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Multiplicative Relaxation with respect to Thompson's Metric

Relajamiento multiplicativo con respecto a la métrica de Thompson

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Abstract. We give a condition so that certain mixed monotone mappings on function spaces have a contractive multiplicative relaxation with respect to Thompson's metric. The corresponding fixed point theorem can be applied to special types of integral equations, for example.

Key words and phrases. Thompson metric, Mixed monotone mappings, Fixed points, Contraction, Relaxation.

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Resumen. Damos una condición para que ciertas aplicaciones monótonas mixtas sobre espacios de funciones tengan una relajación multiplicativa con respecto a las métricas de Thompson. El correspondiente teorema de punto fijo puede ser aplicado a tipos especiales de ecuaciones integrales, por ejemplo.

Palabras y frases clave. Metrica de Thompson, aplicación mixta monotona, puntos fijos, contracción, relajación.

1. Introduction

In [3] Guo considered conditions for mixed monotone mappings leading to contractions on cones with respect to Thompson's metric. Guo's result was generalized in several directions, see [2], [4, Ch.3.3], [5], [6, Th.3.8], [7], and the references given there. In this note we consider a multiplicative relaxation for mixed monotone mappings on certain function spaces. The additive relaxation method, also known as Krasnoselskij iteration, is a well known method for nonexpansive and one-sided Lipschitz continuous functions in Banach spaces [1, Ch.3].

Let $X \neq \varnothing$ be a set, and let $l^\infty(X, \mathbb{R})$ denote the Banach algebra of all bounded functions $u : X \to \mathbb{R}$, endowed with the supremum norm $\|\cdot\|$. Moreover let E be a closed subalgebra of $l^\infty(X,\mathbb{R})$ which contains at least one function $u_0 : X \to \mathbb{R}$ such that $\inf_{x\in X} u_0(x) > 0$. If X is a metric space, an admissible subalgebra of $l^\infty(X,\mathbb{R})$ would be the space $E = C_b(X,\mathbb{R})$ of all continuous and bounded functions on X, for example.

We consider E as ordered by the cone

$$K := \big\{u \in E : u(x) \geq 0 (x \in X)\big\}, \quad u \leq v :\iff v - u \in K.$$

The cone K has nonempty interior K°, since

$$u_0 \in \Big\{u \in E : \inf_{x\in X} u(x) > 0\Big\} = K^\circ,$$

and we set $u \ll v :\iff v - u \in K^\circ$.

We consider K° as metric space endowed with the Thompson metric

$$d : K^\circ \times K^\circ \to \mathbb{R}, \qquad d(u,v) = \sup_{x\in X} \left|\log\left(\frac{u(x)}{v(x)}\right)\right| = \big\|\log(u/v)\big\|.$$

We have

$$0 \leq u \leq v \implies \|u\| \leq \|v\|,$$

thus K is a normal cone, and [8, Lemma 3] proves that (K°, d) is a complete metric space. Now consider a function $g : (0,\infty) \times K^\circ \to K^\circ$ and

$$f : K^\circ \to (0,\infty)^X, \qquad f(u)(x) = g\big(u(x), u\big)(x). \tag{1}$$

We are interested in fixed points of f, and we will give a mixed monotonicity condition such that a suitable relaxation of f of the form

$$u \mapsto \big(f(u)\big)^{1/(q+1)}(u)^{q/(q+1)}, \qquad q > 0$$

is a contraction with respect to the Thompson's metric, whereas f itself is not contractive, in general.

2. A Fixed Point Theorem

We consider the following conditions for $g : (0,\infty) \times K^\circ \to K^\circ$ and f defined by (1):

$$\forall u \in K^\circ : \quad 0 < s \leq t \;\Rightarrow\; g(t,u) \leq g(s,u), \tag{2}$$

$$\forall t \in (0,\infty) : \quad 0 \ll u \leq v \;\Rightarrow\; g(t,u) \leq g(t,v), \tag{3}$$

$$\exists q > 0 \; \forall (t,u) \in (0,\infty) \times K^\circ \; \forall \alpha \geq 1 : g(\alpha t, u) \geq \frac{1}{\alpha^{2q}} g(t,u), \tag{4}$$

$$\exists p > 0 \; \forall (t,u) \in (0,\infty) \times K^\circ \; \forall \alpha \geq 1 : \; g(t, \alpha u) \leq \alpha^p g(t,u), \tag{5}$$

$$f(K^\circ) \subseteq E. \tag{6}$$

Theorem 2.1. *Let $g : (0,\infty) \times K^\circ \to K^\circ$ and f satisfy (2) − (6) for some $p \in (0,1)$ and some $q > 0$, and let $F : K^\circ \to (0,\infty)^X$ be defined by*

$$F(u)(x) = \big(f(u)(x)\big)^{1/(q+1)} \big(u(x)\big)^{q/(q+1)}.$$

Then $f(K^\circ) \subseteq K^\circ$ and $F(K^\circ) \subseteq K^\circ$. Moreover F is a contraction with respect to d with contraction constant $(p+q)/(1+q)$, and F and f have the same unique fixed point in K°.

Proof. First note that

$$u^r \in K^\circ, \qquad (u \in K^\circ,\ r > 0),$$

since E is a Banach algebra. Let $u \in K^\circ$ and $\sigma \coloneqq \sup_{x\in X} u(x)$. We have

$$f(u)(x) = g\big(u(x), u\big)(x) \geq_{(2)} \ g(\sigma, u)(x) \qquad (x \in X).$$

Thus $f(u) \in K^\circ$ since $f(u) \in E$ and $g(\sigma, u) \in K^\circ$, and so $F(u) \in K^\circ$.

Now, let $u, v \in K^\circ$, fix $x \in X$ and first assume that $v(x) \leq u(x)$. Then

$$\alpha \coloneqq \sup_{y\in X} \frac{u(y)}{v(y)} \geq 1 \ \text{ and } \ u \leq \alpha v.$$

We have

$$\begin{aligned}
\log\left(\frac{F(u)(x)}{F(v)(x)}\right) &= \frac{1}{1+q} \log\left(\frac{g\big(u(x), u\big)(x)}{g\big(v(x), v\big)(x)} \cdot \frac{(u(x))^q}{(v(x))^q}\right) \\
&\leq_{(2),(3)} \ \frac{1}{1+q} \log\left(\frac{g\big(v(x), \alpha v\big)(x)}{g\big(v(x), v\big)(x)} \cdot \frac{\big(u(x)\big)^q}{\big(v(x)\big)^q}\right) \\
&\leq_{(5)} \ \frac{1}{1+q} \log\left(\alpha^p \frac{\big(u(x)\big)^q}{\big(v(x)\big)^q}\right) \\
&= \frac{p}{1+q} \log(\alpha) + \frac{q}{1+q} \log\left(\frac{u(x)}{v(x)}\right) \\
&= \frac{p}{1+q} \sup_{y\in X} \log\left(\frac{u(y)}{v(y)}\right) + \frac{q}{1+q} \log\left(\frac{u(x)}{v(x)}\right) \\
&\leq \frac{p+q}{1+q} \sup_{y\in X} \left|\log\left(\frac{u(y)}{v(y)}\right)\right| \\
&= \frac{p+q}{1+q} d(u,v).
\end{aligned}$$

Next, we have

$$\begin{aligned}
\log\left(\frac{F(u)(x)}{F(v)(x)}\right) &= \frac{1}{1+q}\log\left(\frac{g\big(\big(u(x)/v(x)\big)v(x),u\big)(x)}{g\big(v(x),v\big)(x)}\cdot\frac{\big(u(x)\big)^q}{\big(v(x)\big)^q}\right)\\
&\geq_{(4)}\ \frac{1}{1+q}\log\left(\frac{g(v(x),u)(x)}{g(v(x),v)(x)}\cdot\frac{(v(x))^q}{(u(x))^q}\right)\\
&=: c.
\end{aligned}$$

Now, if $v \leq u$ then

$$\begin{aligned}
c &\geq_{(3)} \frac{1}{1+q}\log\left(\frac{\big(v(x)\big)^q}{\big(u(x)\big)^q}\right)\\
&= -\frac{q}{1+q}\log\left(\frac{u(x)}{v(x)}\right)\\
&\geq -\frac{q}{1+q}d(u,v)\\
&\geq -\frac{p+q}{1+q}d(u,v).
\end{aligned}$$

If $v \not\leq u$ we set

$$\beta := \sup_{y\in X}\frac{v(y)}{u(y)}.$$

Then $\beta \geq 1$, $v \leq \beta u$, and we get

$$\begin{aligned}
c &\geq_{(3)}\ \frac{1}{1+q}\log\left(\frac{g\big(v(x),u\big)(x)}{g\big(v(x),\beta u\big)(x)}\cdot\frac{\big(v(x)\big)^q}{\big(u(x)\big)^q}\right)\\
&\geq_{(5)}\ \frac{1}{1+q}\log\left(\frac{1}{\beta^p}\frac{\big(v(x)\big)^q}{\big(u(x)\big)^q}\right)\\
&= -\frac{1}{1+q}\log\left(\beta^p\right)-\frac{q}{1+q}\log\left(\frac{u(x)}{v(x)}\right)\\
&= -\frac{p}{1+q}\sup_{y\in X}\log\left(\frac{v(y)}{u(y)}\right)-\frac{q}{1+q}\log\left(\frac{u(x)}{v(x)}\right)\\
&\geq -\frac{p+q}{1+q}\sup_{y\in X}\left|\log\left(\frac{v(y)}{u(y)}\right)\right|\\
&= -\frac{p+q}{1+q}d(u,v).
\end{aligned}$$

Summing up we have

$$\left|\log\left(\frac{F(u)(x)}{F(v)(x)}\right)\right| \leq \frac{p+q}{1+q}d(u,v). \tag{7}$$

By interchanging u and v we see that (7) holds in case that $u(x) \le v(x)$, as well. Since $x \in X$ was arbitrary we get

$$d\big(F(u), F(v)\big) = \sup_{x \in X} \left| \log\left(\frac{F(u)(x)}{F(v)(x)}\right) \right| \le \frac{p+q}{1+q} d(u,v),$$

and

$$\frac{p+q}{1+q} < 1.$$

According to Banach's Fixed Point Theorem F has a unique fixed point $w \in K^\circ$, which is the unique fixed point of f, since

$$F(u) = u \iff \big(f(u)\big)^{1/(q+1)} = (u)^{1-q/(q+1)} \iff f(u) = u. \qquad ☑$$

3. Example

Let $X = [0,1]$ and $E = C\big([0,1], \mathbb{R}\big)$, let $k : [0,1] \times [0,1] \to (0,\infty)$ be continuous, and let $n \in \mathbb{N}$ and $a_0, \dots, a_n \in K$ with

$$0 \ll \sum_{k=0}^{n} a_k.$$

We look for continuous positive solutions of the integral equation

$$\int_0^1 k(x,\xi) \sqrt{u(\xi)}\, d\xi = \sum_{k=0}^{n} a_k(x) \big(u(x)\big)^{k+1}. \tag{8}$$

By setting $g : (0,\infty) \times K^\circ \to K^\circ$,

$$g(t,u)(x) = \frac{1}{\sum_{k=0}^{n} a_k(x) t^k} \int_0^1 k(x,\xi) \sqrt{u(\xi)}\, d\xi$$

we find that $u \in K^\circ$ is a solution of (8) if and only if

$$f(u)(x) = g\big(u(x), u\big)(x) = u(x) \qquad \big(x \in [0,1]\big).$$

Moreover g and f satisfy (2)-(6) with $p = 1/2$ and $q = n/2$. Thus f has a unique fixed point in K° and

$$u \mapsto F(u) = \big(f(u)\big)^{2/(n+2)} u^{n/(n+2)}$$

is a contraction with respect to d and with contraction constant

$$\frac{1+n}{2+n}.$$

For example the case $n = 2$, $k \equiv 1$, $a_0 = a_1 = 0$, $a_2 = 1$ leads to

$$f(u)(x) = \frac{1}{\big(u(x)\big)^2} \int_0^1 \sqrt{u(\xi)}\, d\xi,$$

which is not a contraction with respect to d, since the successive approximations starting with $u(x) = \gamma\big(x \in [0,1]\big)$ diverge in (K°, d) if $\gamma \in (0,\infty) \setminus \{1\}$.

Figure 1 shows the numerically computed positive solution of

$$\int_0^1 \exp(x\xi)\sqrt{u(\xi)}\, d\xi = \big(\sin(2\pi x)\big)^2 u(x)^3 + \big(\cos(2\pi x)\big)^2 u(x)^4$$

by iteration of F.

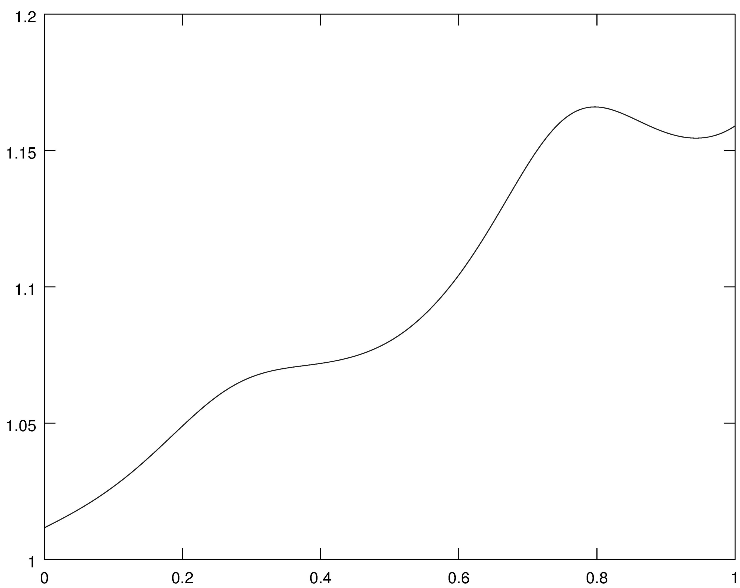

FIGURE 1

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