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Uniform Dimension over Skew PBW Extensions

Dimensión uniforme de las extensiones PBW torcidas

Armando Reyes[a]

Universidad Nacional de Colombia, Bogotá, Colombia

Dedicated to my dear grandfather Luis María

Abstract. The aim of the present paper is to show that, under some conditions, the uniform dimension of a ring R is the same as the uniform dimension of a skew Poincaré-Birkhoff-Witt extension built on R.

Key words and phrases. Non-commutative rings, Filtered and graded rings, PBW extensions, Uniform dimension, Nonsingular modules.

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Resumen. El propósito de este artículo es mostrar que bajo ciertas condiciones, la dimensión uniforme de un anillo R coincide con la dimensión uniforme de una extensión Poincaré-Birkhoff-Witt torcida de R.

Palabras y frases clave. Anillos no conmutativos, anillos filtrados y graduados, extensiones PBW, dimensión uniforme, módulos no singulares.

1. Introduction

A basic tool in the study of Noetherian rings and modules is the uniform dimension (also known as Goldie dimension), noted rudim$(-)$ for the right dimension (similarly ludim$(-)$ for the left dimension). The basic idea of this dimension is that one measures the "size" of a module M by finding out how big a direct sum of nonzero submodules M can contain. For modules over a division ring, uniform dimension is just the usual vector space dimension as defined in linear algebra.

For polynomial rings, Shock in 1972 ([15], Theorem 2.6) proved that if B is a ring having finite left uniform dimension, then the left uniform dimension

[a]Seminario de Álgebra Constructiva - SAC2.

of $B[x]$ is equal to the left uniform dimension of B (see also Goodearl [3], Theorem 3.23). In the case of noncommutative rings, and more specifically skew polynomial rings, we can include (in chronological order) the following works: In 1988, Grzeszczuk [5] proved that if B is a semiprime left Goldie ring equipped with a derivation δ, then the Goldie dimension of $B[y;\delta]$ is equal to the Goldie dimension of B. In fact, he proved that $B[y;\delta]_{B[y;\delta]}$ and B_B have the same uniform dimension if B is right nonsingular, or if B is a $\mathbb{Q}$-algebra with the descending chain condition on right annihilators ([5], Corollary 4). The same year, Quinn ([12], Theorem 15) showed that if B is a $\mathbb{Q}$-algebra and δ is locally nilpotent, then $B[y;\delta]_{B[y;\delta]}$ and B_B have the same uniform dimension. This result cannot hold in general; the classical example is given by $B = \Bbbk[x]/\langle x^2\rangle$ and $\delta = \frac{d}{dx}$, where $\Bbbk$ is a field of characteristic 2, in which case $\mathrm{rudim}(B_B) = 1$ and $\mathrm{rudim}(B[y;\delta]_{B[y;\delta]}) = 2$ ([4], p. 851). In 1995, Matczuk [9] proved that if B is a semiprime left Goldie ring equipped with an automorphism σ and σ-derivation δ, then the Goldie dimension of $B[x;\sigma,\delta]$ is equal to the Goldie dimension of B. In 2005, Leroy and Matczuk [7] generalized this result to the case where σ is an injective endomorphism. A similar remark can be established for the results presented by Mushrub [11] and Sigurdsson [16].

In this paper we present sufficient conditions to guarantee that a ring R and a skew Poincaré Birkhoff Witt extension A built on R have the same uniform dimension. Since skew PBW extensions introduced in [2] are a generalization of PBW extensions, the results established here are more general than the result presented in [1]. In this way this paper continues with the study of several dimensions of skew PBW extensions presented in [8], Section 4, [13] and [14], Chapter 4. The techniques used here are fairly standard and follow the same path as other text on the subject. The results presented are new for skew PBW extensions and all they are similar to others existing in the literature.

The paper is organized as follows. Section 2 contains the definition and some of the properties of the objects we are going to study. In Section 3 we establish an upper bound for the uniform dimension of skew PBW extensions, and in Section 4 we present sufficient conditions under which passing from R to A preserves the dimension. For example, if M is a nonsingular right R-module, or if each nonzero submodule of M contains a nonzero element whose annihilator in R is (Σ,Δ)-invariant, then $M\otimes_R A$ has the same uniform dimension as M. When R is right Noetherian ring and tame as a right module over itself and with prime annihilator ideals under certain conditions of stability, we show that the uniform dimension of both A_A and R_R coincides.

Throughout this paper the rings and algebras are associative with unit, and all modules are unital right modules.

2. Definitions and Elementary Properties

In this section we recall the definition of skew PBW extensions presented in [2] and we also present some key properties of these extensions. The content and proofs of this introductory section can be found in [8], Sections 1 and 2, or [14], Chapter 1. From Definition 2.1 we can see that skew PBW extensions are a generalization of PBW extensions defined by Bell and Goodearl in [1] (see [2] for more details).

Definition 2.1 ([2] Definition 1)**.** Let R and A be rings. We say that A *is a skew* PBW *extension of* R (also called a σ-PBW *extension of* R) if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exist elements $x_1, \ldots, x_n \in A \setminus R$ such that A is a left free R-module, with basis the basic elements

$$\mathrm{Mon}(A) := \{x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} : \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1}$$

(iv) For any elements $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \tag{2}$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$.

***Remark* 2.2** ([2], Remark 2)**.**

(i) Since $\mathrm{Mon}(A)$ is a left R-basis of A, the elements $c_{i,r}$ and $c_{i,j}$ in Definition 2.1 are unique.

(ii) In Definition 2.1 (iv), $c_{i,i} = 1$. This follows from $x_i^2 - c_{i,i}\, x_i^2 = s_0 + s_1 x_1 + \cdots + s_n x_n$, with $s_i \in R$, which implies $1 - c_{i,i} = 0 = s_i$.

(iii) Let $i < j$. By (2) there exist elements $c_{j,i}, c_{i,j} \in R$ such that $x_i x_j - c_{j,i} x_j x_i \in R + R x_1 + \cdots + R x_n$ and $x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n$, and hence $1 = c_{j,i} c_{i,j}$, that is, for each $1 \leq i < j \leq n$, $c_{i,j}$ has a left inverse and $c_{j,i}$ has a right inverse. In general, the elements $c_{i,j}$ are not two sided invertible. For instance, $x_1 x_2 = c_{2,1} x_2 x_1 + p = c_{21}(c_{1,2} x_1 x_2 + q) + p$, where $p, q \in R + R x_1 + \cdots + R x_n$, so $1 = c_{2,1} c_{1,2}$, since $x_1 x_2$ is a basic element of $\mathrm{Mon}(A)$. Now, $x_2 x_1 = c_{1,2} x_1 x_2 + q = c_{1,2}(c_{2,1} x_2 x_1 + p) + q$, but we cannot conclude that $c_{12} c_{21} = 1$ because $x_2 x_1$ is not a basic element of $\mathrm{Mon}(A)$ (we recall that $\mathrm{Mon}(A)$ consists of the standard monomials).

(iv) Each element $f \in A \setminus \{0\}$ has a unique representation as $f = c_1 X_1 + \cdots + c_t X_t$, with $c_i \in R \setminus \{0\}$ and $X_i \in \mathrm{Mon}(A)$ for $1 \leq i \leq t$.

The next proposition justifies the notation and the name of the skew PBW extensions.

Proposition 2.3 ([2], Proposition 3)**.** *Let A be a skew PBW extension of R. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r)x_i + \delta_i(r), \qquad \textit{for each} \quad r \in R. \tag{3}$$

A particular case of skew PBW extension is considered when derivations δ_i are zero for all i. A remarkable case is presented when all endomorphisms σ_i are isomorphisms. These observations are formulated in the next definition.

Definition 2.4 ([2], Definition 4)**.** Let A be a skew PBW extension of R.

(a) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.1 are replaced by

(iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that
$$x_i r = c_{i,r} x_i; \tag{4}$$

(iv') for any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that
$$x_j x_i = c_{i,j} x_i x_j. \tag{5}$$

(b) A is called *bijective* if σ_i is bijective for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Example 2.5. A considerable number of examples of skew PBW extensions are presented in [8], Section 3 and [14], Chapter 2. These examples include PBW extensions and many other algebras of interest for modern mathematical physicists which are not PBW extensions. Some of these algebras are group rings of polycyclic-by-finite groups, Ore algebras, operator algebras, diffusion algebras, quantum algebras, quadratic algebras in 3 variables, Clifford algebras among many others.

Definition 2.6 ([2], Definition 6)**.** Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 2.3.

(i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^\alpha := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) For $X = x^\alpha \in \text{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ will denote a total order defined on $\text{Mon}(A)$ (a total order on $\mathbb{N}_0^n$). For an element $x^\alpha \in \text{Mon}(A)$, $\text{Mon}(x^\alpha) := \alpha \in \mathbb{N}_0^n$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. If $f = c_1 X_1 + \cdots + c_t X_t \in A$, $c_i \in R \setminus \{0\}$, with

$X_1 \succ \cdots \succ X_t$, then $\mathrm{lm}(f) := X_1$ is the *leading monomial* of f, $\mathrm{lc}(f) := c_1$ is the *leading coefficient* of f, $\mathrm{lt}(f) := c_1X_1$ is the *leading term* of f, $\exp(f) := \exp(X_1)$ is the *order* of f, and $E(f) := \{\exp(X_i) : 1 \leq i \leq t\}$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. For a detailed description of monomial orders in skew PBW extensions, see [2, Section 3].

(iii) If f is an element as in Remark 2.2 (iv), then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{t}$.

Skew PBW extensions can be characterized as the following theorem shows.

Theorem 2.7 ([2], Theorem 7)**.** *Let A be a polynomial ring over R with respect to $\{x_1, \dots, x_n\}$. A is a skew PBW extension of R if and only if the following conditions are satisfied:*

(i) *for each $x^{\alpha} \in \mathrm{Mon}(A)$ and every nonzero element r of R, there exist unique elements $r_{\alpha} := \sigma^{\alpha}(r) \in R \setminus \{0\}$, $p_{\alpha,r} \in A$ such that*

$$x^{\alpha}r = r_{\alpha}x^{\alpha} + p_{\alpha,r}, \tag{6}$$

where $p_{\alpha,r} = 0$ or $\deg(p_{\alpha,r}) < |\alpha|$ if $p_{\alpha,r} \neq 0$. If r is left invertible, so is r_{α}.

(ii) *For each $x^{\alpha}, x^{\beta} \in \mathrm{Mon}(A)$ there exist unique elements $c_{\alpha,\beta} \in R$ and $p_{\alpha,\beta} \in A$ such that*

$$x^{\alpha}x^{\beta} = c_{\alpha,\beta}x^{\alpha+\beta} + p_{\alpha,\beta}, \tag{7}$$

where $c_{\alpha,\beta}$ is left invertible, $p_{\alpha,\beta} = 0$ or $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ if $p_{\alpha,\beta} \neq 0$.

In the noncommutative setting an *integral domain*, briefly called a *domain*, is defined as a ring in which the product of any two nonzero elements is nonzero. With this in mind, if A is a skew PBW extension of a domain R, then so is A ([8, Proposition 4.1]).

Skew PBW extensions are filtered rings. We recall the definition of these rings.

Definition 2.8. A *filtered ring* is a ring B with a family $FB = \{F_nB : n \in \mathbb{Z}\}$ of additive subgroups of B where we have the ascending chain $\cdots \subset F_{n-1}B \subset F_nB \subset \cdots$ such that $1 \in F_0B$ and $F_nBF_mB \subseteq F_{n+m}B$ for all $n, m \in \mathbb{Z}$. The filtration FB is called *separated* if $\bigcap_{n\in\mathbb{Z}} F_nB = 0$ and *exhaustive* if $\bigcup_{n\in\mathbb{Z}} F_nB = B$.

From a filtered ring B it is possible to construct its associated graded ring $G(B)$ which is known in the literature as the *associated graded ring* of B.

The first key theorem computes the graduation of a general skew PBW extension of a ring R.

Theorem 2.9 ([8], Theorem 2.2). *Let A be an arbitrary skew PBW extension of R. Then, A is a filtered ring with filtration given by*

$$F_m A := \begin{cases} R, & \text{if} \quad m = 0; \\ \{f \in A : \deg(f) \leq m\}, & \text{if} \quad m \geq 1. \end{cases} \tag{8}$$

and the corresponding graded ring $G(A)$ is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then $G(A)$ is a quasi-commutative bijective skew PBW extension of R.

Next we recall the Hilbert's Basis theorem for skew PBW extensions.

Theorem 2.10 ([8], Corollary 2.4). *Let A be a bijective skew PBW extension of R. If R is a left (right) Noetherian ring, then A is also a left (right) Noetherian ring.*

The next theorem is also very useful in the following section.

Proposition 2.11. *If A is a bijective skew PBW extension of a prime ring R, then A is also a prime ring.*

Proof. Theorem 2.9 shows that $G(A)$ is a quasi-commutative skew PBW extension of R, and by assumption $G(A)$ is also bijective. By [8, Theorem 2.3], we know that $G(A)$ is isomorphic to an iterated skew polynomial ring $R[z_1; \theta_1] \cdots [z_n; \theta_n]$ where θ_i is bijective for $1 \leq i \leq n$. The result follows from [10, Theorem 1.2.9 and Proposition 1.6.6]. ☑

3. Uniform Dimension over Skew *PBW* Extensions I

In this section we establish a relation between the uniform dimensions of a ring R and a skew PBW extension A built on R. If A is a bijective skew PBW extension of a right Noetherian domain R we will show that rudim $A =$ rudim $R = 1$. In a more general case, we prove that if A is a bijective skew PBW extension of a prime right Goldie ring R, then the uniform dimension of A is bounded by the uniform dimension of R.

Definition 3.1. Let B be a ring. If N is a submodule of a right B-module M such that, for all nonzero submodules X of M, one has $N \cap X \neq 0$, then N is an *essential submodule* of M, and M is an *essential extension* of N. We write $N \triangleleft_e M$.

A module U is *uniform* if $U \neq 0$ and each nonzero submodule of U is an essential submodule. This is equivalent to U not containing a direct sum of nonzero submodules. For example, if B is an integral domain, then B_B is uniform if and only if B is a right Ore domain. We recall that a module M is said to have *finite uniform dimension* if it contains no infinite direct sum of nonzero submodules. This is true of any uniform module and of any Noetherian module. Note that a module with Krull dimension has finite uniform dimension ([10, Lemma 6.2.6]). Because bijective skew PBW extensions have Krull dimension ([8, Section 4]) these extensions have uniform dimension.

Since a right Noetherian domain has right uniform dimension 1, Theorem 2.10 and [8, Proposition 4.1], yield the following proposition.

Proposition 3.2. *If A is a bijective skew PBW extension of a right Noetherian domain R, then the uniform dimension of A is 1, that is,* $\operatorname{rudim} A = 1$.

A more general result than Proposition 3.2 is established in Theorem 3.5.

Proposition 3.3 ([7], Theorem 3.4)**.** *If B is a semiprime right Goldie ring and σ is injective, then the Ore extension $B[x; \sigma, \delta]$ is also semiprime right Goldie and both rings have the same right uniform dimension.*

In order to determine an upper bound for a bijective skew PBW extension we need the following lemma. We thank professor Huishi Li for a personal communication with a simplification of our original proof. Before, we recall that if B is a filtered ring with filtration $FB = \{F_nB\}_{n\in\mathbb{Z}}$ and M is a right B-module, the induced filtration $FM = \{F_nM\}_{n\in\mathbb{Z}}$ on M from FB is given by $F_0M \coloneqq M_0 = \{X\}_{F_0B}$, and $F_nM \coloneqq M_0F_nB$, where X is any system of generators of M.

Lemma 3.4. *Let B be a filtered ring and M a right B-module. Suppose that the induced filtration FM on M is separated and exhaustive. If* $\operatorname{rudim}\big(Gr(M)\big) =$ *s, then* $\operatorname{rudim}(M) \leq s$. *In particular, if B is filtered with separated and exhaustive filtration, then* $\operatorname{rudim}\ B \leq \operatorname{rudim}\ G(B)$.

Proof. Let B be a filtered ring with filtration $FB = \{F_nB\}_{n\in\mathbb{Z}}$. Consider $G(B) = \bigoplus_{n\in\mathbb{Z}} G(B)_n$, the associated graded ring of B, where we know that $G(B)_n = F_nB/F_{n-1}B$. Note that every B-module M can be equipped with a $\mathbb{Z}$-filtration FM such that it is turned into a filtered B-module. Suppose that $N = \bigoplus_{i\in I} N_i$ is a direct sum of nonzero submodules of M. Considering the filtration FN_i of each N_i induced by FM, i.e., $F_nN_i = N_i \cap F_nM$, $n \in \mathbb{Z}$. We define the filtration FN of N by putting $F_nN = \bigoplus_{i\in I} F_nN_i, n \in \mathbb{Z}$, or equivalently, $F_nN = N \cap F_nM$, $n \in \mathbb{Z}$.

Since $G(N)_n = \frac{F_nN}{F_{n-1}N} = \frac{\bigoplus_{i\in I} F_nN_i}{\bigoplus_{i\in I} F_{n-1}N_i} = \bigoplus_{i\in I} \frac{F_nN_i}{F_{n-1}N_i} = \bigoplus_{i\in I} G(N_i)_n$ we get

$$G(N) = \bigoplus_{n\in\mathbb{Z}} G(N)_n = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{i\in I} G(N_i)_n = \bigoplus_{i\in I} \bigoplus_{n\in\mathbb{Z}} G(N_i)_n = \bigoplus_{i\in I} G(N_i).$$

For elements $r \in G(R)_n$ and $y \in G(N_i)_m$ we define $(r + F_{n-1}R)(y + F_{m-1}N_i) = ry + F_{n+m-1}$ and thus $G(N_i)$ is a graded submodule of $G(M)$ which gives rise to a direct sum of graded submodules of $G(M)$. If the filtration FM is separated and exhaustive, then $G(N_i) = 0$ if and only if $N_i = 0$. The result follows from [10, Theorem 2.2.9]. ☑

Theorem 3.5. *Let R be a prime right Goldie ring. If A is a bijective skew PBW extension of R, then uniform dimension of A is less or equal than uniform dimension of R.*

Proof. By Lemma 3.4 we obtain rudim $A \leq$ rudim $G(A)$. Theorem 2.11 and Proposition 3.3 (this last says that the uniform dimension is preserved by iterated polynomial rings of automorphism type), imply that rudim $G(A) =$ rudim R. ☑

3.1. Uniform Dimension over Skew Quantum Polynomials

In this section we compute the uniform dimension of skew quantum polynomials introduced in [8].

Definition 3.6 ([8], Example 3.2)**.** Let R be a ring with a fixed matrix of parameters $\mathbf{q} := [q_{ij}] \in M_n(R)$, $n \geq 2$, such that $q_{ii} = 1 = q_{ij}q_{ji} = q_{ji}q_{ij}$ for every $1 \leq i, j \leq n$, and suppose that automorphisms $\sigma_1, \ldots, \sigma_n$ of R are also given. The ring of *skew quantum polynomials over* R, denoted by $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big]$ or $Q^{r,n}_{\mathbf{q},\sigma}(R)$ is defined as the ring satisfying the relations:

(i) $R \subseteq R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big]$;

(ii) $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big]$ is a free left R-module with basis

$$\big\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} : \alpha_i \in \mathbb{Z} \text{ for } 1 \leq i \leq r \text{ and } \alpha_i \in \mathbb{N} \text{ for } r+1 \leq i \leq n\big\}; \quad (9)$$

(iii) the variables $x_1, \ldots, x_n$ satisfy the defining relations

$$x_i x_i^{-1} = 1 = x_i^{-1} x_i, \qquad 1 \leq i \leq r, \quad (10)$$
$$x_j x_i = \sigma_j(x_i) x_j = q_{ij} x_i x_j, \qquad 1 \leq i, j \leq n, \quad (11)$$
$$x_j r = \sigma_j(r) x_j, \qquad r \in R, \quad 1 \leq j \leq n. \quad (12)$$

***Remark* 3.7.** $R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big]$ can be viewed as a localization of a skew PBW extension. For the quasi-commutative bijective skew PBW extension $A := \sigma(R)\langle x_1, \ldots, x_n\rangle$, with $x_i r = \sigma_i(r) x_i$ and $x_j x_i = q_{ij} x_i x_j$, $1 \leq i, j \leq n$. If we set $S := \{r x^{\alpha} : r \in R^*, \ x^{\alpha} \in \text{Mon}\{x_1, \ldots, x_r\}\}$, then S is a multiplicative subset of A and we have the isomorphism $S^{-1}A \cong R_{\mathbf{q},\sigma}\big[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\big]$. See [8, Example 3.2] or [13, Remark 21], for more details.

Examples 3.8. Particular examples of skew polynomial rings include *quantum polynomials*, *algebra of skew quantum polynomials*, *algebra of quantum polynomials*, the *n-multiparametric skew quantum space*, *n-multiparametric skew quantum torus*, *skew Laurent polynomial ring*, *n-multiparametric skew quantum torus*, etc. For a detailed description of these rings and algebras, see [8, Example 3.2] or [13, Remark 22].

Lemma 3.9 ([10], Lemma 2.2.12)**.** *Let S be a left Ore set of regular elements of a ring B. Then* $\text{rudim}_S\ B = \text{rudim}\ B$.

Proposition 3.10. *If R is a right Noetherian domain, then* $\text{ludim}\ Q^{r,n}_{\boldsymbol{q},\sigma}(R) = 1$.

Proof. The assertion follows from Remark 3.7, Proposition 3.2 and Lemma 3.9. ☑

Proposition 3.11. *If R is a prime right Goldie ring, then* $\text{rudim}\ Q^{r,n}_{\boldsymbol{q},\sigma}(R) \leq \text{rudim}\ R$.

Proof. The result follows from Remark 3.7, Theorem 3.5 and Lemma 3.9. ☑

4. Uniform Dimension over Skew *PBW* Extensions II

In this section we establish sufficient conditions under which passing from R to A preserves the uniform dimension for A a bijective skew PBW extension of R.

Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of a ring R. By Proposition 2.3 we know that $x_i r - \sigma_i(r)x_i = \delta_i(r)$ for all $r \in R$, where σ is an injective endomorphism of R and δ_i is a σ_i-derivation of R for each $1 \leq i \leq n$. Let $\Sigma \coloneqq \{\sigma_1, \dots, \sigma_n\}$ and $\Delta \coloneqq \{\delta_1, \dots, \delta_n\}$. We say that the pair (Σ, Δ) is induced by the variables $x_1, \dots, x_n$. If I is an ideal of R, I is called Σ-invariant (Δ-invariant) if it is invariant under each injective endomorphism (σ-derivation) of Σ (Δ), that is, $\sigma_i(I) \subseteq I$ ($\delta_i(I) \subseteq I$) for $1 \leq i \leq n$. If I is both Σ and Δ-invariant ideal we say that I is (Σ, Δ)-invariant. We consider a (Σ, Δ)-invariant ideal I of R to be (Σ, Δ)-prime if whenever a product of two (Σ, Δ)-invariant ideals is contained in I, one of these ideals is contained in I. R is a (Σ, Δ)-prime ring if the ideal 0 is (Σ, Δ)-prime.

The next proposition is very useful for computing uniform dimension of skew PBW extensions.

Proposition 4.1. *Let R be a right Noetherian ring and let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a bijective skew PBW extension of R. If I is a nonzero (Σ, Δ)-invariant ideal of R then $IA = AI$ is an ideal of A with $IA \cap R = I$, R/I embeds in A/IA and A/IA is a skew PBW extension of R/I.*

Proof. Since I is a (Σ, Δ)-invariant ideal of R it follows that $IA = AI$ is an ideal of A with $IA \cap R = I$. Let us see that A/IA is a skew PBW extension of R/I.

(i) It is clear that $R/I \subseteq A/IA$.

(ii) It is also clear that A/IA is a left R/I-module with generating set $\text{Mon}(A/IA)$. Next we show that A/IA is a left free R/I-module. Consider the expression $\overline{r_1}\ \widetilde{X_1} + \cdots + \ \overline{r_n}\ \widetilde{X_n} = 0 + IA$ where $X_i \in \text{Mon}(A)$ for each i. Let us see that $\overline{r_i} = 0 + I$ for each i. By definition above we have $\widetilde{r_1 X_1} + \cdots + \widetilde{r_n X_n} = 0 + IA$, that is $r_1X_1 + \cdots + r_nX_n \in IA$. Since A is a left free R-module, by order conditions on X_i using notation in Definition 2.6 we can write

$$r_1X_1 + \cdots + r_nX_n = m_1X_1 + \cdots + m_nX_n, \quad m_i \in I, \quad i = 1, \dots, n$$

or, equivalently, $(r_1 - m_1)X_1 + \cdots + (r_n - m_n)X_n = 0$. Thus we obtain that $r_i = m_i$ for all i which implies that $r_i \in I$ and thus $\overline{r_i} = 0 + I$ for $i = 1, \dots, n$. Therefore A/IA is a left free R/I-module.

(iii) Let $\overline{r} \neq 0 + I$. We have $\widetilde{x_i}\widetilde{r} = \widetilde{x_i r} \neq 0 + AI$ since $r \notin I$. Then $x_i r \notin IA$ for each i. By Proposition 2.3 we know that $x_i r = c_{i,r}x_i + \delta_i(r)$ for all $r \in R$ and each i. Since R is left Noetherian, for every $\sigma \in \Sigma$ we obtain $I = \sigma(I)$. Then, if $r \notin I$ it follows that $c_{i,r} = \sigma_i(r) \notin I$. In this way $c_{i,r}x_i \notin IA$ whence $\delta_i(r) \notin IA$ which yields $\delta_i(r) \notin I$ for $1 \leq i \leq n$. Therefore we consider $\widetilde{x_i}\overline{r} = \overline{c_{i,r}}\widetilde{x_i} + \overline{\delta_i(r)}$, $i = 1, \dots, n$. Since $\text{Mon}(A/IA)$ is a R/I basis of A/IA then $\overline{c_{i,r}}$ is unique (Remark 2.2).

(iv) Note that $\widetilde{x_j}\widetilde{x_i} \neq 0 + IA$ since $x_jx_i \notin IA$ for $1 \leq i < j \leq n$. By assumption, the elements $c_{i,j}$ are left invertible in R which implies that $c_{i,j} \notin I$ and thus $c_{i,j}x_ix_j \notin IA$ for $1 \leq i < j \leq n$. Hence $x_jx_i - c_{i,j}x_ix_j = \sum_{t=1}^{n} r_tx_t \notin IA$, where $r_t \in R$. Since A is a left free R-module, there exists $j \in \{1, \dots, n\}$ with $r_j \notin I$ and thus $r_jx_j \notin IA$. Thus $\sum_{t\neq j}^{n} r_tx_t \notin IA$. Continuing this way we can see that $r_t \notin I$ for all $t = 1, \dots, n$, and we obtain the equality $\widetilde{x_j}\widetilde{x_i} = \overline{c_{ij}}\widetilde{x_i}\widetilde{x_j} + \sum_{t=1}^{n} \overline{r_t}\widetilde{x_t}$, where $\overline{c_{i,j}} \neq 0 + I, \widetilde{x_i}\widetilde{x_j} \neq 0 + IA$ and $\overline{r_t} \neq 0 + I$ for all $1 \leq i < j \leq n$ and $t = 1, \dots, n$, respectively. Since $\text{Mon}(A/IA)$ is a R/I basis of A/IA the elements $\overline{c_{i,j}}$ are unique (see Remark 2.2).

In this way A/IA is a skew PBW extension of R/I. We keep the variables $x_1, \dots, x_n$ of extension A of the extension A/IA hoping that this will not cause confusion. ☑

If M is a right R-module, and T is a nonzero A-submodule of $M \otimes_R A$, since ${}_RA$ is free, whence faithfully flat, given any right R-modules $N \leq M$, we may identify $N \otimes_R A$ with its image in $M \otimes_R A$. The module $M \otimes_R A$ is called the *induced module*. Observe that $M \otimes_R A$ is, as an abelian group, the direct sum of the subgroups $M \otimes X_i$ for each $X_i \in \mathrm{Mon}(A)$. In this way, any nonzero element $f \in M \otimes_R A$ may be uniquely expressed in the form

$$f = (m_0 \otimes 1) + (m_1 \otimes X_1) + \cdots + (m_t \otimes X_t) \tag{13}$$

where $m_i \in M$ for each i, $m_t \neq 0$, and $\exp(X_i) \prec \exp(X_t)$, $1 \leq i \leq t-1$. We shall usually abbreviate such an expression to

$$f = m_0 + m_1 X_1 + \cdots + m_t X_t. \tag{14}$$

Definition 4.2. A B-module M is a *rational extension* of a submodule N, denoted $N \leq_r M$, provided that $\mathrm{Hom}_B(L/N, M) = 0$ for any submodule L of M that contains N. Equivalently, if these are right modules, $N \leq_r M$ if and only if whenever $x, y \in M$ with $x \neq 0$, there exists $r \in R$ such that $xr \neq 0$ and $yr \in N$ ([3, Proposition 2.25]).

Lemma 4.3. *Let A be a bijective skew PBW extension of a ring R. If $N \leq_r M$ are right R-modules, then $N \otimes_R A \leq_r M \otimes_R A$ as R-modules and hence also as A-modules.*

Proof. Let $x, y \in M \otimes_R A$ with $x \neq 0$. Consider the elements

$$x = (x_0 \otimes 1) + (x_1 \otimes X_1) + (x_2 \otimes X_2) + \cdots + (x_t \otimes X_t) \tag{15}$$

and

$$y = (y_0 \otimes 1) + (y_1 \otimes X_1') + (y_2 \otimes X_2') + \cdots + (y_s \otimes X_s') \tag{16}$$

where $x_i, y_j \in M$, $x_t, y_s \neq 0$, $\exp(x) \coloneqq \exp(X_t)$, and $\exp(y) \coloneqq \exp(X_s')$. For $k = s, s-1, \dots, 0$, the idea is to show that there exists $r_k \in R$ such that $x_t r_k \neq 0$ and

$$yr_k \in (M \otimes 1) + (M \otimes X_1) + \cdots + (M \otimes X_{k-1}) + (N \otimes X_k') + \cdots + (N \otimes X_s').$$

With this in mind, since $N \leq_r M$ there exists $r_s \in R$ such that $x_t r_s \neq 0$ and $y_s r_s \in N$. Because A is bijective, let $r_s' \coloneqq \sigma^{-\exp(X_s')}(r_s)$. Following notation (14), Theorem 2.7 (i) yields

$$
\begin{aligned}
yr'_s &= y_0r'_s + y_1\left[\sigma^{\exp(X'_1)}(r'_s)X'_1 + p_{\exp(X'_1),r'_s}\right] + \\
&\qquad y_2\left[\sigma^{\exp(X'_2)}(r'_s)X'_2 + p_{\exp(X'_2),r'_s}\right] + \cdots + \\
&\qquad\qquad y_s\left[\sigma^{\exp(X'_s)}\left(\sigma^{-\exp(X'_s)}(r_s)\right)X'_s + p_{\exp(X'_s),r'_s}\right] \\
&= y_0r'_s + y_1\left[\sigma^{\exp(X'_1)}(r'_s)X'_1 + p_{\exp(X'_1),r'_s}\right] + \\
&\qquad y_2\left[\sigma^{\exp(X'_2)}(r'_s)X'_2 + p_{\exp(X'_2),r'_s}\right] + \cdots + \\
&\qquad\qquad y_s[r_sX'_s + p_{\exp(X'_s),r'_s}] \\
&= y_0r'_s + y_1\sigma^{\exp(X'_1)}(r'_s)X'_1 + y_1p_{\exp(X'_1),r'_s} + \\
&\qquad y_2\sigma^{\exp(X'_2)}(r'_s)X'_2 + y_2p_{\exp(X'_2),r'_s} + \cdots + y_sr_sX'_s + y_sp_{\exp(X'_s),r'_s}
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
yr'_s = y_0r'_s + y_1\sigma^{\exp(X'_1)}(r'_s)X'_1 + y_2\sigma^{\exp(X'_2)}(r'_s)X'_2 + \cdots + \\
y_sr_sX'_s + \sum_{l=1}^{s} y_lp_{\exp(X'_l),r'_s} \qquad (17)
\end{aligned}
$$

with $p_{\exp(X'_l),r'_s} \in A$ for all $l = 1, \dots, t$, and $p_{\exp(X'_l),r'_s} = 0$, or $\deg\left(p_{\exp(X'_l),r'_s}\right) < \left|\exp(X'_l)\right|$ if $p_{\exp(X'_l),r'_s} \neq 0$. For every l, consider $p_{\exp(X'_l),r'_s} := d_{l,0} + d_{l,1}X'_{l,1} + \cdots + d_{l,h(l)}X'_{l,h(l)}$, with $\exp\left(p_{\exp(X'_l),r'_s}\right) := \exp\left(X'_{l,h(l)}\right)$, and the d_l's are elements of R, the X_l's are basic elements of $\mathrm{Mon}(A)$, and the value $h(l)$ depends of the polynomial l. Then

$$
\sum_{l=1}^{s} y_lp_{\exp(X'_l),r'_s} = \sum_{l=1}^{s}\left[y_ld_{l,0} + y_ld_{l,1}X'_{l,1} + \cdots + y_ld_{l,h(l)}X'_{l,h(l)}\right].
$$

In this way, from (17)

$$
\begin{aligned}
yr'_s &= y_0r'_s + y_1\sigma^{\exp(X'_1)}(r'_s)X'_1 + y_2\sigma^{\exp(X'_2)}(r'_s)X'_2 + \cdots + y_sr_sX'_s + \\
&\qquad \sum_{l=1}^{s}\left[y_ld_{l,0} + y_ld_{l,1}X'_{l,1} + \cdots + y_ld_{l,h(l)}X'_{l,h(l)}\right] \\
&= \left(y_0r'_s + \sum_{l=1}^{s} y_ld_{l,0}\right) + y_1\sigma^{\exp(X'_1)}(r'_s)X'_1 + y_2\sigma^{\exp(X'_2)}(r'_s)X'_2 + \\
&\qquad \cdots + y_sr_sX'_s + \sum_{l=1}^{s}\left[y_ld_{l,1}X'_{l,1} + \cdots + y_ld_{l,h(l)}X'_{l,h(l)}\right].
\end{aligned}
$$

This shows that for the element yr'_s we have the sets of basic monomials given by $\left\{X'_1, X'_2, \dots, X'_s\right\}$, $\left\{X'_{1,1}, X'_{1,2}, \dots, X'_{1,h(1)}\right\}$, $\left\{X'_{2,1}, X'_{2,2}, \dots, X'_{2,h(2)}\right\}$,

$\ldots$, $\{X'_{s,1}, X'_{s,2}, \ldots, X'_{s,h(s)}\}$. Of course, these sets are not necessarily disjoint (note that $\exp(X'_s)$ is greater than others basic elements of yr'_s). If we consider the union

$$\{X'_1, X'_2, \ldots, X'_s\} \cup \bigcup_{l=1}^{s} \{X'_{l,1}, X'_{l,2}, \ldots, X'_{l,h(l)}\}$$

after suppressing possible repetitions of basic monomials, we have a finite number of monomials $X'_1, \ldots, X'_{v-1}, X'_s$, say, if no confusion arises with (16). So, from the last expression for yr'_s above, we obtain

$$yr_s \in (M \otimes 1) + \cdots + (M \otimes X'_{v-1}) + (N \otimes X'_s).$$

Let $0 < k \leq s$. Suppose that there exists $r_k \in R$ which satisfies the required properties. Consider the expression

$$yr_k = (z_0 \otimes 1) + (z_1 \otimes X'_1) + \cdots + (z_s \otimes X'_s),$$

with $z_0, \ldots, z_{k-1} \in M$ and $z_k, \ldots, z_s \in N$. There exists $p \in R$ such that $x_t r_k p \neq 0$ and $z_{k-1}p \in N$. Therefore the element $r_{k-1} = r_k p$ has the required properties. In this way we complete the inductive step. Then $x_t r_0 \neq 0$ which implies $xr_0 \neq 0$ and $yr_0 \in N \otimes_R A$. We conclude that $N \otimes_R A \leq_r M \otimes_R A$ as R-modules and it follows that $N \otimes_R A \leq_R M \otimes_R A$. ☑

***Remark* 4.4.** In the proof of Lemma 4.3 we assume that the skew PBW extension is bijective. Nevertheless, we only used the fact that the injective endomorphisms σ of Proposition 2.3 are bijective, that is, we do not require that the elements $c_{i,j}$ are invertible.

For the next lemma consider a bijective skew PBW extension A of a ring R, M a right R-module, and T a nonzero A-submodule of $M \otimes_R A$.

Lemma 4.5. *If f is a nonzero element of T of minimal monomial order* $\exp(X_t) = \alpha_t$ *among all elements of T (f is expressed as in* (13)*), then* $\sigma^{-\alpha_t}\big(\mathrm{rann}_R\left(\mathrm{lc}(f)\right)\big)A = \mathrm{rann}_A(f)$*. Thus* $fA \cong \mathrm{lc}(f)R \otimes_R A$ *as right A-modules.*

***Proof*.** Consider f a nonzero element of T of minimal monomial order. Following the notation (14), we write $f = m_0 + m_1X_1 + \cdots + m_tX_t$ where $m_i \in M$, $m_t \neq 0$, $X_j \in \mathrm{Mon}(A)$ and $\exp(X_j) \prec \exp(X_t) = \alpha_t$ for all $1 \leq j \leq t-1$. By definition of the right annihilator, $\mathrm{rann}_R\left(\mathrm{lc}(f)\right) = \{r \in R : m_t r = 0\}$. For $r \in R$, consider the element fr. Theorem 2.7 establishes that

$$fr = m_0 r + m_1 X_1 r + \cdots + m_t\big(\sigma^{\alpha_t}(r)X_t + p_{\alpha_t,r}\big),$$

where $p_{\alpha_t,r} = 0$ or $\deg(p_{\alpha_t,r}) < \deg(X_t)$ if $p_{\alpha_t,r} \neq 0$. If $r \in \sigma^{-\alpha_t}\big(\mathrm{rann}_R\left(\mathrm{lc}(f)\right)\big)$, then $\sigma^{\alpha_t}(r) \in \mathrm{rann}_R\left(\mathrm{lc}(f)\right)$ which yields $\deg(fr) <$

$\deg(X_t)$. Because $fr \in T$, then $fr = 0$. Thus, $f\sigma^{-\alpha_t}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big) = 0$ and $f\sigma^{-\alpha_t}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big)A = 0$. Hence $\sigma^{-\alpha_t}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big)A \subseteq \operatorname{rann}_A(f)$.

Let us see now that $\operatorname{rann}_A(f) \subseteq \sigma^{-1}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big)A$. Let $u = r_0 + r_1Y_1 + \cdots + r_kY_k$ an element of $\operatorname{rann}_A(f)$. Then

$$fu = \big(m_0 + m_1X_1 + \cdots + m_tX_t\big)\big(r_0 + r_1Y_1 + \cdots + r_kY_k\big) = 0,$$

which implies that $m_tX_tr_kY_k = 0$, whence $m_t\sigma^{\alpha_t}(r_k)X_tY_k = 0$, i.e., $m_t\sigma^{\alpha_t}(r_k) = 0$, and $\sigma^{\alpha_t}(r_k) \in \operatorname{rann}_R(m_t) = \operatorname{rann}_R\big(\operatorname{lc}(f)\big)$, that is, $r_k \in \sigma^{-\alpha_t}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big)$. In this way $r_kY_k \in \sigma^{-\alpha_t}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big)A \subseteq \operatorname{rann}_A(f)$ (by the proof above). Because $u \in \operatorname{rann}_A(f)$, $u - r_kY_k \in \operatorname{rann}_A(f)$. Repeating this process we show that the summands $r_{k-1}Y_{k-1}$, $r_{k-2}Y_{k-2}$, $\dots$, r_0 are elements of $\sigma^{-\alpha_t}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big)A$ which yields that $u \in \sigma^{-\alpha_t}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big)A$ and hence we prove the inclusion $\operatorname{rann}_A(f) \subseteq \sigma^{-\alpha_t}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big)A$. Then $\operatorname{rann}_A(f) = \sigma^{-\alpha_t}\big(\operatorname{rann}_R\big(\operatorname{lc}(f)\big)\big)A$ and $fA \cong \operatorname{lc}(f)R \otimes_R A$ as right A-modules. ☑

Definition 4.6. If M is a right module over a ring B, an element of $m \in M$ is said to be a *singular element* of M if the right ideal $\operatorname{rann}_B(m)$ is essential in B_B. The set of all singular elements of M is denoted by $\mathcal{Z}(M)$. M_B is a *singular* (*nonsingular*) module if $\mathcal{Z}(M) = M$ ($\mathcal{Z}(0) := 0$).

We have the following key result.

Proposition 4.7. *Let A be a bijective skew PBW extension of a ring R and let M be a nonsingular right R-module. If either R is a right Noetherian ring or M is a Noetherian module, then*

$$\operatorname{rudim}_R(M) = \operatorname{rudim}_A(M \otimes_R A).$$

Proof. If R is a right Noetherian ring or M is a Noetherian module, then every nonzero submodule of M contains a uniform Noetherian submodule. This implies that M contains an essential submodule N which is a direct sum of uniform Noetherian submodules. Since M is nonsingular, $N \leq_r M$ and so by Lemma 4.3, $N \otimes_R A \leq_r M \otimes_R A$ which implies that $\operatorname{rudim}_R(N \otimes_R A) = \operatorname{rudim}(M \otimes_R A)$.

In this way we have to show that if M is a nonsingular uniform Noetherian module, then $M \otimes_R A$ is uniform. Since $M \otimes_R A$ is Noetherian, it contains a uniform submodule T. Consider an element nonzero f of T of minimal monomial order as in Lemma 4.5, Lemmas 4.3 and 4.5 imply that

$$fA \cong \operatorname{lc}(f)R \otimes_R A \leq_r M \otimes_R A.$$

Since fA is uniform then $M \otimes_R A$ is uniform. ☑

The next proposition establishes that nonsingularity is preserved for induced modules.

Proposition 4.8. *Let A be a bijective skew PBW extension of a ring R and let M be a right R-module. If M_R is nonsingular, then $(M \otimes_R A)_A$ is nonsingular. Conversely, if R_R is nonsingular and $(M \otimes_R A)_A$ is nonsingular, then M_R is nonsingular.*

Proof. Suppose that M_R is nonsingular. Let T be the singular submodule of $M \otimes_R A$. If $T \neq 0$, let $f \in T$ be nonzero with minimal monomial order as in Lemma 4.5. We obtain that $\operatorname{rann}_A(f) = \operatorname{rann}_R\big(\operatorname{lc}(f)\big)A$,and since M is nonsingular, there is a nonzero right ideal I of R with $\operatorname{rann}_R\big(\operatorname{lc}(f)\big) \cap I = 0$. Hence $\operatorname{rann}_R\big(\operatorname{lc}(f)\big)A \cap IA = 0$ which implies that $\operatorname{rann}_A(f)$ is not an essential right ideal of A, which contradicts the definition of T. We conclude that $T = 0$.

Finally suppose that R_R and $(M \otimes_R A)_A$ are nonsingular. Let m be an element of M with $I = \operatorname{rann}_R(m)$. If I is an essential right ideal of R, then $I_R \leq_r R_R$ and hence $IA_A \leq_r A_A$. The fact $(m \otimes 1)IA = 0$ implies that $m = 0$ which shows that M_R is nonsingular. ☑

Definition 4.9 ([1], Section 2)**.** Let B be a right Noetherian ring and let U be a uniform right B-module. Then there is a unique prime ideal P of B which is the largest annihilator of any nonzero submodule of U. This prime ideal is called the *assassinator* of U, and U is called *tame* if it contains a copy of a nonzero right ideal of B/P.

Alternatively, U is tame if and only if the submodule $\operatorname{rann}_U(P)$ is torsion free as an (B/P)-module. An arbitrary right B-module M is tame if all of its uniform submodules are tame, and we denote the set of assassinator prime ideals of uniform submodules of M by $\operatorname{ass}(M)$.

Proposition 4.10. *Let A be a bijective skew PBW extension of a right Noetherian ring, let (Σ, Δ) be the pair induced by $x_1, \dots, x_n$ and let M be a tame right R-module such that each member of* $\operatorname{ass}(M)$ *is* (Σ, Δ)*-invariant. Then* $\operatorname{rudim}_R(M) = \operatorname{rudim}_A(M \otimes_R A)$.

Proof. Let E be the injective hull of M. Since

$$\operatorname{rudim}_R(E) = \operatorname{rudim}_R(M) \leq \operatorname{rudim}_A(M \otimes_R A) \leq \operatorname{rudim}_A(E \otimes_R A),$$

it is sufficient to show that $\operatorname{rudim}_R(E) = \operatorname{rudim}_A(E \otimes_R A)$. Since R is right Noetherian, E is a direct sum of uniform (indecomposable) injective submodules. Using the fact that the tensor product preserves direct sums, it is enough to prove the assertion with E uniform ([6, Theorem 3.48 and Corollary 6.10]). We also note that neither the tameness of M nor the set $\operatorname{ass}(M)$ is changed by passing to an essential extension or an essential submodule of M ([1, p. 20]). In this way, following Definition 4.9 we may consider the case where $M = E(U)$ is the injective hull of a uniform right ideal U of some factor ring R/P with P a (Σ, Δ)-invariant prime ideal of R.

Let $E_0 = \operatorname{ann}_E(P)$. Then E_0 is the (R/P)-injective hull of U, and E_0 is torsionfree and uniform as an (R/P)-module, so by Proposition 4.7 the module $E_0 \otimes_{R/P} (A/PA) \cong E_0 \otimes_R A$ is uniform as a right A-module (note that A/PA is a skew PBW extension of R/P by Proposition 4.1). In this way, to conclude the proof we have to show that $E_0 \otimes_R A \leq_e E \otimes_R A$. By contradiction, suppose that $E_0 \otimes_R A$ is not essential in $E \otimes_R A$. Then there is a nonzero element $a \in E \otimes_R A$ of minimal monomial order such that $aA \cap (E_0 \otimes_R A) = 0$. Following (13) we have the expression

$$a = (a_0 \otimes 1) + (a_1 \otimes X_1) + \cdots + (a_m \otimes X_m),$$

where $a_i \in E$ for each i, $a_m \neq 0$, $\exp(X_i) \prec \exp(X_m)$, $1 \leq i \leq m-1$, and the element a satisfies the conditions of the Lemma 4.5. Since E_0 is essential in E, there exists $r \in R$ such that $a_m r \in E_0$ and a_m is nonzero. We may replace a by ar and then without lost of generality we suppose that $a_m \in E_0$. In this way $a_m P = 0$, and using the fact that P is (Σ, Δ)-invariant and part (i) of Theorem 2.7 we have that $(a_m \otimes X_m)P = 0$. Now, the equality $aA \cap (E_0 \otimes_R A) = 0$ implies $aPA \cap (E_0 \otimes_R A) = 0$, and using the minimality of m we obtain that $aP = 0$ whence $\big(a - (a_m \otimes X_m)\big)P = 0$. Thus $a_{m-1}P = 0$. Continuing this way we can see that $a_i P = 0$ for every a_i, but this means that $a \in E_0 \otimes_R A$, which contradicts $a \neq 0$. So, $E_0 \otimes_R A \leq_e E \otimes_R A$ and the assertion follows. ☑

Next theorem establishes conditions under which passing from R to A preserves the dimension where A is a skew PBW extension of R.

Theorem 4.11. *Let A be a bijective skew PBW extension of a right Noetherian ring. Suppose that R is tame as a right R-module over itself and that any prime annihilator ideal in R is (Σ, Δ)-invariant. Then* $\operatorname{rudim}_R(R) = \operatorname{rudim}_A(A)$.

Proof. The assertion follows from Definition 4.9 and Proposition 4.10. ☑

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Departamento de Matemáticas
Universidad Nacional de Colombia
Facultad de Ciencias
Carrera 30, calle 45
Bogotá, Colombia
e-mail: mareyesv@unal.edu.co