strong map of Ω-spectra $f : \mathbb{E} \longrightarrow \mathbb{F}$ is a sequence of pointed maps $f_n : E_n \longrightarrow F_n$ compatible with the structure maps. We denote by SPECTRA the category of Ω-spectra and strong maps. Recall that the homotopy groups of a spectrum $\mathbb{E}$ are defined by

$$\pi_i(\mathbb{E}) = \text{colim } \pi_{i+k}(E_k)$$

where the structure maps are given as follows:

$$\pi_{i+k}(E_k) \xrightarrow{\wedge id} \pi_{i+k+1}(E_k \wedge S^1) \xrightarrow{\sigma_{k_*}} \pi_{i+k+1}(E_{k+1}).$$

Definition 3.3. A contravariant (covariant) spectrum over the small category $\mathcal{C}$ is a contravariant (covariant) functor $\mathbb{E} : \mathcal{C} \longrightarrow$ SPECTRA.

Let us recall the following

Definition 3.4. Let (X, A) be a $\mathcal{C}$-pair of the same variance of the $\mathcal{C}$-spectrum $\mathbb{E}$. We define the cohomology groups $E^p_{\mathcal{C}}(X, A)$ for a pair (X, A) with coefficients in the spectrum $\mathbb{E}$, by

$$E^p_{\mathcal{C}}(X, A) = \pi_{-p}\big(\hom_{\mathcal{C}}(X \cup_A \text{Cone}(A), \mathbb{E})\big).$$

If $A = \varnothing$, we just drop A from the notation above.

We now discuss an algebraic version of the previous constructions.

Definition 3.5. Let $\mathcal{C}$ be a small category and let R be a commutative ring. A contravariant (covariant) $R\mathcal{C}$-module is a contravariant (covariant) functor from $\mathcal{C}$ to the category of R-modules. A contravariant (covariant) R$\mathcal{C}$-chain complex is a functor from $\mathcal{C}$ to the category of R-chain complexes.

A contravariant $R\mathcal{C}$-module is free if it is isomorphic to an $R\mathcal{C}$-module of the shape

$$\bigoplus_{i \in I} R\big[\text{mor}_{\mathcal{C}}(-, c_i)\big]$$

for some index set I and objects $c_i \in \mathcal{C}$.

Given a covariant $\mathcal{C}$-module A and a contravariant $\mathcal{C}$-module B, the tensor product is defined to be the R-module

$$\bigoplus_{c \in \text{Ob}(\mathcal{C})} A(c) \otimes B(c) / \sim$$

where $\sim$ is generated by the typical tensor relation $mf \otimes n = m \otimes fn$.

Given two R$\mathcal{C}$-modules A, B of the same variance, the abelian group

$$\hom_{R\mathcal{C}}(A, B)$$

is the $\mathbb{Z}$-module of natural transformations of functors from $\mathcal{C}$ to R-modules.

The following construction will be needed later:

Definition 3.6. Given a category $\mathcal{C}$, and an object c, the category under c, denoted by $c \downarrow \mathcal{C}$, is the category where the objects are morphisms $\varphi : c \to c_0$ and a morphism between $\varphi_0 : c \to c_0$ and $\varphi_1 : c \to c_1$ is a morphism ψ in $\mathcal{C}$ such that $\psi \circ \varphi_0 = \varphi_1$.

Consider the contravariant $\mathcal{C}$-space $\mathrm{B}c \downarrow \mathcal{C}$ which assigns to an object c in $\mathcal{C}$ the geometric realization of the category $c \downarrow \mathcal{C}$.

The contravariant, free $\mathbb{Z}\mathcal{C}$-chain complex $C_*^{\mathbb{Z}}(\mathcal{C})$ is defined as the cellular $\mathbb{Z}$-chain complex of the $\mathcal{C}$-space $\mathrm{B}c \downarrow \mathcal{C}$. In Symbols

$$C_*^{\mathbb{Z}}(c) = C^*(\mathrm{B}c \downarrow \mathcal{C}).$$

Definition 3.7. Given a $\mathcal{C}$-space (X, A) and a $\mathcal{C}$-$\mathbb{Z}$ module M of the same variance, the n-th $\mathcal{C}$-cohomology of (X, A) with coefficients in M, denoted by $H^n_{\mathbb{Z}\mathcal{C}}\big((X, A); M\big)$, is the n-th cohomology of the $\mathcal{C}$-cochain complex obtained by taking the $\mathbb{Z}$-module of $\mathbb{Z}\mathcal{C}$ maps between the cellular $\mathcal{C}$-chain complex of a $\mathcal{C}$-CW approximation $(X', A') \to (X, A)$ and M, in symbols

$$H^n_{\mathbb{Z}\mathcal{C}}\big((X, A); M\big) := H^n\Big(\hom_{\mathbb{Z}\mathcal{C}}\big(C_*(X', A'), M\big)\Big).$$

For the orbit category $\mathrm{Or}(G)$, the $\mathrm{Or}(G)$-cohomology with coefficients in an $\mathrm{Or}(G)$-module is known as Bredon cohomology [3].

Remark **3.8.** Definition 3.1 specializes to constructions which have been considered by other authors [13]. For the case $X = \{\bullet\}$, $A = \varnothing$, this is known as the cohomology of a category.

In another direction, specializing to the category with one object and just the identity morphism gives the cellular cohomology of a CW-approximation of spaces.

4. Representability

In this section we will prove:

Theorem 4.1. *Let $\mathcal{H}^*_{\mathcal{C}} : \mathcal{C} - Pairs \to \mathbb{Z} - Mod$ be a $\mathcal{C}$-cohomology theory defined on contravariant $\mathcal{C}$-spaces in the sense of Definition 3.1. Then, there exists a contravariant $\mathcal{C}$-Ω-spectrum $\mathbb{E}_{\mathcal{H}_{\mathcal{C}}}$ and a natural transformation of $\mathcal{C}$-cohomology theories*

$$\mathcal{H}^n_{\mathcal{C}}(X) \longrightarrow \big[X, E_{\mathcal{H}_{\mathcal{C}}}(n)\big]_{\mathcal{C}}$$

consisting of group isomorphisms.

We introduce the notion of a double-sided map cylinder in the context of (pointed) $\mathcal{C}$-spaces.

Definition 4.2. Let $f, g : X \to Y$ be two pointed maps between pointed $\mathcal{C}$-spaces. A double-sided mapping cylinder for f and g is a pointed $\mathcal{C}$-space Z

together with a natural transformation $p : Y \to Z$ with the property that for any map $j : Y \to W$ satisfying $[j \circ f] = [j \circ g]$, one has a map $j' : Z \to W$ such that $[j] = \left[j' \circ p\right]$.

There exists a concrete model for the double-sided mapping cylinder of two pointed $\mathcal{C}$-maps $f, g : X \to Z$, denoted by C_g^f, and defined as the quotient space

$$X \wedge I_+ \coprod Z/(x,0) \sim f(x) \qquad (x,1) \sim g(x) \qquad + \wedge I_+ \sim +.$$

An easy consequence of the exact sequence property for pairs in reduced $\mathcal{C}$-cohomology theories is the following fact.

Lemma 4.3. *Let $T : \mathcal{C} - Spaces \longrightarrow \mathbb{Z} - Mod$ be a $\mathcal{C}$-cohomology theory. Let $j : Y \to Z$ be the canonical inclusion of Y into the double-sided mapping cylinder for the $\mathcal{C}$-maps $f, g : X \to Y$. Then, for every element $w \in T(Y)$ satisfying that $T[g](w) = T[f](w)$ in $T(X)$, there exists a $v \in T(Z)$ with $T[j](v) = w$.*

The following result is crucial for the representability theorem proved in this section. One usual reference is [9, page 61].

Lemma 4.4 (Yoneda)**.** *Let $T : \mathcal{C} \longrightarrow Set$ be a contravariant functor defined on the small category $\mathcal{C}$ with values in the category of sets. Then, for every object c there is a bijection*

$$\left\{\textit{Natural transformations} \quad \mathrm{mor}_{\mathcal{C}}(-,c) \xrightarrow{\bullet} T(-)\right\} \cong \left\{\textit{Elements in} \quad T(c)\right\}$$

Moreover, the bijection is given by assigning to a natural transformation $t : mor_{\mathcal{C}}(-,c) \to T$ the image of the map induced by the identity $t(i_c) : T(c) \to T(c)$. The inverse map is given by assigning to an element $u \in T(c)$ the natural transformation $\varphi_u : mor_{\mathcal{C}}(-,c) \to T(-)$ which assigns a morphism $f : d \to c$ the evaluation $T(f)(u) \in T(d)$.

Definition 4.5. A functor $T : \mathcal{C} \to$ Set naturally equivalent to $\mathrm{mor}_{\mathcal{C}}(\quad, c)$ for a fixed object $c \in \mathcal{C}$ is called *representable*. An element $u \in T(c)$ associated to id_c under such a natural correspondence is called *universal element*. In case of a functor T defined on the category of $\mathcal{C}$-spaces, and a $\mathcal{C}$-space Y representing T, the $\mathcal{C}$-space Y is said to be a *classifying object*.

Definition 4.6. Let T be a contravariant, $\mathcal{C}$-homotopy invariant functor defined on the category of $\mathcal{C}$-pairs, and taking values in the category of abelian groups.

- T is said to have the exact sequence property if the following holds: For any sequence of $\mathcal{C}$-pairs, $A \xrightarrow{i} X \xrightarrow{j} (X, A)$ the induced sequence

  $$T(X, A) \xrightarrow{T(j)} T(X) \xrightarrow{T(i)} T(A)$$

  is exact.

- T is said to satisfy the wedge axiom if for any family of $\mathcal{C}$-spaces X_i, the inclusions $X_i \to \vee X_i$ induce abelian group isomorphisms $T(\vee X_i) \cong \prod T(X_i)$.

Notice in particular that this is the case for a $\mathcal{C}$-cohomology group in a fixed degree $T := \mathcal{H}^q_{\mathcal{C}}(, \varnothing)$.

Since we are dealing with functors defined on the homotopy category of $\mathcal{C}$-spaces, the morphism sets appearing on the left hand side of the isomorphism described in the Yoneda Lemma are given by $\mathcal{C}$-homotopy classes of maps, denoted by $[-,-]_{\mathcal{C}}$. Moreover, since we are dealing mostly with free $\mathcal{C}$-complexes, we will detect the isomorphism of the functor with a representable functor on free cells.

Definition 4.7. Given a group-valued functor T satisfying the exact sequence property and the wedge axiom defined in the category of contravariant $\mathcal{C}$-spaces and a $\mathcal{C}$-space Y, an element $u \in T(Y)$ is said to be n-universal if the function $\varphi_u : \big[\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q, Y)\big]_{\mathcal{C}} \to T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q\big)$ given by the Yoneda Lemma is an isomorphism for $q < n$ and an epimorphism for $q = n$ and all objects $c \in \mathcal{C}$.

We describe briefly the strategy for the proof of Theorem 4.1. We will use the Yoneda Lemma, the wedge and the exact sequence property to construct $\mathcal{C}$-spaces Y_i, as well as elements u_i, which are i-universal for all i. This will provide a free $\mathcal{C}$-CW complex Y with an element u, which gives via the Yoneda Lemma an isomorphism of the functor $T = \mathcal{H}^q_{\mathcal{C}}(, \varnothing)$ with the representable functor $[-,Y]_{\mathcal{C}}$. We will be able to verify that the suspension axiom gives a structure of a contravariant Ω-spectrum on this $\mathcal{C}$-space.

Lemma 4.8. *Let Y be a $\mathcal{C}$-space, let T be a functor satisfying the exact sequence property and the wedge axiom. Pick up an element $u \in T(Y)$. Then, there is a $\mathcal{C}$-CW complex $Y_1 = Y \vee_\alpha mor_{\mathcal{C}}(-,c)_+ \wedge S^1$ obtained by attaching pointed $\mathcal{C}$-cells to Y and a 1-universal element $u_1 \in T(Y_1)$ with $u_1|_Y = u \in T(Y)$.*

Proof. We denote by $Y_1 = Y \vee_\alpha \mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^1$ the space obtained by attaching a copy of the pointed 1-cell for every element $\alpha \in T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^1\big)$.

From the wedge axiom one gets $T(Y_1) \cong T(Y) \times \Pi_\alpha T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^1\big)$. Take the element $u_1 \in T(Y_1)$ which maps under this equivalence onto $\big(u, (\alpha)\big)$. Clearly, $u_1|_Y = u$ and notice that the natural transformation induced by u_1 takes.

$$\big[S^0 \wedge \mathrm{mor}_{\mathcal{C}}(-,c)_+, Y_1\big]_{\mathcal{C}} \to \big[S^0, Y_1(c)\big]_+ = \{\bullet\}$$

to the set $T\big(S^0 \wedge \mathrm{mor}_{\mathcal{C}}(-,c)_+\big) \cong \{\bullet\}$ bijectively for every object c.

The natural transformation $\left[\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^1, Y_1\right]_{\mathcal{C}} \longrightarrow T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^1\big)$ is surjective. To see this, let $\alpha \in T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^1\big)$. Let $f_\alpha : \mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^1 \longrightarrow \vee_\alpha \mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^1$ be the inclusion. The class $[f_\alpha]$ maps under the transformation to α. ☑

Lemma 4.9. *Given a $\mathcal{C}$-space Y and an element $u \in T(Y)$, there is a space Y_n obtained by attaching pointed cells of dimension less than or equal to n and an n-universal element $u'_n \in T(Y_n)$ with $u'_n|_Y = u$.*

Proof. We assume inductively that we constructed Y_{n-1} with an element u'_{n-1} with the above described property for $n-1$ instead of n. As before, for $\beta \in T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^n\big)$, we consider a copy of $\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^n$ and we put $Y'_n = Y_{n-1} \vee \big(\vee_\beta \mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^n\big)$. The wedge axiom gives $T(Y'_n) \cong T(Y_{n-1}) \times \Pi_\beta T(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^n)$. We select the element u'_n which maps to $\big(u'_{n-1}, (\beta)\big)$ under this equivalence. As in the previous result, the corresponding map $\varphi_{u'_n}$: $\left[\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^n, Y'_n\right] \longrightarrow T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^n\big)$ is surjective. We select a representative f_α of every element $\alpha \in \left[\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^{n-1}, Y'_n\right]$ with $\varphi_{u'_n}(\alpha) = 0 \in T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^{n-1}\big)$. We attach an n-cell of type $\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge D^n$ with f_α as attaching map and obtain the space Y_n.

The space Y'_n together with the inclusion $j : Y'_n \hookrightarrow Y_n$ is a double-sided mapping cylinder for the diagram $\vee_\alpha \mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^{n-1} \underset{i}{\overset{c}{\rightrightarrows}} Y'_n \hookrightarrow Y_n$, where i is the map given by f_α on each summand α and c is the map given by $\vee f_\alpha$. Notice that u'_n satisfies $T[c](u'_n) = T[i](u'_n)$. From Lemma 4.3, one gets an element $u_n \in T(Y_n)$ satisfying $u_n|_{Y_{n-1}} = u_{n-1}$. We claim that u_n has the desired property.

The diagram commutes

$$\begin{tikzcd}
\left[\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q, Y_{n-1}\right]_{\mathcal{C}} \arrow[rr, "j_*|_{Y_{n-1}}"] \arrow[dr, "\varphi_{u_{n-1}}"'] & & \left[\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q, Y_n\right]_{\mathcal{C}} \arrow[dl, "\varphi_{u_n}"] \\
& T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q\big), &
\end{tikzcd}$$

with j_* an isomorphism for $q \leq n-2$, since the cell structure in lower dimensions remains unaffected. Thus, in that range, $\varphi_{u_{n-1}}$ is an isomorphism, as well as φ_{u_n}. This is actually the situation for $q = n-1$. The surjectivity of the map is clear. Now, let $f : \mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^{n-1} \longrightarrow Y_n$ be a representative of an element in $\ker \varphi_{u_n}$. Because of the surjectivity of j_*, there is a map $\alpha : \mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^{n-1} \to Y_{n-1}$ such that $j_*\big([\alpha]\big) = f$. Then, $\varphi_{u_{n-1}}(\alpha) = 0$, and α represents one of the attaching maps used to define Y_n. It follows that $j_*(\alpha) = 0$ and f is nullhomotopic. The surjectivity in the case $q = n$ is a consequence of the corresponding property for $\varphi_{u'_n}$. ☑

***Remark* 4.10.** Notice that the construction proposed here depends on the choice of maps f_{α_i} representing elements α_i for $T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^i\big)$, giving the attaching maps to obtain Y_{i+1} out of Y_i, as well as the subsequent choice of elements $\beta \in \ker j^* : T(Y_i) \to T(Y_{i-1})$.

Corollary 4.11. *Given a $\mathcal{C}$-space Y and an element $u \in T(Y)$, there is a $\mathcal{C}$-CW complex Y' obtained from Y by attaching cells, together with an ∞-universal element $u' \in T(Y')$ satisfying $u'|_Y = u$.*

***Proof*.** From Lemma 4.9, we get a sequence of spaces Y_n, linked with maps $i_n : Y_n \longrightarrow Y_{n+1}$, one each obtained from the previous one after an attachment of pointed cells. The space $Y = \underset{n}{\mathrm{hocolim}} Y_n$ is a pointed $\mathcal{C}$-CW-complex. We get also n-universal elements u_n one each extending the previous one. From Lemma 4.3 for the pairs (Y_{n+1}, Y_n), there exists an element $u \in T(Y)$ satisfying $u|_{Y_n} = y_n$. The morphism $\varphi_u : \big[\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q, Y\big] \longrightarrow T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q\big)$ is an isomorphism for every q. ☑

The following result analyses uniqueness of the spaces obtained with this construction

Proposition 4.12. *Let Y and Y' be free $\mathcal{C}$-CW complexes with ∞-universal elements $u \in T(Y)$, $u' \in T(Y')$. Then there is a (weak) $\mathcal{C}$-homotopy equivalence $h : Y \longrightarrow Y'$.*

***Proof*.** Let $Y_0 = Y \vee Y'$. From the wedge axiom one gets $T(Y_0) \cong T(Y) \times T(Y')$. There exists a unique element $u_0 \in T(Y_0)$ being mapped into (u, u') via the wedge isomorphism and restricting to u_0. From Corollary 4.11 we get a space Y'' with an ∞-universal element u''. We denote by j the composition of the inclusion $Y \hookrightarrow Y_0 \hookrightarrow Y''$. In the commutative diagram

$$\begin{tikzcd}
\big[\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q, Y\big]_{\mathcal{C}} \arrow[rr, "j_*"] \arrow[dr, "\varphi_{u''}"'] & & \big[\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q, Y''\big]_{\mathcal{C}} \arrow[dl, "\varphi_u"] \\
& T\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q\big), &
\end{tikzcd}$$

the descending arrows are isomorphisms. It follows that the homotopy sets are isomorphic for every q. Then, as a consequence of Theorem 2.4 one gets a $\mathcal{C}$-weak homotopy equivalence $Y \to Y''$ and hence a $\mathcal{C}$-homotopy equivalence between Y and Y'. ☑

Finally, we state the following technical results, which are the last requirements to finish the proof.

Lemma 4.13. *Let (X, A) be a $\mathcal{C}$-CW pair. Then, for a map $g : A \longrightarrow Y$, an universal ∞-element $u \in T(Y)$ and an element $v \in T(X)$ with $T[g](u) = v|_A$, there exists an extension $f : X \longrightarrow Y$ with $T[f](u) = v$.*

Proof. We consider $X \vee Y$ and $(v, u) \in T(X \vee Y)$ under the canonical equivalence. Then, if Z is a double-sided mapping cylinder for the maps $A \xrightarrow{k} X \to X \vee Y$, $A \xrightarrow{g} Y \to X \vee Y$ with structural map $j : X \vee Y \longrightarrow Z$, we get from Lemma 4.3 an element $w \in T(Z)$ satisfying $T[j](w) = (v, u)$. We can apply Corollary 4.11 to obtain a pointed $\mathcal{C}$-CW complex Y' obtained from Z and an ∞-universal element extending w. Due to Proposition 4.12, there exists a map $h : Y' \longrightarrow Y$ being a $\mathcal{C}$-homotopy equivalence. Now define f' as the composition $X \xrightarrow{i} X \vee Y \xrightarrow{j} Z \xrightarrow{i'} Y' \xrightarrow{h} Y$ and notice that the maps $f' \circ i$ and g are homotopic, due to the property of double-sided mapping cylinders. The map $A \hookrightarrow X$ is a $\mathcal{C}$-cofibration. In particular, if H is any homotopy between $f' \circ k$ and g, the problem

$$\begin{tikzcd}
 & X \arrow[dr, "i_0"] \arrow[rrd, bend left, "f'"] & & \\
A \arrow[ur, "i"] \arrow[dr, "id\wedge i_0"'] & & X \wedge I \arrow[r, dashed, "\overline{H}"] & Y \\
 & A \wedge I \arrow[ur, "k\wedge id"] \arrow[rru, bend right, "H"'] & &
\end{tikzcd}$$

admits a solution $\overline{H}$. We define $f(t) = \overline{H}(\{1\}) \times t)$ and $T[f](u) = v$. ☑

Proposition 4.14. *Given a $\mathcal{C}$-space Y, and $u \in T(Y)$, which is an ∞-universal element, then φ_u induces a natural isomorphism $[X, Y] \cong T(X)$.*

Proof. We prove that the morphism φ_u is surjective. Let $v \in T(X)$. Then we apply Lemma 4.13 to the $\mathcal{C}$-pair $(X, +)$ and the map $\rho : + \longrightarrow Y$ to get a map $f : X \longrightarrow Y$ with $T[f](v) = u$. Now, let us prove the injectivity. Suppose we have $[g_0]$, $[g_1] \in [X, Y]_{\mathcal{C}}$ with $\varphi_u([g_0]) = \varphi_u([g_1])$. We consider the space $X' = X \wedge I_+/\{+\} \wedge I_+$. Now we consider the subspace $A = X \wedge \partial I_+/\{+\} \wedge I$. The space A is homeomorphic to $X \vee X$. Define the map $g : A \longrightarrow Y$ by $g = g_1 \vee g_0$. We consider the element v' naturally assigned to $\big(T[g_0](u), T[g_1])(u)\big) \in T(X) \times T(X) \cong T(X \vee X)$ and u. We apply then Lemma 4.13 to this situation. One gets a map $f : X' \longrightarrow Y$ extending g with the property that $T[f](u) = \big(T[g_0](u), T[g_1](u)\big)$. Let $\rho : X \wedge I_+ \longrightarrow X'$ be the quotient map and define $H : X \wedge I_+ \longrightarrow X$ by the composition $f \circ \rho$. This gives a $\mathcal{C}$-homotopy between g_0 and g_1. ☑

We now collect all results in this section, leading to a proof of Theorem 4.1.

End of the proof of Theorem 4.1

Let $\mathcal{H}^*_{\mathcal{C}}$ be a $\mathcal{C}$-cohomology theory and let $\alpha^q \in \mathcal{H}^q_{\mathcal{C}}\big(\mathrm{mor}(-,c)_+ \wedge S^0\big)$ be an arbitrary element. Consider the space W_0 obtained as the wedge $\bigvee_{\alpha^q} \mathrm{mor}(-,c)_+ \wedge S^q$ and let $i_{\alpha^q} : \mathrm{mor}(-,c)_+ \wedge S^q \to W_0$ the inclusion of the summand indicated by α^q. Because of the wedge axiom, there exists an element $u_0 \in \mathcal{H}^q_{\mathcal{C}}\big(\mathrm{mor}(-,c)_+ \wedge S^0\big)$ such that $i^*_\alpha(u_0) = \alpha^q$. Applying Corollary 4.11 and Proposition 4.12 to the element u_0, the space W_0 and the functor $\mathcal{H}^q_{\mathcal{C}}$ gives a contravariant $\mathcal{C}$-space $Y_{\mathcal{H}^q_{\mathcal{C}}}$.

The $\mathcal{C}$-spaces $\big(E_{\mathcal{H}_{\mathcal{C}}}(n)(c)\big) \coloneqq Y_{\mathcal{H}^n_{\mathcal{C}}}$ obtained by the previous construction give rise to $\mathcal{C}$-Ω-spectra. To check this, notice that for any object c, the naturality of the transformation associates to the suspension isomorphism $\sigma_X : \mathcal{H}^q_{\mathcal{C}}\big(\mathrm{mor}_{\mathcal{C}}(-,c)\big) \to \mathcal{H}^{q+1}_{\mathcal{C}}\big(\Sigma\mathrm{mor}_{\mathcal{C}}(-,c)\big)$ a natural isomorphism of representable functors

$$\Big[\mathrm{mor}_{\mathcal{C}}(-,c)_+, Y_{\mathcal{H}^q_{\mathcal{C}}}\Big]_{\mathcal{C}} \to \Big[\mathrm{mor}_{\mathcal{C}}(-,c)_+, \Omega Y_{\mathcal{H}^{q+1}_{\mathcal{C}}}\Big]_{\mathcal{C}}.$$

It follows that there exists a weak $\mathcal{C}$-homotopy equivalence $Y_{\mathcal{H}^q_{\mathcal{C}}} \to \Omega Y_{\mathcal{H}^{q+1}_{\mathcal{C}}}$. We pick a choice of such maps for all q and $c \in \mathcal{C}$. This finishes the proof of Theorem 4.1.

Natural Transformations of $\mathcal{C}$-Cohomology Theories

In this section we will analyse the behaviour of the previous construction under natural transformations.

Definition 4.15. Let $F : \mathcal{C} \to \mathcal{D}$ be a a covariant functor between small categories, let $\mathcal{H}^*_{\mathcal{C}}$ and $\mathcal{K}^*_{\mathcal{D}}$ be $\mathcal{C}$- respectively $\mathcal{D}$- cohomology theories defined on contravariant $\mathcal{C}$, respectively $\mathcal{D}$-spaces. Given integer numbers n, k, an operation of type $(F, \mathcal{H}, \mathcal{K}, n, k)$ is a natural transformation $\Theta_X : \mathcal{K}^n_{\mathcal{D}}(F_*X) \longrightarrow \mathcal{H}^k_{\mathcal{C}}(X)$ consisting of natural group homomorphisms, which are compatible with long exact sequences, boundary maps and suspension isomorphisms. In other words, for every $\mathcal{C}$-map $f : X \longrightarrow Y$ the diagram

$$\begin{tikzcd}
\mathcal{H}^k_{\mathcal{C}}(X) & & \mathcal{H}^k_{\mathcal{C}}(Y) \arrow[ll, "f^*"'] \\
\mathcal{K}^n_{\mathcal{D}}(F_*X) \arrow[u, "\Theta_X"] & & \mathcal{K}^n_{\mathcal{D}}(F_*Y) \arrow[ll, "\mathcal{K}^n_{\mathcal{D}}(F_*(f))"] \arrow[u, "\Theta_Y"']
\end{tikzcd}$$

commutes and it is compatible with long exact sequences, boundary operators and suspension isomorphisms.

We will investigate the behaviour of the representing objects with respect to operations. First, we notice that the classifying object for the functor $\mathcal{K}^q_{\mathcal{D}}\big(F_*(\ )\big)$

defined on the category of $\mathcal{C}$-spaces is given by the $\mathcal{C}$-space $Y_{\mathcal{K}^r_{\mathcal{D}}(F_*(\quad))}$. This follows from the fact that universal elements are unique, Proposition 4.12, as well as the adjunctions described in Lemma 2.7.

Lemma 4.16. *Let $\Theta : \mathcal{K}^*_{\mathcal{D}}(F_* \quad) \longrightarrow \mathcal{H}^*_{\mathcal{C}}(\quad)$ be an operation of type $(F, \mathcal{H}, \mathcal{K}, r, r)$. Then there is a cellular map $F_\Theta : F^*Y_{\mathcal{K}^r_{\mathcal{D}}} \longrightarrow Y_{\mathcal{H}^r_{\mathcal{C}}}$, well defined up to $\mathcal{C}$-homotopy inducing Θ.*

Proof. For simplicity, we denote the $\mathcal{C}$-spaces $Y_{\mathcal{H}^r_{\mathcal{C}}}$, respectively $Y_{\mathcal{K}^r_{\mathcal{D}}(F_*(\quad))}$ by $Y_{\mathcal{H}_{\mathcal{C}}}$, respectively $F^*Y_{\mathcal{K}_{\mathcal{D}}}$.

We construct the map inductively on the cell skeleton. Let C be an object in $\mathcal{C}$. The map $F : \operatorname{mor}_{\mathcal{C}}(-, c)_+ \to \operatorname{mor}_{\mathcal{D}}\big(F(-), F(c)\big)_+$ assigning to a morphism ψ in $\mathcal{C}$ the morphism $F(\psi)$ gives a map $F_{\Theta_0} := f^0 : F^*Y^0_{\mathcal{K}_{\mathcal{D}}} \to Y^0_{\mathcal{H}_{\mathcal{C}}}$.

We assume inductively that we constructed natural transformations $f^q := F_{\Theta_q} : F^*Y^q_{\mathcal{K}_{\mathcal{D}}} \longrightarrow Y^q_{\mathcal{H}_{\mathcal{C}}}$ for $q = 1, \dots, n$ such that the diagrams

$$\begin{tikzcd}
\pi_q\big(Y^q_{\mathcal{H}_{\mathcal{C}}}(c)\big) \arrow[r] & \mathcal{H}^r_{\mathcal{C}}\big(\operatorname{mor}_{\mathcal{C}}(-, c)_+ \wedge S^q\big) \\
\pi_q\big(F^*Y^q_{\mathcal{K}_{\mathcal{D}}}(F(c))\big) \arrow[r] \arrow[u, "F_{\Theta_q *}"] & \mathcal{K}^r_{\mathcal{D}}\big(F_*\operatorname{mor}_{\mathcal{C}}(-, c)_+ \wedge S^q\big) \arrow[u, "\Theta"']
\end{tikzcd}$$

commute.

Recall that in the proof of Lemma 4.9 we used the intermediate space

$$X^q_{\mathcal{H}} = Y^{q-1}_{\mathcal{H}_{\mathcal{C}}} \bigvee_{\alpha \in \mathcal{H}^r_{\mathcal{C}}(\operatorname{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q)} \operatorname{mor}_{\mathcal{C}}(-, c)_+ \wedge S^q$$

and group homomorphisms

$$\begin{aligned}
\alpha_{\mathcal{H}} : \pi_q\big(X^q_{\mathcal{H}_{\mathcal{C}}}(c)\big) &\longrightarrow \mathcal{H}^r_{\mathcal{C}}\big(\operatorname{mor}_{\mathcal{C}}(-, c)_+ \wedge S^q\big) \\
\beta_{\mathcal{K}} : \pi_q\big(F^*X^q_{\mathcal{K}_{\mathcal{D}}}\big(F(c)\big)\big) &\longrightarrow \mathcal{K}^r_{\mathcal{D}}\big(F_*\operatorname{mor}_{\mathcal{D}}(-, c)_+ \wedge S^q\big)
\end{aligned}$$

obtained by the q-universality of elements in $\mathcal{H}^r_{\mathcal{C}}\big(Y^q_{\mathcal{H}_{\mathcal{C}}}\big)$, respectively $\mathcal{K}^r_{\mathcal{C}}\big(F_*\big(Y^q_{\mathcal{K}_{\mathcal{C}}}\big)\big)$. We obtained $Y^{q+1}_{\mathcal{H}^r_{\mathcal{C}}}$, respectively $Y^{q+1}_{\mathcal{K}^r_{\mathcal{D}}}$ by attaching cells by means of the maps in the kernel of these homomorphisms. Notice that our inductive hypothesis and the fact that the operation is a group homomorphism imply that $\ker \Theta_{\operatorname{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q} \circ \beta_{\mathcal{K}} \subset \ker \alpha_{\mathcal{K}} \circ f_{q*}$, since the diagram above commutes. Let us define the map f_{q+1} as the dotted arrow in the following diagram

$$\begin{tikzcd}
 & \vee_{\beta_{\mathcal{H}}}\big(F^*\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q\big) \arrow[rr] \arrow[dd, bend right] & & \vee_{\beta_{\mathcal{H}}}\big(F^*\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge D^{q+1}\big) \arrow[dd, bend left] \\
\vee_{\alpha_{\mathcal{H}}}\big(\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q\big) \arrow[ur, "g_q|"] \arrow[rr] \arrow[dd, bend right] & & \vee_{\alpha}\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge D^{q+1} \arrow[ur, "g_q"'] \arrow[dd, bend left] & \\
 & F^*\big(Y^q_{\mathcal{K}_{\mathcal{D}}}\big) \arrow[rr] & & F^*\Big(Y^{q+1}_{\mathcal{K}_{\mathcal{D}}}\Big) \\
Y^q_{\mathcal{H}_{\mathcal{C}}} \arrow[ur, "f_q"] \arrow[rr] & & Y^{q+1}_{\mathcal{H}_{\mathcal{C}}} \arrow[ur, dashed, "f_{q+1}"'] &
\end{tikzcd} .$$

Here, the map g^q maps all wedge factors α to the factor β defined by the homotopy class of the constant map $\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge S^q, \to Y_{\mathcal{H}}$ with image on the basis point. In this factor, the map g^q is defined to be the map $F \wedge \mathrm{id}$: $\mathrm{mor}_{\mathcal{C}}(-,c)_+ \wedge D^{q+1} \to \mathrm{mor}_{\mathcal{D}}\big(F(-),F(c)\big)_+ \wedge D^q$. This finishes the inductive definition of f. ☑

5. Equivariant Cohomology Theories and Natural Transformations

The study of $\mathcal{C}$-spaces and $\mathcal{C}$-cohomology Theories was motivated by equivariant Algebraic Topology, particularly by the examples addressed in Definition 2.10 and Definition 2.9. The notion of $\mathcal{C}$-cohomology theory and the notion of an operation generalize the notion of an induction structure as well as some operations in equivariant cohomology theories.

Equivariant Cohomology Theories.

Definition 5.1. Let X be a pointed space with a base point preserving action of the discrete group G. Recall that a pointed G-CW complex structure on (X,A) consists of a filtration of the G-space $X = \cup_{0\leq n} X_n, X_0 = A$ together with a choice of a G-fixed base point $\{\bullet\} \subset A$ and for which every space is inductively obtained from the previous one by attaching cells in pushout diagrams consisting of pointed maps

$$\begin{tikzcd}
\coprod_i S^{n-1} \wedge G/H_{i_+} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_i D^n \wedge G/H_{i_+} \arrow[r] & X_n
\end{tikzcd}$$

There exist functors $^{-} : G - \mathrm{Spaces}_+ \to \mathrm{Or}(G) - \mathrm{Spaces}_+$, called the fixed point system and $\widehat{} : \mathrm{Or}(G) - \mathrm{Spaces}_+ \to G - \mathrm{Spaces}_+$, called the coalescence functor, between the categories of pointed spaces over the orbit category $\mathrm{Or}(G)$ and the category of pointed G-spaces. They assign to a G-space the contravariant space of Definition 2.10 and to a contravariant $\mathrm{Or}(G)$-space the space $\widehat{X} \coloneqq X \underset{\mathrm{Or}(G)}{\otimes} \nabla$, where ∇ is the covariant $\mathrm{Or}(G)$-space defined as G/H_+ on every orbit G/H with the action of G induced from the left translation on G/H. These functors are adjoint and take pointed G-CW complexes to pointed $\mathrm{Or}(G)$-complexes, giving a bijection between cells of type G/H in Y and pointed cells in the $\mathrm{Or}(G)$-space Y^- based at the object G/H. Compare [4, Theorem 7.4, p. 250] for the unpointed version.

Recall the notion of an equivariant cohomology theory [7].

Definition 5.2. Let G be a group and fix an associative ring with unit R. A G-cohomology theory with values in R-modules is a collection of contravariant functors $\mathcal{H}_G^n$ indexed by the integer numbers $\mathbb{Z}$ from the category of G-CW pairs together with natural transformations $\partial_G^n : \mathcal{H}_G^n(A) \coloneqq \mathcal{H}_G^n(A, \varnothing) \to \mathcal{H}_G^{n+1}(X, A)$, such that the following axioms are satisfied:

(1) If f_0 and f_1 are G-homotopic maps $(X, A) \to (Y, B)$ of G-CW pairs, then $\mathcal{H}_G^n(f_0) = \mathcal{H}_G^n(f_1)$ for all n.

(2) Given a pair (X, A) of G-CW complexes, there is a long exact sequence

$$\cdots \xrightarrow{\mathcal{H}_G^{n-1}(i)} \mathcal{H}_G^{n-1}(A) \xrightarrow{\partial_G^{n-1}} \mathcal{H}_G^n(X, A) \xrightarrow{\mathcal{H}_G^n(j)}$$
$$\mathcal{H}_G^n(X) \xrightarrow{\mathcal{H}_G^n(i)} \mathcal{H}_G^n(A) \xrightarrow{\partial_G^n} \mathcal{H}_G^{n+1}(X, A) \xrightarrow{\mathcal{H}_{n+1}(j)} \cdots$$

where $i : A \to X$ and $j : X \to (X, A)$ are the inclusions.

(3) Let (X, A) be a G-CW pair and $f : A \to B$ be a cellular map. The canonical map $(F, f) : (X, A) \to (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}_G^n\big(X \cup_f B, B\big) \xrightarrow{\cong} \mathcal{H}_G^n(X, A).$$

(4) Let $\{X_i : i \in \mathcal{I}\}$ be a family of G-CW-complexes and denote by $j_i : X_i \to \coprod_{i\in\mathcal{I}} X_i$ the inclusion map. Then the map

$$\Pi_{i\in\mathcal{I}} \mathcal{H}_G^n(j_i) : \mathcal{H}_G^n\left(\coprod_i X_i\right) \xrightarrow{\cong} \Pi_{i\in\mathcal{I}} \mathcal{H}_G^n(X_i)$$

is bijective for each $n \in \mathbb{Z}$.

Let $\alpha : H \to G$ be a group homomorphism and X be a H-CW complex. The induced space $\mathrm{ind}_\alpha X$, is defined to be the G-CW complex given as the quotient space $G \times X$ by the right H-action given by $(g,x)\cdot h = \big(g\alpha(h), h^{-1}x\big)$.

An equivariant cohomology Theory consists of a family of G-Cohomology Theories $\mathcal{H}^*_G$ together with natural group homomorphisms

$$\mathrm{ind}_\alpha : \mathcal{H}^n_G\big(\mathrm{ind}_\alpha(X,A)\big) \longrightarrow \mathcal{H}^n_H(X,A)$$

satisfying the following conditions:

(1) ind_α is an isomorphism whenever $\ker\alpha$ acts freely on X.

(2) For any n, $\partial^n_G \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial^n_G$.

(3) For any group homomorphism $\beta : G \to K$ such that $\ker\beta\circ\alpha$ acts freely on X, one has

$$\mathrm{ind}_{\alpha\circ\beta} = \mathcal{H}^n_K(f_1 \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha) : \mathcal{H}^n_K\big(\mathrm{ind}_{\beta\circ\alpha}(X,A)\big) \to \mathcal{H}^n_H(X,A)$$

where $f_1 : \mathrm{ind}_\beta \mathrm{ind}_\alpha \to \mathrm{ind}_{\beta\circ\alpha}$ is the canonical G-homeomorphism.

(4) For any $n \in \mathbb{Z}$, any $g \in G$, the homomorphism

$$\mathrm{ind}_{c_g:G\to G} : \mathcal{H}^n_G\big(\mathrm{ind}_{c_g:G\to G}(X,A)\big) \to \mathcal{H}^n_G(X,A)$$

agrees with the map $\mathcal{H}^n_G(f_2)$, where $f_2 : (X,A) \to \mathrm{ind}_{c(g):G\to G}(X,A)$ sends x to $\big(1, g^{-1}x\big)$ and $c(g)$ is the conjugation isomorphism in G.

We explain the relation of these notions to the naturality considerations in the previous section.

In [8, Example 1.7, p. 1030], an equivariant cohomology theory is constructed given a contravariant functor $\mathbf{E}$ from the category of small groupoids and injective homomorphisms to the category of Ω-spectra, under the assumption that equivalences of groupoids are sent to weak equivalences of spectra. The idea is the following. Given a G-set S, the transport groupoid $\mathcal{G}^G(S)$ has as objects the elements of S. The set of morphisms from s_0 to s_1 consists of the elements in G which satisfy $gs_0 = s_1$, composition comes from the multiplication in G. By assigning to an homogeneous space G/H the transport groupoid we obtain a covariant functor $\mathrm{Or}(G) \to \mathrm{Groupoids}$. The equivariant cohomology theory with coefficients in $\mathbf{E}$ is defined as

$$H^p_G(X,A,\mathbf{E}) \coloneqq \pi_{-p}\Big(\mathrm{hom}_{\mathrm{Or}(G)}\Big(X^-_+ \underset{A^-}{\cup} \mathrm{cone}A^-_+, \mathbf{E}\circ\mathcal{G}^G\Big)\Big).$$

The construction in Section 4 of a homotopy class of a weak map between spectra realizing an operation defined on cohomology theories over different categories gives a partial converse to this construction.

Corollary 5.3. *Let $\mathcal{H}^*_-$ be an equivariant cohomology theory and let G be a discrete group. Let $\mathcal{H}^*_{\mathrm{Or}(G)}$ be the $\mathrm{Or}(G)$-cohomology theory defined on $\mathrm{Or}(G)$ spaces by applying to a $\mathrm{Or}(G)$ pair (X, A) a cellular approximation $(X', A') \to (X, A)$, followed by the coalescence functor. In symbols,*

$$\mathcal{H}^*_{\mathrm{Or}\,G}(X, A) = \mathcal{H}^*_G\Big(\widehat{(X', A')}\Big).$$

For any $p \in \mathbb{Z}$, the classifying object construction $G \mapsto Y_{\mathcal{H}^p_{\mathrm{Or}(G)}}(G/G)$ sends a group isomorphism to a weak homotopy equivalence.

Proof. Let $\alpha : H \to G$ be a group isomorphism. The induction structure of $\mathcal{H}^*_-$ together with the adjunctions in 2.7 give natural transformations of representable functors

$$\begin{tikzcd}
 & \Big[\alpha_*\big(\mathrm{mor}_{\mathrm{Or}(H)}(-,c)\big)_+ \wedge S^n, Y_{\mathcal{H}^p_{\mathrm{Or}(G)}}\Big]_{\mathrm{Or}(G)} \arrow[dr] & \\
\Big[\mathrm{mor}_{\mathrm{Or}(H)}(-,c)_+ \wedge S^n, \alpha^*\Big(Y_{\mathcal{H}^p_{\mathrm{Or}(G)}}\Big)\Big]_{\mathrm{Or}(H)} \arrow[ur] & & \Big[\mathrm{mor}_{\mathrm{Or}(H)}(-,c)_+ \wedge S^n, Y_{\mathcal{H}^p_{\mathrm{Or}(H)}}\Big]_{\mathrm{Or}\,(H)}
\end{tikzcd}$$

consisting of isomorphisms. Moreover, these can be realized up to homotopy by a $\mathrm{Or}(H)$-map $\alpha^* Y_{\mathcal{H}^p_{\mathrm{Or}(G)}} \to Y_{\mathcal{H}^p_{\mathrm{Or}(H)}}$. On the other hand α induces homeomorphisms of $\mathrm{Or}(H)$-spaces

$$\mathrm{mor}_{\mathrm{Or(H)}}(-, c)_+ \wedge D^r \to \mathrm{mor}_{\mathrm{Or(G)}}\big(\alpha(-), \alpha(c)\big)_+ \wedge D^r$$

which fit into a cellular map $\alpha : Y_{\mathcal{H}^p_{\mathrm{Or}(H)}} \to \alpha^* Y_{\mathcal{H}^p_{\mathrm{Or}(G)}}$ which is seen to be a weak $\mathrm{Or}(H)$-equivalence inverse to the previous map. Evaluation at H/H, respectively G/G gives a weak homotopy equivalence. ☑

***Remark* 5.4.** The construction in Section 4 and the consequence in Corollary 5.3 do not give a functor from the category of small groupoids to the category of spectra and strong maps. All relevant maps, even the described weak equivalence, are only defined up to weak $\mathcal{C}$-homotopy.

Operations in Bredon Cohomology with Local Coefficients

We will now introduce an example of a $\mathcal{C}$-cohomology theory, Bredon cohomology with local coefficients. Bredon cohomology with local coefficients was introduced by Moerdijk and Svensson in [13], and with an equivalent approach by Mukherjee and Pandey [14].

We describe some categories and notations which are relevant to this construction.

Let X be a compactly generated, Hausdorff space. The category of equivariant simplices of X, denoted by $\Delta_G(X)$ has as objects continuous maps $\sigma : G/H \times \Delta^n \to X$, where $\Delta^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \Sigma x_i = 1, x_i \geq 0\}$ is the canonical n-simplex. A morphism in $\Delta_G(X)$ between the objects $\sigma_1 : G/H_1 \times \Delta^n \to X$ and $\sigma_2 : G/H_2 \times \Delta^m \to X$ consists of a pair (φ, α), where $\varphi : G/H_1 \to G/H_2$ is a G-equivariant map, $\alpha : \Delta^m \to \Delta^n$ is a simplicial operator and $\sigma_1 = \sigma_2 \circ (\varphi, \alpha)$.

The equivariant fundamental category of X, $\pi_{\mathrm{Or}(G)}(X)$ is the category where the objects are G-maps $x_H : G/H \to X$ and where a morphism consists of a pair $(\varphi, [H])$ where $\varphi : G/H_1 \to G/H_2$ is a G-map and $[H]$ is the homotopy class of a G-homotopy $H : I \times G/H_1 \to X$ between x_{H_1} and $x_{H_2} \circ \varphi$. Notice the projection functor $p : \Delta_G(X) \to \pi_{\mathrm{Or}(G)}(X)$ given by assigning to a higher dimensional simplex $\Delta^n \to X$ the restriction to the last n-th vertex $G/H \times e^n_n \to X$ in a fixed ordering $e^n_0, \dots, e^n_n$.

A local coefficient system with values in R-modules is a contravariant functor $M : \pi_{\mathrm{Or}(G)}(X) \to R - \mathrm{Mod}$. Given a ring R, a discrete group G and a G-space X, the singular chain complex of X, $C^{\mathrm{sing}}_*(\Delta_G(X))$ is the free $\Delta_G(X)$-chain complex which is given on every object C as the cellular chain complex of the canonical $\Delta_G(X)$-cellular approximation of the constant functor $\{\bullet\}$

Definition 5.5. Let G be a discrete group and X be a G-space. The Bredon cohomology groups of X with coefficients in the local coefficient system M are defined to be the $\Delta_G(X)$-cohomology groups of the cochain complex of the chain complex of natural transformations between the cellular chain complex of the canonical $\Delta_G(X)$-cellular approximation of the constant functor $\{\bullet\}$ and the functor p^*M obtained by composing the functor M with the projection functor $p : \Delta_G(X) \to \pi_{\mathrm{Or}(G)}(X)$. In symbols,

$$H^n_{\mathbb{Z}\Delta_G(X)}(X, M) \coloneqq H^n\big(\hom_{\mathbb{Z}\Delta_G(X)}\big(C^{\mathrm{sing}}_*\big(\Delta_G(X)\big), p^*M\big)\big).$$

Recall the category $\mathcal{E}$ of Definition 2.8.

Definition 5.6. Given a contravariant $\mathrm{Or}(G) \times \mathcal{E}^{op}$-space E, the contravariant $\mathrm{Or}(G)$-space B defined by restricting to the full subcategory $\mathrm{Or}(G) \times s$ is called the basis of E. Notice that there exists a diagram of $\mathrm{Or}(G)$-spaces $p : E \to B$.

Given a local coefficient system M on X, Basu and Sen [2] used equivariant versions of constructions of classifying spaces of crossed complexes to promote the $\mathrm{Or}(G)$-space X^- to an $\mathrm{Or}(G) \times \mathcal{E}^{op}$-space with basis denoted by $\Phi K(\underline{\pi}, 1)$.

The following result is proved in [2, Theorem 6.3, p. 24], and it is an explicit approach to the representability of a particular $\mathcal{C}$-cohomology theory.

Theorem 5.7. *There exists a contravariant functor* $\mathcal{E}_n^{\mathcal{M}} : \mathrm{Or}(G) \times \mathcal{E}^{op} \to \Omega - SPECTRA$ *with basis* $\Phi K(\underline{\pi}, 1)$ *such that given a local coefficient system M on X, the n-th Bredon cohomology groups with coefficients in a local coefficient system M for a G-space X are classified by* $\mathrm{Or}(G) \times \mathcal{E}^{op}$*-maps*

$$\left[X^-_{+\Phi K(\underline{\pi},1)}, \mathcal{E}_n^{\mathcal{M}}\right]_{\mathrm{Or}(G)\times\mathcal{E}^{op}}.$$

The previous theorem has the immediate consequence that Bredon cohomology with local coefficients is an $\mathrm{Or}(G) \times \mathcal{E}^{op}$-cohomology theory. We will examine some natural transformations defined on it.

Given a local coefficient system consisting of $\mathbb{Z}/2$-modules, Steenrod operations $\cup_i : H^*_{\mathbb{Z}\Delta_G(X)}(X, M) \to H^{*+i}_{\mathbb{Z}\Delta_G(X)}(X, M)$ on Bredon cohomology with local coefficients were introduced by Ginot [5, Theorem 4.1, p. 246], and with an alternative approach by Mukherjee-Sen [15]. Steenrod operations induce natural transformations

$$Sq^i : H^*_{\mathbb{Z}\Delta_G(X)}(X, M) \to H^{*+i}_{\mathbb{Z}\Delta_G(X)}(X, M)$$

which satisfy Cartan and Adem relations, generalize cup products, and $Sq^i(f) = 0$ holds whenever $f \in H^m_{\mathbb{Z}\Delta_G(X)}(X, M)$ with $i > m$.

Corollary 5.8. *Let M be a local coefficient system with values on* $\mathbb{Z}/2$*-modules and* $H^*_{\mathbb{Z}\,\mathrm{Or}(G)}(\quad, M)$ *be Bredon cohomology with coefficients in M. The Steenrod square operations* Sq_k *correspond to* $\mathrm{Or}(G)^{op}\times\mathcal{E}$*-homotopy classes of* $\mathrm{Or}(G)\times \mathcal{E}$*- maps*

$$Sq_k \in \left[Y_{H_{\mathbb{Z}\Delta_G(X)^n, M}}, Y_{H_{\mathbb{Z}\Delta_G(X)^{n+k}, M}}\right]_{\mathrm{Or}(G)^{op}\times\mathcal{E}}$$

between the representing objects constructed either in Theorem 4.1 or [2].

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