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A Generalization for the Riesz p-Variation

Una generalización de la p-variación de Riesz

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Abstract. In this paper we introduce a generalization of the concept of Riesz p-variation and construct a function space which is normalizable and moreover is a Banach space as well as a Banach algebra. Furthermore, using Medved'ev approach we obtain an integral characterization of the functions in this function space.

Key words and phrases. Riesz p-variation, (ϕ, α)-Bounded variation, Bounded variation.

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Resumen. En este artículo se introduce una generalización del concepto de p-variación de Riesz y se construye un espacio de funciones que es normalizable y además es tanto espacio de Banach como un álgebra de Banach. Adicionalmente, usando el enfoque dado por Medved'ev se obtiene una caracterización integral de las funciones en dicho espacio funcional.

Palabras y frases clave. p-Variación de Riesz, variación (ϕ, α)-acotada, variación acotada.

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1. Introduction

Two centuries ago, around 1880, C. Jordan introduced the notion of a function of bounded variation in [8] and established the relation between those functions and monotonic ones when he was studying convergence of Fourier series. Later on the concept of bounded variation was generalized in various directions by many mathematicians, such as F. Riesz, N. Wiener, R. E. Love, H. Ursell, L. C. Young, W. Orlicz, J. Musielak, L. Tonelli, L. Cesari, R. Caccioppoli, E. de Giorgi, O. Oleinik, E. Conway, J. Smoller, A. Vol'pert, S. Hudjaev, L. Ambrosio, G. Dal Maso, among many others. It is noteworthy to mention that many of these generalizations where motivated by problems in such areas as calculus of variations, convergence of Fourier series, geometric measure theory, mathematical physics, etc. For many applications of functions of bounded variation in mathematical physics see, e.g., the monograph [14]. For a thorough exposition regarding bounded variation spaces and related topics, see the recent monograph [1].

In his 1910 paper [13], F. Riesz defined the concept of bounded p-variation and proved that, for $1 < p < \infty$, this class coincides with the class of absolutely continuous functions with derivative in $L^p[a,b]$. Moreover the p-variation of a function f on $[a,b]$ is given by

$$\mathsf{V}_p\big(f,[a,b]\big) = \mathsf{V}_p(f) = \big\|f'\big\|^p_{L^p[a,b]}.$$

In [3] the first and third named authors generalized the concept of bounded p-variation introducing a strictly increasing continuous function $\alpha : [a,b] \to \mathbb{R}$ and considering the bounded p-variation with respect to α. This new concept was called (p,α)-bounded variation and denoted by $\mathsf{BV}_{(p,\alpha)}[a,b]$.

In this paper we generalize the concept of (p,α)-bounded variation. In order to do that, we take a ϕ-function and then we consider the bounded ϕ-variation with respect to α. We will call this new concept (ϕ,α)-bounded variation in the sense of Riesz and denote it by $\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$. We will show that $\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$ is a modular space and this allows us to construct a vector space $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ generated by $\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$ and thus define a norm on it, the Nakano-Luxemburg norm.

After that, we use the embedding $\mathsf{RBV}_{(\phi,\alpha)}[a,b] \hookrightarrow \mathsf{B}[a,b]$ to show that $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ is a complete space, without assuming the validity of the ∞_1-condition, which is standard to impose in analogous cases. However we will use the ∞_1-condition to avoid that $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ coincide with $\mathsf{BV}[a,b]$. Finally, we obtain the following result: $f \in \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$ if and only if $f \in \alpha$-$\mathsf{AC}[a,b]$ and $\int_a^b \phi\big(\big|f'_\alpha(t)\big|\big)\,\mathrm{d}\alpha(t) < +\infty$ which is a generalization of the Medved'ev lemma, see also [2] for some embedding results in these spaces.

2. Preliminaries

In this section, we gather definitions and notations that will be used throughout the paper. Let α be any strictly increasing, continuous function defined on $[a,b]$. By μ_α we denote the Lebesgue-Stieltjes measure induced by α. For a standard treatment regarding Lebesgue-Stieltjes measure and integral, see e.g. [6].

Definition 2.1. A function $f:[a,b]\to\mathbb{R}$ is said to be *absolutely continuous with respect to* α if, for every $\varepsilon>0$, there exists some $\delta>0$ such that if $\{(a_j,b_j)\}_{j=1}^n$ are disjoint open subintervals of $[a,b]$, then

$$\sum_{j=1}^{n}\bigl|\alpha(b_j)-\alpha(a_j)\bigr|<\delta \quad \text{implies} \quad \sum_{j=1}^{n}\bigl|f(b_j)-f(a_j)\bigr|<\varepsilon.$$

All functions in $\alpha\text{-}\mathsf{AC}[a,b]$ are bounded and form an algebra of functions under pointwise defined standard operations.

Definition 2.2. Suppose f and α are real-valued functions defined on the same open interval I and let $x_0\in I$. We say that f *is α-differentiable at x_0* if exists

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{\alpha(x)-\alpha(x_0)}.$$

In that case we denote its value by $f'_\alpha(x_0)$, which we call the *α-derivative of f at x_0*.

The following lemma connects the $\alpha\text{-}\mathsf{AC}[a,b]$ concept with the concept of α-derivative (for a proof of Lemma 2.3 see e.g. [5, 4, 7, 13]).

Lemma 2.3. *If $f\in\alpha\text{-}\mathsf{AC}[a,b]$, then f'_α exists and is finite* μ_α-a.e. *on $[a,b]$. Moreover, there holds an analogue of the* fundamental theorem of calculus *for $f\in\alpha\text{-}\mathsf{AC}[a,b]$, i.e.* $|f(x)-f(y)|=\left|\int_y^x f'_\alpha\,\mathrm{d}\alpha(t)\right|$.

Definition 2.4. Let $\phi:[0,\infty)\to[0,\infty)$ be a function such that: (i) ϕ is continuous; (ii) ϕ is strictly increasing; (iii) $\phi(0)=0$; and (iv) $\lim_{t\to\infty}\phi(t)=\infty$. Then such a function is known as a *ϕ-function*.

3. Functions of (ϕ,α)-Bounded Variation

Definition 3.1. Let f be a real-valued function on $[a,b]$ and ϕ be a ϕ-function. Let $\Pi=\{a=x_0<x_1<\cdots<x_n=b\}$ be a partition of $[a,b]$. We consider

$$\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi)=\sum_{j=1}^{n}\phi\left(\frac{\bigl|f(x_j)-f(x_{j-1})\bigr|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\bigl(\alpha(x_j)-\alpha(x_{j-1})\bigr)$$

and

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\bigl(f;[a,b]\bigr)=\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)=\sup_{\Pi}\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi),$$

where the supremum is taken over all partitions Π of $[a,b]$. $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$ is called the *Riesz* (ϕ,α)*-variation of* f *on* $[a,b]$. If $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \infty$, we say that f *is a function of Riesz* (ϕ,α)*-bounded variation*. The set of all these functions is denoted by

$$\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b] = \big\{ f : [a,b] \to \mathbb{R} \,\big|\, \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \infty \big\}.$$

Setting $\phi(t) = t^p (1 \leq p < \infty)$ we obtain the concept of (p,α)-bounded variation defined in [3]. Fixing $\alpha(t) = t$, $t \in [a,b]$, the quantity $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\big(f;[a,b]\big)$ is known as the *Riesz-Medvedev total variation*, see [1] for more details.

By $\mathsf{B}[a,b]$ we denote the Banach space of bounded functions on $[a,b]$ with the supremum norm.

Theorem 3.2. *Let* $f : [a,b] \to \mathbb{R}$ *be a function and* ϕ *be a* ϕ*-function. If* $f \in \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$*, then* $f \in \mathsf{B}[a,b]$.

Proof. Let $x \in (a,b)$. Then taking the division $\Pi = (a < x < b)$ we get

$$\phi\bigg(\frac{|f(x)-f(a)|}{\alpha(x)-\alpha(a)}\bigg)\big(\alpha(x)-\alpha(a)\big) \leq M = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \infty. \tag{1}$$

In particular, from (1)

$$\phi\bigg(\frac{|f(x)-f(a)|}{\alpha(x)-\alpha(a)}\bigg)\big(\alpha(x)-\alpha(a)\big) \leq M, \quad \forall x \in \big[(a+b)/2, b\big). \tag{2}$$

Since α is increasing $\alpha\Big(\frac{a+b}{2}\Big) \leq \alpha(x)$, for all $x \in \big[(a+b)/2, b\big)$, from which

$$0 < \alpha\bigg(\frac{a+b}{2}\bigg) - \alpha(a) \leq \alpha(x) - \alpha(a), \quad \forall x \in \big[(a+b)/2, b\big). \tag{3}$$

Since ϕ takes non-negative values, i.e. $\phi(t) \geq 0$, $\forall t \geq 0$, from (2) and (3), we get

$$\phi\bigg(\frac{|f(x)-f(a)|}{\alpha(x)-\alpha(a)}\bigg)\Bigg(\alpha\bigg(\frac{a+b}{2}\bigg) - \alpha(a)\Bigg) \leq M, \quad \forall x \in \big[(a+b)/2, b\big)$$

from which we obtain

$$\phi\bigg(\frac{|f(x)-f(a)|}{\alpha(x)-\alpha(a)}\bigg) \leq \frac{M}{\alpha\Big(\frac{a+b}{2}\Big) - \alpha(a)} = C_1, \quad \forall x \in \big[(a+b)/2, b\big). \tag{4}$$

Since $\phi : [0,\infty) \to [0,\infty)$ is bijective (strictly increasing, so injective, continuous with $\phi(0) = 0$ and $\lim_{t\to\infty}\phi(t) = \infty$), $\phi^{-1} : [0,\infty) \to [0,\infty)$ is bijective strictly increasing, from (4) we deduce $\frac{|f(x)-f(a)|}{\alpha(x)-\alpha(a)} \leq \phi^{-1}(C_1) = L_1$, $\forall x \in \big[(a+b)/2, b\big)$ or

$$|f(x)-f(a)| \leq L_1\big(\alpha(x)-\alpha(a)\big), \quad \forall x \in \big[(a+b)/2, b\big).$$

Then

$$\begin{aligned}|f(x)| &\leq |f(x) - f(a)| + |f(a)| \\ &\leq L_1\big(\alpha(x) - \alpha(a)\big) + f(a) \leq L_1\big(\alpha(b) - \alpha(a)\big) + |f(a)| \quad (5)\end{aligned}$$

for all $x \in \big[(a+b)/2, b\big)$, where the last inequality follows from the fact that α is increasing.

For the case where $x \in (a, (a+b)/2]$ we get

$$|f(x)| \leq L_2\big(\alpha(b) - \alpha(a)\big) + |f(b)|, \quad (6)$$

adapting the previous argument.

From (5) and (6) it follows that f is bounded on (a, b), hence on $[a, b]$. ☑

4. $\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a, b]$ as a Vector Space

In this section we study some properties of the (ϕ, α)-variation. Let us consider $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$ as a functional defined on the set of Riesz (ϕ, α)-bounded variation, that is $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)} : \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a, b] \to [0, +\infty)$ with $f \mapsto \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$.

Theorem 4.1. *Let ϕ be a ϕ-function. Then*

(1) $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(-f) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$ for $f \in \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a, b]$*;*

(2) $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}$ is convex if and only if ϕ is convex*;*

(3) $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) = 0$ if and only if f is a constant function*;*

(4) If ϕ is convex and $0 \leq \lambda \leq 1$, then $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f) \leq \lambda \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$ for $f \in \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a, b]$.

***Proof*.**

(1) Follows directly from the definition of $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$.

(2) Suppose that ϕ is convex. Let $f, g \in \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a, b]$ and $\lambda, \mu \in [0, 1]$ such that $\lambda + \mu = 1$. Let $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Since ϕ is increasing and convex we have

$$
\begin{aligned}
\sigma^{\mathsf{R}}_{(\phi,\alpha)}&\big(\lambda f+\mu g;\Pi\big)\\
&=\sum_{j=1}^{n}\phi\left(\frac{\big|(\lambda f+\mu g)(x_j)-(\lambda f+\mu g)(x_{j-1})\big|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\\
&\leq \lambda\sum_{j=1}^{n}\phi\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\\
&\qquad\qquad+\mu\sum_{j=1}^{n}\phi\left(\frac{\big|g(x_j)-g(x_{j-1})\big|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\\
&=\lambda\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi)+\mu\sigma^{\mathsf{R}}_{(\phi,\alpha)}(g;\Pi)\\
&\leq \lambda\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)+\mu\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(g),
\end{aligned}
$$

which entails $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f+\mu g)\leq\lambda\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)+\mu\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(g)$, i.e. $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}$ is convex on $\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$. Moreover, if $f,g\in\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$ then $\lambda f+\mu g\in\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$ with $\lambda+\mu=1$ and $\lambda,\mu\in[0,1]$. Conversely, suppose that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}$ is convex. Let $s,t\in[0,\infty)$ and define for $x\in[a,b]$, the functions $f(x)=s\alpha(x)$, and $g(x)=t\alpha(x)$. Let $\lambda,\mu\in[0,1]$ with $\lambda+\mu=1$. Let $\Pi=\{a=x_0<x_1<\cdots<x_n=b\}$ be a partition of $[a,b]$, then we have

$$
\begin{aligned}
\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi)&=\sum_{j=1}^{n}\phi\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\\
&=\sum_{j=1}^{n}\phi\left(\frac{|s\alpha(x_j)-s\alpha(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\\
&\qquad\qquad=\phi(s)\big(\alpha(b)-\alpha(a)\big)<\infty,
\end{aligned}
$$

which is true for any partition Π of $[a,b]$.

Therefore $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)=\phi(s)\big(\alpha(b)-\alpha(a)\big)$ and $f\in\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$. In the same way, we have $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(g)=\phi(t)\big(\alpha(b)-\alpha(a)\big)$ and $g\in\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$. Since $f,g\in\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$ and $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}$ is convex, we have that $\lambda f+\mu g\in\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$ with $\lambda,\mu\in[0,1]$ and $\lambda+\mu=1$. On the other hand, observe that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f+\mu g)=\phi(\lambda s+\mu t)\big(\alpha(b)-\alpha(a)\big)$. By hypothesis $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f+\mu g)\leq\lambda\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)+\mu\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(g)$ then $\phi(\lambda s+\mu t)\leq\lambda\phi(s)+\mu\phi(t)$ and thus ϕ is convex on $[0,+\infty)$.

(3) If f is constant on $[a,b]$, then $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)=0$, since $\phi(0)=0$. Next, suppose that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)=0$, then for all partitions Π of $[a,b]$, we have $\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi)=0$. Let now $\Pi=\{a<t<b\}$ be a partition of $[a,b]$, then

$$\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi) = \phi\left(\frac{|f(b)-f(t)|}{\alpha(b)-\alpha(t)}\right)\big(\alpha(b)-\alpha(t)\big) + \phi\left(\frac{|f(t)-f(a)|}{\alpha(t)-\alpha(a)}\right)\big(\alpha(t)-\alpha(a)\big) = 0$$

and hence $\phi\Big(\frac{|f(t)-f(a)|}{\alpha(t)-\alpha(a)}\Big)\big(\alpha(t)-\alpha(a)\big) = 0$ and $\frac{|f(t)-f(a)|}{\alpha(t)-\alpha(a)} = 0$. Therefore $f(t) = f(a)$ for $t \in [a,b]$, that is, f is constant on $[a,b]$.

(4) By (2) and (3) we have $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\big(\lambda f + (1-\lambda)0\big) \le \lambda \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) + (1-\lambda)\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(0) = \lambda \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$. ☑

***Remark* 4.2.** In general $\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$ is *not* a vector space.

The following lemma will be useful in what follows.

Lemma 4.3. *Let* $(\mathbb{R},\mathsf{V},+)$ *be a vector space. Let* $\mathsf{A} \subset \mathsf{V}$ *be a symmetric and convex subset. Let* $[\mathsf{A}]$ *be the vector space generated by* A*. Then* $[\mathsf{A}] = \{v \in \mathsf{V} \mid \exists\lambda > 0 \quad \text{such that} \quad \lambda v \in \mathsf{A}\}$.

Next, we apply Lemma 4.3 to Theorem 4.1 to obtain the following.

Corollary 4.4. *Let* ϕ *be a convex* ϕ*-function, then* $\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b] \subset \mathsf{B}[a,b]$ *is a symmetric and convex set. Moreover* $\big\{f : [a,b] \to \mathbb{R} \,\big|\, \exists\lambda > 0 \quad \text{such that} \quad \lambda f \in \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]\big\}$ *is the vector space generated by* $\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]$.

Definition 4.5. Let ϕ be a convex ϕ-function. Then

$$\big\{f : [a,b] \to \mathbb{R} \,\big|\, \exists\lambda > 0 \quad \text{such that} \quad \lambda f \in \mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]\big\} = \big\{f : [a,b] \to \mathbb{R} \,\big|\, \exists\lambda > 0 \quad \text{such that} \quad \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f) < +\infty\big\}$$

is called the *vector space of* (ϕ,α)*-bounded variation functions in the sense of Riesz* and we denote it by $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$.

We now have that $\mathsf{RBV}_{(\phi,\alpha)}[a,b] = \Big[\mathsf{BV}^{\mathsf{R}}_{(\phi,\alpha)}[a,b]\Big] \subset \mathsf{B}[a,b]$.

5. $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ as a Normed Space

Before introducing a norm in the $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ space, let us show some preliminary results.

Theorem 5.1. *Let* ϕ *be a convex* ϕ*-function and* $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$*. Then*

(1) If $0 < k < k'$, then $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(kf) \le \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(k'f)$;

(2) $\lim_{\beta\to 0} \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\beta f) = 0$;

(3) $\{\varepsilon > 0 \mid \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/\varepsilon) \le 1\} \ne \varnothing$.

Proof.

(1) Let $0 < k < k'$ and $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Then $|kf(x_j) - kf(x_{j-1})| \le |k'f(x_j) - k'f(x_{j-1})|$. Since ϕ is increasing, for $j = 1, \dots, n$, we have

$$\phi\left(\frac{\left|kf(x_j) - kf(x_{j-1})\right|}{\alpha(x_j) - \alpha(x_{j-1})}\right) \le \phi\left(\frac{\left|k'f(x_j) - k'f(x_{j-1})\right|}{\alpha(x_j) - \alpha(x_{j-1})}\right)$$

and thus $\sigma^{\mathsf{R}}_{(\phi,\alpha)}(kf;\Pi) \le \sigma^{\mathsf{R}}_{(\phi,\alpha)}(k'f;\Pi)$, which yields the desired result.

(2) Since $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$, there exists $\lambda > 0$ such that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f) < \infty$. Let $0 < \beta \le \lambda$, then by Theorem 4.1(4) we have $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\beta f) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{\beta}{\lambda}\lambda f\right) \le \frac{\beta}{\lambda}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f) < \infty$, therefore $0 \le \lim_{\beta\to 0} \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\beta f) \le \lim_{\beta\to 0} \frac{\beta}{\lambda}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\lambda f) = 0$.

(3) From part (2) there exists $\beta > 0$ such that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(\beta f) \le 1$, which implies the assertion (3). ☑

Definition 5.2. Let ϕ be a convex ϕ-function. Then

$$\mathsf{RBV}^0_{(\phi,\alpha)}[a,b] = \left\{f : [a,b] \to \mathbb{R} \;\middle|\; f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b] \quad \text{and} \quad f(a) = 0\right\}$$

is the vector space of functions of bounded Riesz (ϕ,α)-variation that vanish at a.

We now define the functional $|\bullet|^{\mathsf{R}}_{(\phi,\alpha)} : \mathsf{RBV}^0_{(\phi,\alpha)}[a,b] \to \mathbb{R}^+$ by $|f|^{\mathsf{R}}_{(\phi,\alpha)} = \inf\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/\varepsilon) \le 1\}$, which is well-defined by Theorem 5.1(3).

We show that $|\bullet|^{\mathsf{R}}_{(\phi,\alpha)}$ defines a norm on $\mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$, the so-called *Nakano-Luxemburg norm*. Before doing that, we need the following result.

Lemma 5.3. *Let ϕ be a convex ϕ-function and $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. Then*

(1) $|f|^{\mathsf{R}}_{(\phi,\alpha)} \ne 0$ implies $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(f/|f|^{\mathsf{R}}_{(\phi,\alpha)}\right) \le 1$;

(2) $|f|^{\mathsf{R}}_{(\phi,\alpha)} < k$ is equivalent to $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k) \le 1$, $k > 0$;

(3) $0 \le |f|^{\mathsf{R}}_{(\phi,\alpha)} \le 1$ implies $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \le |f|^{\mathsf{R}}_{(\phi,\alpha)}$;

(4) $\left\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/\varepsilon) \le 1\right\} = \left(|f|^{\mathsf{R}}_{(\phi,\alpha)}, +\infty\right)$.

Proof.

(1) Let $k > |f|^{\mathsf{R}}_{(\phi,\alpha)}$ and $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Then $\sigma^{\mathsf{R}}_{(\phi,\alpha)}\big(f/k;\Pi\big) \le \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k) \le 1$ and $\sigma^{\mathsf{R}}_{(\phi,\alpha)}\big(f/|f|^{\mathsf{R}}_{(\phi,\alpha)};\Pi\big) \le \lim_{k\to|f|^{\mathsf{R}}_{(\phi,\alpha)}} \sigma^{\mathsf{R}}_{(\phi,\alpha)}\big(f/k;\Pi\big) \le 1$ which gives $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\big(f/|f|^{\mathsf{R}}_{(\phi,\alpha)}\big) \le 1$.

(2) Let $|f|^{\mathsf{R}}_{(\phi,\alpha)} < k$. If $|f|^{\mathsf{R}}_{(\phi,\alpha)} = 0$. Then there exists k' such that $0 < k' < k$ and $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k') \le 1$. Since $1/k < 1/k'$ by Theorem 4.1(4) we have $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k) \le \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k') \le 1$. If $0 < |f|^{\mathsf{R}}_{(\phi,\alpha)} < k$, then $1/k < 1/|f|^{\mathsf{R}}_{(\phi,\alpha)}$ and again by Theorem 4.1(4) and Lemma 5.3(1) we obtain $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k) \le \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\Big(f/|f|^{\mathsf{R}}_{(\phi,\alpha)}\Big) \le 1$.

Conversely, $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k) \le 1$ implies that $k \in \big\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/\varepsilon) \le 1\big\}$ which gives that $k > |f|^{\mathsf{R}}_{(\phi,\alpha)}$.

(3) If $|f|^{\mathsf{R}}_{(\phi,\alpha)} = 0$, then for $k > 0$ we have that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k) \le 1$, that is, $k \in \big\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/\varepsilon) \le 1\big\}$. Let $0 < k \le 1$. Then, by Theorem 4.1(4), we have $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\big(k(f/k)\big) \le k\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k) \le k$; thus $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) = 0$ and hence the required inequality holds.

If $0 < |f|^{\mathsf{R}}_{(\phi,\alpha)} \le 1$ by Theorem 4.1(4) we have that

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\Big(|f|^{\mathsf{R}}_{(\phi,\alpha)}\Big(f/|f|^{\mathsf{R}}_{(\phi,\alpha)}\Big)\Big) \le |f|^{\mathsf{R}}_{(\phi,\alpha)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\Big(f/|f|^{\mathsf{R}}_{(\phi,\alpha)}\Big)$$

which implies $\frac{1}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \le \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\Big(f/|f|^{\mathsf{R}}_{(\phi,\alpha)}\Big) \le 1$ and by (1) we get $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \le |f|^{\mathsf{R}}_{(\phi,\alpha)}$.

(4) $k \in \big\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/\varepsilon) \le 1\big\} \Leftrightarrow \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f/k) \le 1$ which, by (2), is equivalent to $|f|^{\mathsf{R}}_{(\phi,\alpha)} < k$ which is equivalent to $k \in \Big(|f|^{\mathsf{R}}_{(\phi,\alpha)}, \infty\Big)$. ☑

Theorem 5.4. *Let ϕ be a convex ϕ-function. Then $|\bullet|^{\mathsf{R}}_{(\phi,\alpha)}$ is a norm on* $\mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$

$$|f|^{\mathsf{R}}_{(\phi,\alpha)} = \inf\left\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{\varepsilon}\right) \le 1\right\}$$

for $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$.

Proof. We first show the *positive definiteness.* If $f = 0$, then $|0|^{\mathsf{R}}_{(\phi,\alpha)} = \Big\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\big(\frac{0}{\varepsilon}\big) \le 1\Big\} = 0$. If $|f|^{\mathsf{R}}_{(\phi,\alpha)} = 0$, then by Lemma 5.3(3), $0 \le \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \le |f|^{\mathsf{R}}_{(\phi,\alpha)} = 0$. Thus $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) = 0$ and by Theorem 4.1(3) we conclude that f is a constant function. Then, since $f(a) = 0$ then necessarily $f \equiv 0$.

To prove the *absolute homogeneity*, we first suppose that $m \neq 0$. Then

$$\begin{aligned} |mf|^{\mathsf{R}}_{(\phi,\alpha)} &= \inf\left\{\varepsilon > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{mf}{\varepsilon}\right) \leq 1\right\} \\ &= |m| \inf\left\{\delta > 0 : \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{\delta}\right) \leq 1\right\} = |m||f|^{\mathsf{R}}_{(\phi,\alpha)}. \end{aligned}$$

If $m = 0$ both sides are equal to zero.

We now show the validity of the *triangle inequality*. Lef $f, g \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. If $f = 0$ or $g = 0$ then trivially we see that $|f+g|^{\mathsf{R}}_{(\phi,\alpha)} = |f|^{\mathsf{R}}_{(\phi,\alpha)} + |g|^{\mathsf{R}}_{(\phi,\alpha)}$. If $f \neq 0$ and $g \neq 0$, by the convexity of $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}$ we have

$$\begin{aligned} &\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f+g}{|f|^{\mathsf{R}}_{(\phi,\alpha)} + |g|^{\mathsf{R}}_{(\phi,\alpha)}}\right) \\ &\quad = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{|f|^{\mathsf{R}}_{(\phi,\alpha)}}{|f|^{\mathsf{R}}_{(\phi,\alpha)} + |g|^{\mathsf{R}}_{(\phi,\alpha)}}\frac{f}{|f|^{\mathsf{R}}_{(\phi,\alpha)}} + \frac{|g|^{\mathsf{R}}_{(\phi,\alpha)}}{|f|^{\mathsf{R}}_{(\phi,\alpha)} + |g|^{\mathsf{R}}_{(\phi,\alpha)}}\frac{g}{|g|^{\mathsf{R}}_{(\phi,\alpha)}}\right) \\ &\leq \frac{|f|^{\mathsf{R}}_{(\phi,\alpha)}}{|f|^{\mathsf{R}}_{(\phi,\alpha)} + |g|^{\mathsf{R}}_{(\phi,\alpha)}}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\right) + \frac{|g|^{\mathsf{R}}_{(\phi,\alpha)}}{|f|^{\mathsf{R}}_{(\phi,\alpha)} + |g|^{\mathsf{R}}_{(\phi,\alpha)}}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{g}{|g|^{\mathsf{R}}_{(\phi,\alpha)}}\right) \\ &\qquad \leq \frac{|f|^{\mathsf{R}}_{(\phi,\alpha)}}{|f|^{\mathsf{R}}_{(\phi,\alpha)} + |g|^{\mathsf{R}}_{(\phi,\alpha)}} + \frac{|g|^{\mathsf{R}}_{(\phi,\alpha)}}{|g|^{\mathsf{R}}_{(\phi,\alpha)} + |g|^{\mathsf{R}}_{(\phi,\alpha)}} = 1 \end{aligned}$$

where the last inequality is due to Lemma 5.3(1). Thus, by Lemma 5.3(2) we have

$$|f+g|^{\mathsf{R}}_{(\phi,\alpha)} \leq |f|^{\mathsf{R}}_{(\phi,\alpha)} + |g|^{\mathsf{R}}_{(\phi,\alpha)}. \tag{7}$$

In this way we obtain the triangle inequality (7) for $f, g \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. ☑

***Remark* 5.5.** In what follows, we could had appeal to the theory of modular spaces, see e.g. [11, Theorem 1.5], but we preferred to use a direct approach.

In the following we are going to define a norm on the space $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$.

Definition 5.6. Let ϕ be a convex ϕ-function. We define the functional $\|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}$: $\mathsf{RBV}_{(\phi,\alpha)}[a,b] \to \mathbb{R}$ by $f \mapsto |f(a)| + \left|f - f(a)\right|^{\mathsf{R}}_{(\phi,\alpha)}$.

Theorem 5.7. *Let ϕ be a convex ϕ-function. Then $\|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}$ is a norm on $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$.*

Proof. To show the *positive definiteness* we take a function $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ with $\|f\|^{\mathsf{R}}_{(\phi,\alpha)} = 0$; then $|f(a)| = 0$ and $\left|f - f(a)\right|^{\mathsf{R}}_{(\phi,\alpha)} = 0$. From this we have $f(a) = 0$ and by Theorem 5.4 we get $f - f(a) = 0$.

The *absolute homogeneity* holds, since taking $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ and $c \in \mathbb{R}$, we have $\|cf\|^{\mathsf{R}}_{(\phi,\alpha)} = |cf(a)| + \left|cf - (cf)(a)\right|^{\mathsf{R}}_{(\phi,\alpha)} = |c|\|f\|^{\mathsf{R}}_{(\phi,\alpha)}$.

The *triangular inequality* follows from the fact that, if $f, g \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ then

$$
\begin{aligned}
\|f+g\|^{\mathsf{R}}_{(\phi,\alpha)} &= |(f+g)(a)| + \left|(f+g) - (f+g)(a)\right|^{\mathsf{R}}_{(\phi,\alpha)} \\
&\le |f(a)| + |g(a)| + \left|f - f(a)\right|^{\mathsf{R}}_{(\phi,\alpha)} + \left|g - g(a)\right|^{\mathsf{R}}_{(\phi,\alpha)} \\
&= \|f\|^{\mathsf{R}}_{(\phi,\alpha)} + \|g\|^{\mathsf{R}}_{(\phi,\alpha)}.
\end{aligned}
$$

☑

***Remark* 5.8.** If ϕ is a convex ϕ-function, then

(1) $\left(\mathsf{RBV}^0_{(\phi,\alpha)}[a,b], |\bullet|^{\mathsf{R}}_{(\phi,\alpha)}\right)$ is a normed space;

(2) $\left(\mathsf{RBV}_{(\phi,\alpha)}[a,b], \|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}\right)$ is a normed space.

6. $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ as a Banach Space

In what follows we are going to prove the completeness of the normed space constructed in Section 5. First of all we will need the following well-known result.

Lemma 6.1. *Let ϕ be a convex ϕ-function defined on $[0,\infty)$. Then the function $\psi : (0,\infty) \to \mathbb{R}$, defined by $x \mapsto \phi(x)/x$, is non-decreasing.*

In what follows, we do not use the ∞_1-condition as was used in [9] to prove that, for $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$, there exists M such that $\|f\|_\infty \le M|f|^{\mathsf{R}}_{(\phi,\alpha)}$.

Lemma 6.2. *Let ϕ be a convex ϕ-function. If $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$, then $\|f\|_\infty \le M|f|^{\mathsf{R}}_{(\phi,\alpha)}$ with*

$$
M = \max\left\{ \frac{1}{\big(\alpha(b)-\alpha(a)\big)\phi\Big(\frac{1}{\alpha(b)-\alpha(a)}\Big)}, \big(\alpha(b)-\alpha(a)\big)\phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right) \right\}.
$$

Proof. If $|f|^{\mathsf{R}}_{(\phi,\alpha)} = 0$, the result is trivial. Next, we assume that $|f|^{\mathsf{R}}_{(\phi,\alpha)} \ne 0$ and let us consider

$$
E = \left\{ x \in (a,b] : \left| \frac{f(x)}{|f|^{\mathsf{R}}_{(\phi,\alpha)}} \right| \ge \frac{\alpha(x)-\alpha(a)}{\alpha(b)-\alpha(a)} \right\}.
$$

Let $x \in E$. Then

$$\frac{1}{\alpha(b)-\alpha(a)} \leq \Big(\big|f(x)/|f|^{\mathsf{R}}_{(\phi,\alpha)}\big|\Big)\Big/\big(\alpha(x)-\alpha(a)\big),$$

by Lemma 6.1

$$\phi\bigg(\frac{1}{\alpha(b)-\alpha(a)}\bigg)\bigg/\bigg(\frac{1}{\alpha(b)-\alpha(a)}\bigg) \leq \phi\left(\frac{\big|f(x)/|f|^{\mathsf{R}}_{(\phi,\alpha)}\big|}{\alpha(x)-\alpha(a)}\right)\Bigg/\left(\frac{\big|f(x)/|f|^{\mathsf{R}}_{(\phi,\alpha)}\big|}{\alpha(x)-\alpha(a)}\right)$$

then by Lemma 5.3(1) we have

$$\big(\alpha(b)-\alpha(a)\big)\left|\frac{f(x)}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\right|\phi\bigg(\frac{1}{\alpha(b)-\alpha(a)}\bigg) \leq \phi\left(\frac{\big|f(x)-f(a)\big|/|f|^{\mathsf{R}}_{(\phi,\alpha)}}{\alpha(b)-\alpha(a)}\right)\big(\alpha(x)-\alpha(a)\big) \leq \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\Big(f/|f|^{\mathsf{R}}_{(\phi,\alpha)}\Big) \leq 1.$$

From this we obtain

$$|f(x)| \leq \frac{|f|^{\mathsf{R}}_{(\phi,\alpha)}}{\big(\alpha(b)-\alpha(a)\big)\phi\Big(\frac{1}{\alpha(b)-\alpha(a)}\Big)}. \tag{8}$$

Let $x \in (a,b] \setminus E$. Then

$$\left|\frac{f(x)}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\right| < \frac{\alpha(x)-\alpha(a)}{\alpha(b)-\alpha(a)} \leq 1.$$

Next, let us consider

$$\phi\left(\frac{\left|\frac{f(x)}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\right|}{\alpha(b)-\alpha(a)}\right) \leq \frac{\alpha(x)-\alpha(a)}{\alpha(b)-\alpha(a)}\phi\left(\frac{\left|\frac{f(x)}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\right|}{\alpha(x)-\alpha(a)}\right) = \frac{1}{\alpha(b)-\alpha(a)}\phi\left(\frac{\left|\frac{f(x)-f(a)}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\right|}{\alpha(x)-\alpha(a)}\right)\big(\alpha(x)-\alpha(a)\big).$$

Since $\frac{\alpha(x)-\alpha(a)}{\alpha(b)-\alpha(a)} \leq 1$, ϕ is convex and $\phi(0)=0$.

By the previous estimate, Lemma 5.3(1) and taking into account the fact that $f(a) = 0$ and the partition $a < x < b$ of $[a,b]$ we have

$$\phi\left(\frac{\left|\frac{f(x)}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\right|}{\alpha(b)-\alpha(a)}\right) \leq \frac{1}{\alpha(b)-\alpha(a)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\Big(f/|f|^{\mathsf{R}}_{(\phi,\alpha)}\Big) \leq \frac{1}{\alpha(b)-\alpha(a)},$$

from which we obtain

$$\frac{\left|\frac{f(x)}{|f|^{\mathsf{R}}_{(\phi,\alpha)}}\right|}{\alpha(b)-\alpha(a)} \leq \phi^{-1}\big(1/\big(\alpha(b)-\alpha(a)\big)\big)$$

and thus

$$|f(x)| \leq \big(\alpha(b)-\alpha(a)\big)\phi^{-1}\big(1/\big(\alpha(b)-\alpha(a)\big)\big)|f|^{\mathsf{R}}_{(\phi,\alpha)} \tag{9}$$

for $x \notin E$. By (8) and (9) we have

$$|f(x)| \leq$$
$$\max\left\{\frac{1}{\big(\alpha(b)-\alpha(a)\big)\phi\Big(\frac{1}{\alpha(b)-\alpha(a)}\Big)}, \big(\alpha(b)-\alpha(a)\big)\phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right)\right\}|f|^{\mathsf{R}}_{(\phi,\alpha)}$$

for all $x \in (a,b]$.

Let $M = \max\left\{\frac{1}{(\alpha(b)-\alpha(a))\phi\left(\frac{1}{\alpha(b)-\alpha(a)}\right)}, \big(\alpha(b)-\alpha(a)\big)\phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right)\right\}$. Then we have $\|f\|_\infty \leq M|f|^{\mathsf{R}}_{(\phi,\alpha)}$ for $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. ☑

Theorem 6.3. *Let ϕ be a convex ϕ-function. Then $\Big(\mathsf{RBV}^0_{(\phi,\alpha)}[a,b], |\bullet|^{\mathsf{R}}_{(\phi,\alpha)}\Big)$ is a complete space.*

Proof. Let $\{f_n\}_{n\in\mathbb{R}}$ be a Cauchy sequence in $\mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. Given $\varepsilon > 0$ we might select $\varepsilon' = \varepsilon M$, and thus there exists $n \in \mathbb{N}$ such that, for all $p, q > N$ we have

$$|f_p - f_q|^{\mathsf{R}}_{(\phi,\alpha)} < \frac{\varepsilon'}{M} = \varepsilon.$$

By Lemma 6.2 we obtain $\|f_p - f_q\|_\infty < \varepsilon'$. This last inequality implies that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\big(\mathsf{B}[a,b], \|\bullet\|_\infty\big)$ and hence converges to f in norm $\|\bullet\|_\infty$. Let us now define $f : [a,b] \to \mathbb{R}$ by $x \mapsto \lim_{n\to\infty} f_n(x)$ if $x \neq a$ and $f(a) = 0$. We need to show that

(1) $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$, and

(2) $\{f_n\}_{n\in\mathbb{N}}$ converges to f in norm $|\bullet|^{\mathsf{R}}_{(\phi,\alpha)}$.

From Lemma 5.3(2) we see that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f_p-f_q}{\varepsilon}\right) \leq 1$.

Let $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Then

$$\sigma^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f_p-f}{\varepsilon};\Pi\right) = \sigma^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f_p-\lim_{q\to\infty} f_q}{\varepsilon};\Pi\right)$$
$$= \lim_{q\to\infty}\sigma^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f_p-f_q}{\varepsilon};\Pi\right) \leq \lim_{q\to\infty}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f_p-f_q}{\varepsilon}\right) \leq 1$$

for all partitions Π of $[a,b]$, which yields

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f_p-f}{\varepsilon}\right) = \sup_{\Pi}\sigma^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f_p-f}{\varepsilon};\Pi\right) \leq 1.$$

for $p > N$. Therefore $f_p - f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. Since $\mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ is a linear space, we conclude that the function $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. Moreover, since $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{f_p-f}{\varepsilon}\right) \leq 1$, from Lemma 5.3(2), we conclude that $|f_p - f|^{\mathsf{R}}_{(\phi,\alpha)} < \varepsilon$ if $p > N$, which means that $\{f_n\}_{n\in\mathbb{N}}$ converges to f in $|\bullet|^{\mathsf{R}}_{(\phi,\alpha)}$-norm . ☑

Theorem 6.4. *Let ϕ be a convex ϕ-function. Then $\left(\mathsf{RBV}_{(\phi,\alpha)}[a,b], \|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}\right)$ is a complete space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_p - f_q\|^{\mathsf{R}}_{(\phi,\alpha)} < \varepsilon$ whenever $p, q > N$, that is $\big|(f_p - f_q)(a)\big| + \big|(f_p - f_q) - (f_p - f_q)(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}$ whenever $p, q > N$. Let $q_p = f_p - f_p(a)$, $p \in \mathbb{N}$. Since $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ if and only if $f - f(a) \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ then $g \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. Then $|g_q - g_q|^{\mathsf{R}}_{(\phi,\alpha)} < \varepsilon$ whenever $p, q > N$ and thus $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $\left(\mathsf{RBV}^0_{(\phi,\alpha)}[a,b], |\bullet|^{\mathsf{R}}_{(\phi,\alpha)}\right)$ which is complete, and therefore converges in norm $|\bullet|^{\mathsf{R}}_{(\phi,\alpha)}$ to a function $g \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$.

On the other hand, since $\big|f_p(a) - f_q(a)\big| < \varepsilon$ whenever $p, q > N$, we have that $\big\{f_n(a)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and hence converges to a function $f_0 \in \mathbb{R}$. Let $f = g + f_0$, then $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$. Since g and f_0 are constant functions they have (ϕ,α)-bounded variation in the Riesz sense and $f(a) = (g + f_0)(a) = g(a) + f_0 = f$ and $g = f - f(a)$. Moreover

$$\big\|(f_n - f)\big\|^{\mathsf{R}}_{(\phi,\alpha)} = \big|(f_n - f)(a)\big| + \big|(f_n - f) - (f_n - f)(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}$$
$$= \big|f_n(a) - f(a)\big| + \big|g_n - g\big|^{\mathsf{R}}_{(\phi,\alpha)}.$$

Since $f_n(a) \to f(a)$ and $g_n \to g$, we have the result. ☑

7. $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ as a Banach Algebra

In their 1987 paper [9] L. Maligranda and W. Orlicz gave a lemma which supplies a test to check if some function space is a Banach algebras, namely

Lemma 7.1 (Maligranda–Orlicz criterion)**.** *Let* $(\mathsf{X}, \|\bullet\|)$ *be a Banach space whose elements are bounded functions and the space is closed under multiplication of functions. Let us assume that* $f \bullet g \in \mathsf{X}$ *and* $\|fg\| \leq \|f\|_\infty \bullet \|g\| + \|f\| \bullet \|g\|_\infty$ *for any* $f, g \in \mathsf{X}$*. Then the space* X *equipped with the norm* $\|f\|_1 = \|f\|_\infty + \|f\|$ *is a normed Banach algebra. Also, if* $\mathsf{X} \hookrightarrow \mathsf{B}[a,b]$*, then the norms* $\|\bullet\|_1$ *and* $\|\bullet\|$ *are equivalent. Moreover, if* $\|f\|_\infty \leq M\|f\|$ *for* $f \in \mathsf{X}$*, then* $(\mathsf{X}, \|\bullet\|_2)$ *is a normed Banach algebra with* $\|f\|_2 = 2M\|f\|$*,* $f \in \mathsf{X}$ *and the norms* $\|\bullet\|_2$ *and* $\|\bullet\|$ *are equivalent.*

To begin with, we are going to show that the space $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ is closed under multiplication of functions.

Theorem 7.2. *Let* ϕ *be a convex* ϕ*-function. If* $f, g \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$*, then* $fg \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ *and* $\big|fg - (fg)(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} \leq \|f\|_\infty \big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} + \|g\|_\infty \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}$.

Proof. If $f = f(a) = const$ or $g = g(a) = const$, the result is obvious. Assume that $f \neq const$ and $g \neq const$. Since the functions are bounded (see Theorem 3.2) we write $\lambda_1 = \|f\|_\infty$, $\lambda_2 = \|g\|_\infty$ and $\lambda = \lambda_1 \big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} + \lambda_2 \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}$. Let $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Since $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ if and only if $f - f(a) \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ we have

$$\begin{aligned}
&\sigma^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{fg - (fg)(a)}{\lambda}; \Pi\right) \\
&\quad = \sigma^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{fg}{\lambda}; \Pi\right) \sum_{j=1}^{n} \phi\left(\frac{\big|(fg)(x_j) - (fg)(x_{j-1})\big|}{\lambda\big(\alpha(x_j) - \alpha(x_{j-1})\big)}\right)\big(\alpha(x_j) - \alpha(x_{j-1})\big) \\
&\quad \leq \sum_{j=1}^{n} \phi\left(\frac{\lambda_2\big|f(x_j) - f(x_{j-1})\big| + \lambda_1\big|g(x_j) - g(x_{j-1})\big|}{\lambda\big(\alpha(x_j) - \alpha(x_{j-1})\big)}\right)\big(\alpha(x_j) - \alpha(x_{j-1})\big)
\end{aligned}$$

$$= \sum_{j=1}^{n} \phi \left(\frac{\lambda_2 \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \frac{\big|f(x_j) - f(x_{j-1})\big|}{\big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} \big(\alpha(x_j) - \alpha(x_{j-1})\big)} + \right.$$

$$\left. \frac{\lambda_1 \big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \frac{\big|g(x_j) - g(x_{j-1})\big|}{\big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} \big(\alpha(x_j) - \alpha(x_{j-1})\big)} \right) \big(\alpha(x_j) - \alpha(x_{j-1})\big)$$

$$\leq \sum_{j=1}^{n} \left[\frac{\lambda_2 \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \phi \left(\frac{\big|f(x_j) - f(x_{j-1})\big|}{\big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} \big(\alpha(x_j) - \alpha(x_{j-1})\big)} \right) + \right.$$

$$\left. \frac{\lambda_1 \big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \phi \left(\frac{\big|g(x_j) - g(x_{j-1})\big|}{\big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} \big(\alpha(x_j) - \alpha(x_{j-1})\big)} \right) \right] \big(\alpha(x_j) - \alpha(x_{j-1})\big)$$

$$= \frac{\lambda_2 \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \sigma^{\mathsf{R}}_{(\phi,\alpha)} \left(\frac{f - f(a)}{\big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}; \Pi \right) + \frac{\lambda_1 \big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \sigma^{\mathsf{R}}_{(\phi,\alpha)} \left(\frac{g - g(a)}{\big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}; \Pi \right)$$

$$\leq \frac{\lambda_2 \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)} \left(\frac{f}{\big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}} \right) + \frac{\lambda_1 \big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)} \left(\frac{g}{\big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}} \right).$$

Since $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ if and only if $f - f(a) \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ we have

$$= \frac{\lambda_2 \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)} \left(\frac{f - f(a)}{\big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}} \right) + \frac{\lambda_1 \big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)} \left(\frac{g - g(a)}{\big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}} \right).$$

Since $f - f(a)$ and $g - g(a)$ belong to $\mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ and by Lemma 5.3(1) we obtain

$$\leq \frac{\lambda_2 \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} + \frac{\lambda_2 \big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}}{\lambda} = 1.$$

This holds for any partition Π of $[a,b]$, then

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{fg}{\lambda}\right) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{fg-(fg)(a)}{\lambda}\right) = \sup_{\Pi} \sigma^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{fg-(fg)(a)}{\lambda};\Pi\right) \leq 1.$$

Since $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}\left(\frac{fg}{\lambda}\right) < \infty$ we conclude that $fg \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ and $fg-(fg)(a) \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$. By Lemma 5.3(2) we have $\big|fg - (fg(a))\big|^{\mathsf{R}}_{(\phi,\alpha)} \leq \lambda$. Replacing λ_1, λ_2 and λ we obtain

$$\big|fg-(fg)(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} \leq \|f\|_\infty \big|g - g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} + \|g\|_\infty \big|f - f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}. \qquad ☑$$

Corollary 7.3. *Let ϕ be a convex ϕ-function. If $f, g \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$, then $fg \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ and*

$$|fg|^{\mathsf{R}}_{(\phi,\alpha)} \leq \|f\|_\infty |g|^{\mathsf{R}}_{(\phi,\alpha)} + \|g\|_\infty |f|^{\mathsf{R}}_{(\phi,\alpha)}.$$

Theorem 7.4. *Let ϕ be a convex ϕ-function. Then*

(1) $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ *with the norm* $\|\bullet\|^{\mathsf{R}}_1 = \|\bullet\|_\infty + \|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}$ *is a Banach algebra;*

(2) $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ *with the norm* $\|\bullet\|^{\mathsf{R}}_2 = 2\max\{1, M\}\|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}$ *is a Banach algebra, where*

$$M = \max\left\{\frac{1}{\big(\alpha(b)-\alpha(a)\big)\phi\Big(\frac{1}{\alpha(b)-\alpha(a)}\Big)}, \big(\alpha(b)-\alpha(a)\big)\phi^{-1}\left(\frac{1}{\alpha(b)-\alpha(a)}\right)\right\}.$$

(3) The norms $\|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}$, $\|\bullet\|^{\mathsf{R}}_1$ *and* $\|\bullet\|^{\mathsf{R}}_2$ *are all equivalent.*

Proof. First of all we need to check the hypothesis from Maligranda-Orlicaz criterion (Lemma 7.1). By Theorem 3.2 we know that $\mathsf{RBV}_{(\phi,\alpha)}[a,b] \subset \mathsf{B}[a,b]$ and by Theorem 6.4 we know that $\big(\mathsf{RBV}_{(\phi,\alpha)}[a,b], \|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}\big)$ is a Banach space which is closed with respect to function multiplication; moreover

$$\begin{aligned}\|fg\|^{\mathsf{R}}_{(\phi,\alpha)} &= \big|(fg)(a)\big| + \big|fg-(fg)(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}\\ &\le 2|f(a)|\,|g(a)| + \|f\|_\infty\bullet\big|g-g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} + \|g\|_\infty\bullet\big|f-f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}\\ &\le \|f\|_\infty|g(a)| + \|g\|_\infty|f(a)| + \|f\|_\infty\bullet\big|g-g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)} +\\ &\qquad\qquad \|g\|_\infty\bullet\big|f-f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}\\ &= \|f\|_\infty\Big(|g(a)| + \big|g-g(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}\Big) + \|g\|_\infty\Big(|f(a)| + \big|f-f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}\Big)\\ &= \|f\|_\infty\|g\|^{\mathsf{R}}_{(\phi,\alpha)} + \|g\|_\infty\|f\|^{\mathsf{R}}_{(\phi,\alpha)}\end{aligned}$$

for $f,g\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$.

Then $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ with the norm $\|f\|^{\mathsf{R}}_1 = \|f\|_\infty + \|f\|^{\mathsf{R}}_{(\phi,\alpha)}$, is a Banach algebra. Moreover, if $f(a)\neq f\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$, then $f-f(a)\in\mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ and by Lemma 6.2 we have that $\big\|f-f(a)\big\|_\infty \le M\big|f-f(a)\big|^{\mathsf{R}}_{(\phi,\alpha)}$ yielding $\|f\|_\infty\le\max\{1,M\}\|f\|^{\mathsf{R}}_{(\phi,\alpha)}$ for $f\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$.

From Lemma 7.1 we deduce that $\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ with the norm $\|f\|^{\mathsf{R}}_2 = 2\max\{1,M\}\|f\|^{\mathsf{R}}_{(\phi,\alpha)}$ is a Banach algebra and the norms $\|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}$ and $\|\bullet\|^{\mathsf{R}}_2$ are equivalent. ☑

Combining Lemma 7.1 and Theorem 7.4 we have

Corollary 7.5. *Let ϕ be a convex ϕ-function. Then*

(1) $\mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ *with the norm* $\|\bullet\|^{\mathsf{R}}_1 = \|\bullet\|_\infty + |\bullet|^{\mathsf{R}}_{(\phi,\alpha)}$ *is a Banach algebra;*

(2) $\mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$ *with the norm* $\|\bullet\|^{\mathsf{R}}_2 = 2\max\{1,M\}|\bullet|^{\mathsf{R}}_{(\phi,\alpha)}$ *where*

$$M=\max\left\{\frac{1}{\big(\alpha(b)-\alpha(a)\big)\phi\Big(\frac{1}{\alpha(b)-\alpha(a)}\Big)},\ \big(\alpha(b)-\alpha(a)\big)\phi^{-1}\Big(\frac{1}{\alpha(b)-\alpha(a)}\Big)\right\}.$$

is a Banach algebra;

(3) The norms $\|\bullet\|^{\mathsf{R}}_{(\phi,\alpha)}$, $\|\bullet\|^{\mathsf{R}}_1$ *and* $\|\bullet\|^{\mathsf{R}}_2$ *are all equivalent.*

8. Medved'ev's Theorem

In what follows, we need to justify why we need to introduce another condition on the function ϕ, the so-called ∞_1-condition. This is done to avoid the trivialization of the theory as stated in Corollary 8.4.

We first show some auxiliary results.

Theorem 8.1. *Let ϕ be a convex ϕ-function. Then* $\mathsf{RBV}_{(\phi,\alpha)}[a,b] \subset \mathsf{B}[a,b]$. *Moreover, we have the following estimate*

$$\mathsf{V}(f,[a,b]) \leq \big(\alpha(b)-\alpha(a)\big) + \frac{1}{\phi(1)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f).$$

***Proof*.** Let $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Note that $\sum_{j=1}^{n}\big|f(x_j)-f(x_{j-1})\big| = \sum_{j=1}^{n}\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}\big(\alpha(x_j)-\alpha(x_{j-1})\big)$. Let $E = \Big\{j \in \{1,2,\ldots,n\} : \frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})} \leq 1\Big\}$. If $j \notin E$, then $\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})} \geq 1$ and by Lemma 6.1 we obtain

$$\frac{\phi(1)}{1} \leq \frac{\phi\Big(\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}\Big)}{\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}}.$$

and thus

$$\frac{\big|f(x_j)-f(x_{j-1})\big|}{\alpha(x_j)-\alpha(x_{j-1})} \leq \frac{1}{\phi(1)}\phi\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{\alpha(x_j)-\alpha(x_{j-1})}\right)$$

for $j \notin E$. Then

$$\begin{aligned}
&\sum_{j=1}^{n}\big|f(x_j)-f(x_{j-1})\big| \\
&\quad = \sum_{j\in E}\big|f(x_j)-f(x_{j-1})\big| + \sum_{j\notin E}\frac{\big|f(x_j)-f(x_{j-1})\big|}{\alpha(x_j)-\alpha(x_{j-1})}\big(\alpha(x_j)-\alpha(x_{j-1})\big) \\
&\leq \sum_{j\in E}\big|f(x_j)-f(x_{j-1})\big| + \frac{1}{\phi(1)}\sum_{j\notin E}\phi\left(\frac{\big|f(x_j)-f(x_{j-1})\big|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big) \\
&\qquad \leq \alpha(b)-\alpha(a) + \frac{1}{\phi(1)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < +\infty
\end{aligned}$$

for all partitions Π of $[a,b]$. Therefore $\mathsf{V}\big(f,[a,b]\big) \leq \alpha(b)-\alpha(a) + \frac{1}{\phi(1)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$. ☑

We now introduce the concept of ∞_1-condition.

Definition 8.2. Let ϕ be a convex ϕ-function. If $\lim_{x\to\infty}\frac{\phi(x)}{x} = +\infty$, then we say that ϕ *satisfies the ∞_1-condition.*

Theorem 8.3. *Let ϕ be a convex ϕ-function which does not satisfies the ∞_1-condition. Then we have* $\mathsf{BV}[a,b] \subset \mathsf{RBV}_{(\phi,\alpha)}[a,b]$, *with* $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \leq r\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$, *where* $\sup_{x\in(0,\infty)}\phi(x)/x = r < +\infty$.

Proof. Let $f \in \mathsf{BV}[a,b]$ and $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Let us consider

$$\frac{\phi\left(\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}\right)}{\frac{|f(x_j)-f(x_{j-1})|}{\alpha(x_j)-\alpha(x_{j-1})}} \le r, \qquad j = 1,2,\dots,n.$$

Then

$$\phi\left(\frac{\left|f(x_j)-f(x_{j-1})\right|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big) \le r\left|f(x_j)-f(x_{j-1})\right|, \quad j = 1,2,\dots,n.$$

and

$$\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi) \le r\sum_{j=1}^{n}\left|f(x_j)-f(x_{j-1})\right|$$

for all partition Π of $[a,b]$. Therefore $\sigma^{\mathsf{R}}_{(\phi,\alpha)}(f;\Pi) \le r\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$ and $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \le r\mathsf{V}\big(f,[a,b]\big)$. Consequently, $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$. ☑

From Theorem 8.1 and 8.3 we deduce the following result.

Corollary 8.4. *Let ϕ be a convex ϕ-function such that* $\lim_{x\to\infty}\frac{\phi(x)}{x} = r < +\infty$. *Then* $\mathsf{RBV}_{(\phi,\alpha)}[a,b] = \mathsf{BV}[a,b]$ *and*

$$\frac{1}{r}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \le \mathsf{V}\big(f,[a,b]\big) \le \alpha(b)-\alpha(a) + \frac{1}{\phi(1)}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f).$$

To avoid this case and the trivialization of the theory, we will impose the ∞_1-condition on the ϕ function hereafter.

Theorem 8.5. *Let ϕ be a convex ϕ-function which satisfies the ∞_1-condition and $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$. Then f is absolutely continuous with respect to α on $[a,b]$, that is,* $\mathsf{RBV}_{(\phi,\alpha)}[a,b] \subset \alpha\text{-}\mathsf{AC}[a,b]$.

Proof. Let $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$. Given $\varepsilon > 0$, let us consider (a_j,b_j), $j = 1,2,\dots,n$ a finite collection of disjoint subintervals contained in $[a,b]$. Let $m > 0$ such that $\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \frac{m\varepsilon}{2}$. Since ϕ satisfy the ∞_1-condition, there exists $x_0 \in (0,\infty)$ such that $\phi(x) \ge mx$ for $x \ge x_0$. Next, let us consider the following set $E = \Big\{j \in \{1,2,\dots,n\} : \frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)} \ge x_0\Big\}$. If $j \in E$, then $x_0 \le \frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)}$ and since ϕ satisfies the ∞_1-condition we have $m\frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)} \le \phi\Big(\frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)}\Big)$ and thus $\left|f(b_j)-f(a_j)\right| \le \frac{1}{m}\phi\Big(\frac{|f(b_j)-f(a_j)|}{\alpha(b_j)-\alpha(a_j)}\Big)\big(\alpha(b_j)-\alpha(a_j)\big)$. From this inequality we obtain

$$\begin{aligned}
\sum_{j=1}^{n}\big|f(b_j)-f(a_j)\big| &= \sum_{j\in E}\big|f(b_j)-f(a_j)\big| + \sum_{j\notin E}\big|f(b_j)-f(a_j)\big| \\
&\le \frac{1}{m}\sum_{j\in E}\phi\left(\frac{\big|f(b_j)-f(a_j)\big|}{\alpha(b_j)-\alpha(a_j)}\right)\big(\alpha(b_j)-\alpha(a_j)\big) + x_0\sum_{j\notin E}\big(\alpha(b_j)-\alpha(a_j)\big) \\
&\le \frac{1}{m}\sum_{j=1}^{n}\phi\left(\frac{\big|f(b_j)-f(a_j)\big|}{\alpha(b_j)-\alpha(a_j)}\right)\big(\alpha(b_j)-\alpha(a_j)\big) + x_0\sum_{j=1}^{n}\big(\alpha(b_j)-\alpha(a_j)\big) \\
&< \frac{1}{m}\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) + x_0\sum_{j=1}^{n}\alpha(b_j)-\alpha(a_j).
\end{aligned}$$

Choose $0<\delta<\varepsilon/(2x_0)$. If $\sum_{j=1}^{n}\alpha(b_j)-\alpha(a_j)<\delta$, then we have $\sum_{j=1}^{n}\big|f(b_j)-f(a_j)\big|<\frac{\varepsilon}{2}+x_0\delta<\varepsilon$.

Finally, collecting all of this information we conclude that, given $\varepsilon>0$ there exists $\delta>0$ such that for all finite family of disjoint subintervals $\big\{(a_j,b_j): j=1,2,\dots,n\big\}$ of $[a,b]$ such that $\sum_{j=1}^{n}\alpha(b_j)-\alpha(a_j)<\delta$, then $\sum_{j=1}^{n}\big|f(b_j)-f(a_j)\big|<\varepsilon$, which means that $f\in\alpha\text{-}\mathsf{AC}[a,b]$. ☑

The coming result is a generalization of the result due to Medved'ev [10] which provide us with a characterization of the (ϕ,α)-bounded variation functions in the sense of Riesz.

Theorem 8.6. *Let ϕ be a convex ϕ-function which satisfies the ∞_1-condition. Let $f:[a,b]\to\mathbb{R}$, then:*

(1) If f is an α-absolutely continuous function on $[a,b]$ and

$$\int_a^b \phi\big(|f'_\alpha(x)|\big)\,\mathrm{d}\mu_\alpha(x)<\infty,$$

then we have that $f\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ and

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)\le\int_a^b \phi\big(|f'_\alpha(x)|\big)\,\mathrm{d}\mu_\alpha(x).$$

(2) If $f\in\mathsf{RBV}_{(\phi,\alpha)}[a,b]$ then f is α-absolutely continuous on $[a,b]$ and

$$\int_a^b \phi\big(|f'_\alpha(x)|\big)\,\mathrm{d}\mu_\alpha(x)\le\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f).$$

Proof. (1) Since $f \in \alpha\text{-}\mathsf{AC}[a,b]$, by Lemma 2.3 there f'_α exists μ_α-a.e. on $[a,b]$. Let $x_1, x_2 \in [a,b]$ with $x_1 < x_2$, then

$$\begin{aligned}
\phi\left(\frac{\left|f(x_2)-f(x_1)\right|}{\alpha(x_2)-\alpha(x_1)}\right)\big(\alpha(x_2)-\alpha(x_1)\big) &= \phi\left(\frac{\left|\int_{x_1}^{x_2} f'_\alpha(x)\,\mathrm{d}\alpha(x)\right|}{\alpha(x_2)-\alpha(x_1)}\right)\big(\alpha(x_2)-\alpha(x_1)\big)\\
&\le \phi\left(\frac{\int_{x_1}^{x_2} \left|f'_\alpha(x)\right|\mathrm{d}\alpha(x)}{\alpha(x_2)-\alpha(x_1)}\right)\big(\alpha(x_2)-\alpha(x_1)\big)\\
&= \phi\left(\frac{\int_{x_1}^{x_2} \left|f'_\alpha(x)\right|\mathrm{d}\alpha(x)}{\int_{x_1}^{x_2}\mathrm{d}\alpha(x)}\right)\big(\alpha(x_2)-\alpha(x_1)\big)\\
&\le \frac{\int_{x_1}^{x_2} \phi\big(\left|f'_\alpha(x)\right|\big)\,\mathrm{d}\alpha(x)}{\int_{x_1}^{x_2}\mathrm{d}\alpha(x)}\big(\alpha(x_2)-\alpha(x_1)\big)\\
&= \int_{x_1}^{x_2} \phi\big(\left|f'_\alpha(x)\right|\big)\,\mathrm{d}\alpha(x),
\end{aligned}$$

where we have used Lemma 2.3 and the generalized Jensen inequality (see, e.g. [12, Theorem 1.2.5]).

Now, let $\Pi = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be an arbitrary partition of $[a,b]$. Then

$$\begin{aligned}
&\sum_{j=1}^{n}\phi\left(\frac{\left|f(x_j)-f(x_{j-1})\right|}{\alpha(x_j)-\alpha(x_{j-1})}\right)\big(\alpha(x_j)-\alpha(x_{j-1})\big)\\
&\qquad\le \sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}\phi\big(\left|f'_\alpha(x)\right|\big)\,\mathrm{d}\alpha(x) = \int_a^b \phi\big(\left|f'_\alpha(x)\right|\big)\,\mathrm{d}\alpha(x) < \infty,
\end{aligned}$$

and hence

$$\mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) \le \int_a^b \phi\big(\left|f'_\alpha(x)\right|\big)\,\mathrm{d}\alpha(x),$$

i.e., $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$.

(2) Let $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$. Then, by Theorem 8.5, f is absolutely continous with respect to α on $[a,b]$ and thus f'_α exists μ_α-a.e. on $[a,b]$. Let $n \in \mathbb{N}$ and $\Pi_n = \{a = x_{0,n} < x_{1,n} < \cdots < x_{n,n} = b\}$ be a partition of $[a,b]$ given by

$$x_{j,n} = a + \frac{j(b-a)}{n}, \qquad j = 1, \ldots, n.$$

Next, let us consider the sequence $\{f_n\}_{n\in\mathbb{N}}$ with $f_n : [a,b] \to \mathbb{R}$ given by

$$f_n(x) = \begin{cases} \frac{f(x_{k+1,n}) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})}, & \text{if} \quad x_{k,n} \le x < x_{k+1,n}; \\ \text{any other reasonable value,} & \text{if} \quad x = b. \end{cases}$$

Claim $\{f_n\}_{n\in\mathbb{N}}$ converge to f'_α on those points where f is α-differentiable and different of $x_{i,n}$, $i = 0, 1, \ldots, n$, that is, on

$$A = \{x \in [a,b] : f'_\alpha(x) \text{ exists}\} \smallsetminus \{x_{i,n} : n \in \mathbb{N}, i = 0, 1, \ldots, n\}.$$

Let $x \in A$. Then, for each $n \in \mathbb{N}$ there exists $k \in \{0, \ldots, n\}$ such that $x_{k,n} \le x < x_{k+1,n}$. Then

$$\begin{aligned} f_n(x) &= \frac{f(x_{k+1,n}) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} \\ &= \frac{f(x_{k+1,n}) - f(x) + f(x) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} \\ &= \frac{\alpha(x_{k+1,n}) - \alpha(x)}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} \frac{f(x_{k+1,n}) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x)} + \\ &\qquad \frac{\alpha(x) - \alpha(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x_{k,n})} \frac{f(x) - f(x_{k,n})}{\alpha(x) - \alpha(x_{k,n})}. \end{aligned}$$

Therefore, $f_n(x)$ is a convex combination of the points

$$\frac{f(x_{k+1,n}) - f(x_{k,n})}{\alpha(x_{k+1,n}) - \alpha(x)} \qquad \text{and} \qquad \frac{f(x) - f(x_{k,n})}{\alpha(x) - \alpha(x_{k,n})}.$$

When $n \to \infty$, then $x_{n,k} \to x$ and $x_{k+1,n} \to x$. Since f is α-differentiable at x, the expressions

$$\frac{f(x_{k+1,n}) - f(x)}{\alpha(x_{k+1,n}) - \alpha(x)} \qquad \text{and} \qquad \frac{f(x) - f(x_{k,n})}{\alpha(x) - \alpha(x_{k,n})}$$

go to the α-derivative, i.e. to f'_α. From this we have that $\lim_{n\to\infty} f_n(x) = f'_\alpha(x)$, $x \in A$ μ_α-a.e on $[a,b]$. Since ϕ is continuous, then $\lim_{n\to\infty} \phi(|f_n(x)|) = \phi(\lim_{n\to\infty} |f_n(x)|) = \phi(|f'_\alpha(x)|)$, $x \in A$. Using Fatou's lemma, we obtain

$$
\begin{aligned}
\int_a^b \phi\big(|f'_\alpha|\big)\,\mathrm{d}\alpha(x) &= \int_a^b \lim_{n\to\infty} \phi\big(|f_n(x)|\big)\,\mathrm{d}\alpha(x) \\
&\leq \liminf_{n\to\infty} \int_a^b \phi\big(|f_n(x)|\big)\,\mathrm{d}\alpha(x) \\
&= \liminf_{n\to\infty} \sum_{i=0}^{n-1} \int_{x_{i,n}}^{x_{i+1,n}} \phi\big(|f_n(x)|\big)\,\mathrm{d}\alpha(x) \\
&= \liminf_{n\to\infty} \sum_{i=0}^{n-1} \phi\left(\frac{\big|f(x_{i+1,n}) - f(x_{i,n})\big|}{\alpha(x_{i+1,n}) - \alpha(x_{i,n})}\right) \int_{x_{i,n}}^{x_{i+1,n}} \mathrm{d}\alpha(x) \\
&= \liminf_{n\to\infty} \sum_{i=0}^{n-1} \phi\left(\frac{\big|f(x_{i+1,n}) - f(x_{i,n})\big|}{\alpha(x_{i+1,n}) - \alpha(x_{i,n})}\right) \big(\alpha(x_{i+1,n}) - \alpha(x_{i,h})\big) \\
&\leq \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f) < \infty. \quad ☑
\end{aligned}
$$

Corollary 8.7. *Let ϕ be a convex ϕ-function such that satisfies the ∞_1-condition. If $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$, then f is α-absolutely continuous on $[a,b]$ and*

$$
\int_a^b \phi\big(|f'_\alpha(x)|\big)\,\mathrm{d}\alpha(x) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f).
$$

Corollary 8.8. *Let ϕ be a convex ϕ-function such that satisfies the ∞_1-condition. Then $f \in \mathsf{RBV}_{(\phi,\alpha)}[a,b]$ if and only if f is α-absolutely continuous on $[a,b]$ and $\int_a^b \phi\big(|f'_\alpha|\big)\,\mathrm{d}\alpha(x) < \infty$. Moreover $\int_a^b \phi\big(|f'_\alpha|\big)\,\mathrm{d}\alpha(x) = \mathsf{V}^{\mathsf{R}}_{(\phi,\alpha)}(f)$.*

Corollary 8.9. *Let ϕ be a convex ϕ-function such that satisfies the ∞_1-condition. Let $f \in \mathsf{RBV}^0_{(\phi,\alpha)}[a,b]$, then*

$$
|f|^{\mathsf{R}}_{(\phi,\alpha)} = \inf\left\{\varepsilon > 0 : \int_a^b \phi\Big(\frac{|f'_\alpha(x)|}{\varepsilon}\Big)\,\mathrm{d}\alpha(x) \leq 1\right\}.
$$

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