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Transitivity of the Induced Map $C_n(f)$

Transitividad de la función inducida $C_n(f)$

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Abstract. A map $f : X \to X$, where X is a continuum, is said to be transitive if for each pair U and V of nonempty open subsets of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$. In this paper, we show relationships between transitivity of f and its induced maps $C_n(f)$ and $F_n(f)$, for some $n \in \mathbb{N}$. Also, we present conditions on X such that given a map $f : X \to X$, the induced function $C_n(f) : C_n(X) \to C_n(X)$ is not transitive, for any $n \in \mathbb{N}$.

Key words and phrases. Transitivity, Induced map, Continua, Hyperspaces of continua, Symmetric products, Continuum of type λ, Dendrites.

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Resumen. Una función continua $f : X \to X$, definida en un continuo X, se dice transitiva si para cada U y V abiertos diferentes del vacío de X, existe $n \in \mathbb{N}$, tal que $f^n(U) \cap V \neq \varnothing$. En este artículo mostramos relaciones entre la transitividad de f y las funciones inducidas $C_n(f)$ y $F_n(f)$, para alguna $n \in \mathbb{N}$. Además, presentamos condiciones sobre X para que dada una función $f : X \to X$, la función inducida $C_n(f) : C_n(X) \to C_n(X)$ no sea transitiva, para ninguna $n \in \mathbb{N}$.

Palabras y frases clave. Transitividad, función inducida, continuos, hiperespacios de continuos, producto simétrico, continuos tipo λ, dendritas.

1. Introduction

A map $f : X \to X$, where X is a continuum, is said to be transitive if for each pair U and V of nonempty open subsets of X, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \varnothing$. In [8], Robert Devaney says that a map $f : X \to X$, where X is a metric space, is chaotic on X provided that: i) f has sensitive dependence on initial conditions, ii) the periodic points of f are dense in X, and iii) f is transitive. In [3], it is shown that if the periodic points of f are dense and f is transitive, then f has sensitive dependence on initial conditions; i. e., condition i) is not necessary in Devaney's definition. Also, it is known that if f is defined on $[0, 1]$, then f is chaotic if and only if f is transitive [4]. Therefore, transitivity is an important property in chaotic dynamical systems.

A *continuum* is a compact, connected and nonempty metric space. Let X be a continuum and let $n \in \mathbb{N}$. The *$n-$fold hyperspace of* X, denoted by $C_n(X)$, is defined as the set $C_n(X) = \{A \subset X : A \text{ is closed, nonempty and has at most } n \text{ components}\}$. The *$n-$fold symmetric product*, denoted by $F_n(X)$, is defined as $F_n(X) = \{A \subset X : A \text{ is nonempty and has at most } n \text{ points}\}$. Given a map $f : X \to X$ and $n \in \mathbb{N}$, it is possible to define the induced maps $C_n(f) : C_n(X) \to C_n(X)$ and $F_n(f) : F_n(X) \to F_n(X)$. In Section 3 of this paper, after the introduction and preliminaries, we study all possible relationships between the following three statements:

(1) f is transitive.

(2) $C_n(f)$ is transitive, for some $n \in \mathbb{N}$.

(3) $F_n(f)$ is transitive, for some $n \in \mathbb{N}$.

In Section 4, we prove that if either X contains a free arc, X is a continuum of type λ or X is a dendrite, then the induced map $C_n(f) : C_n(X) \to C_n(X)$ is not transitive, for any $n \in \mathbb{N}$. The transitivity of $C_1(f)$ was studied by G. Acosta, A. Illanes and H. Mendez in [1].

2. Preliminaries

A *continuum* is a compact, connected and nonempty metric space. An *arc* is any space homeomorphic to the closed interval $[0, 1]$. Also, if $h : [0, 1] \to \alpha$ is a homeomorphism, then $p = h(0)$ and $q = h(1)$ are called *the end points of* the arc α; one says that α is an arc from p to q. Given an arc α with end points p and q in a continuum X, we say that α is a *free arc* if $\alpha \setminus \{p, q\}$ is an open subset of X. A *map* is assumed to be a continuous function. If X is a continuum, then given $A \subset X$, the closure and the interior are denoted by $\overline{A}$ and $\text{Int}(A)$, respectively. A *dendrite* is a locally connected continuum which contains no homeomorphic copy of $S^1 = \{z \in \mathbb{C} : ||z|| = 1\}$. A continuum X is said to be *irreducible* provided that there exist $p, q \in X$ such that no proper subcontinuum of X contains $\{p, q\}$; we say that X is *irreducible between p and q*. A map between continua $f : X \to Y$ is said to be *monotone* provided that $f^{-1}(y)$ is connected for each $y \in Y$. A continuum X which is irreducible between p and q is said to be of *type* λ if there is a monotone map $m : X \to [0, 1]$ such that $m(p) = 0$, $m(q) = 1$ and $\text{Int}\left(m^{-1}(t)\right) = \varnothing$ for each $t \in [0, 1]$ (see [11] for a complete investigation about continua of type λ).

Given a continuum X and a positive integer n, the *$n-$fold hyperspace of* X, denoted by $C_n(X)$, is defined as the set $C_n(X) = \{A \subset X : A$ is closed, nonempty and has at most n components$\}$ topologized by the Hausdorff metric [10, Definition 2.1]. It is well known that $C_n(X)$ is an arcwise connected continuum [13, Corollary 1.8.12]. The *$n-$ fold symmetric product*, denoted by $F_n(X)$, is defined for $F_n(X) = \{A \in C_n(X) : A$ has at most n points$\}$ [6]. $F_n(X)$ is endowed with the relative topology as a subspace of $C_n(X)$.

Let X be a continuum and let $D_1, \ldots, D_k$ be nonempty subsets of X. We define $\langle D_1, \ldots, D_k \rangle = \big\{A \in C_n(X) : A \subset \cup_{i=1}^{k} D_i$ and $A \cap D_i \neq \varnothing$ for each $i \in \{1, \ldots, k\}\big\}$. Let $\mathcal{B} = \big\{\langle U_1, \ldots, U_k \rangle : U_i$ is open and $k \in \mathbb{N}\big\}$; $\mathcal{B}$ is a base for the topology generated by the Hausdorff metric on $C_n(X)$ [13, Theorem 1.8.16].

Let $f : X \to Y$ be a map between continua and let $n \in \mathbb{N}$. Then the function $C_n(f) : C_n(X) \to C_n(Y)$ given by $C_n(f)(A) = f(A)$ for each $A \in C_n(X)$, is called the *induced map between the $n-$fold hyperspaces $C_n(X)$ and $C_n(Y)$*. The map $F_n(f) : F_n(X) \to F_n(Y)$ given by $F_n(f) = C_n(f)|_{F_n(X)}$ is called the *induced map between the $n-$fold symmetric products $F_n(X)$ and $F_n(Y)$*. In [10, p. 188], it is shown that $C_n(f)$ is a map. Regarding induced maps, the reader may see [7, 10, 9, 13].

Given a map $f : X \to X$ and $n \in \mathbb{N}$, f^n means the composition $f \circ f \circ \cdots \circ f$, n times. If $n = 0$, f^0 is the identity map. Let $x \in X$. The *orbit* of x, denoted by $\mathcal{O}(x, f)$, is the set of points $\mathcal{O}(x, f) = \big\{f^n(x) : n \in \mathbb{N} \cup \{0\}\big\}$. The *$\omega-$limit of* x, denoted by $\omega(x, f)$, is given as the set of accumulation points of the sequence $\mathcal{O}(x, f)$. It is easy to see that $\omega(x, f) = \omega(f^k(x), f)$ for each $k \in \mathbb{N}$.

Definition 2.1. Let X be a continuum and let $f : X \to X$ be a map. We say that f is *transitive* provided that for each pair of nonempty open subsets U and V of X, there exists $n \in \mathbb{N}$, such that $f^n(U) \cap V \neq \varnothing$.

Definition 2.2. Let X be a continuum and let $f : X \to X$ be a map. We say that f is *exact* provided that for each nonempty open subset U of X, there exists $n \in \mathbb{N}$, such that $f^n(U) = X$.

The next claim follows easily from Definitions 2.1 and 2.2.

Claim 2.3. Let $f : X \to X$ be a map. If f is exact then f is transitive.

Theorem 2.4. *[5, Proposition 39, p.155] Let X be a continuum and let $f : X \to X$ be a map. Then f is transitive if and only if there exists $x \in X$ such that $\omega(x, f) = X$.*

3. On $C_n(f), F_n(f)$ and f

Given a continuum X and a map $f : X \to X$, we study the relationships between the following three statements:

(1) f is transitive.

(2) $C_n(f)$ is transitive, for some $n \in \mathbb{N}$.

(3) $F_n(f)$ is transitive, for some $n \in \mathbb{N}$.

Lemma 3.1. *Let X be a continuum, let $n \in \mathbb{N}$ and let $f : X \to X$ be an exact map. If $B \in C_n(X)$ is such that $\operatorname{Int}(B) \neq \varnothing$, then $\omega\big(B, C_n(f)\big) = \{X\}$.*

***Proof*.** Since f is exact, there exists $k \in \mathbb{N}$ such that $f^k\big(\operatorname{Int}(B)\big) = X$. Thus, $f^m(B) = X$ for each $m \geq k$. Therefore, $\omega\big(B, C_n(f)\big) = \{X\}$. ☑

Notice that if $f : S^1 \to S^1$ is defined by $f(z) = ze^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then f is transitive and the induced map $F_n(f)$ is not transitive, for any $n \in \mathbb{N} \setminus \{1\}$ [9, Example 3.8]. Therefore, (1) does not imply (3).

Claim 3.2. Let X be a continuum and let $n \in \mathbb{N}$. The family $\mathcal{B}_0 = \big\{\langle U_1, \ldots, U_s\rangle \cap F_n(X) : U_i \text{ is open of } X \text{ and } s \leq n\big\}$, is a base for the topology on $F_n(X)$.

***Proof*.** Let $\langle V_1, \ldots, V_k\rangle$ be an open subset of $C_n(X)$ such that $\langle V_1, \ldots, V_k\rangle \cap F_n(X) \neq \varnothing$. Let $\{x_1, \ldots, x_s\} \in \langle V_1, \ldots, V_k\rangle \cap F_n(X)$. Note that $s \leq n$. Let $U_i = \bigcap \big\{V_j : x_i \in V_j,\ j \in \{1, \ldots, k\}\big\}$, for each $i \in \{1, \ldots, s\}$. It is not difficult to see that $\{x_1, \ldots, x_s\} \in \langle U_1, \ldots, U_s\rangle \cap F_n(X) \subset \langle V_1, \ldots, V_k\rangle \cap F_n(X)$ and the proof is complete. ☑

Proposition 3.3. *Let X be a continuum, let $n \in \mathbb{N}$ and let $f : X \to X$ be a map. If f is exact, then $F_n(f)$ is transitive.*

***Proof*.** Let $\langle U_1, \ldots, U_l\rangle \cap F_n(X)$ and $\langle V_1, \ldots, V_s\rangle \cap F_n(X)$ be open subsets of $F_n(X)$ such that $l, s \leq n$ (Claim 3.2). Suppose that $l \leq s \leq n$. Since f is exact, there exists $k \in \mathbb{N}$ such that $f^k(U_i) = X$ for each $i \in \{1, \ldots, l\}$. Hence, $f^k(U_i) \cap V_j \neq \varnothing$ for each $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, s\}$. Let $x_i \in U_i$ such that $f^k(x_i) \in V_i$, and let $x_j \in U_l$ such that $f^k(x_j) \in V_j$, for each $i \in \{1, \ldots, l\}$ and $j \in \{l+1, \ldots, s\}$. It is clear that $\{x_1, \ldots, x_s\} \in \langle U_1, \ldots, U_l\rangle \cap F_n(X)$ and $F_n(f)^k\big(\{x_1, \ldots, x_s\}\big) \in \langle V_1, \ldots, V_s\rangle \cap F_n(X)$. Therefore, $F_n(f)^k\big(\langle U_1, \ldots, U_l\rangle \cap F_n(X)\big) \cap \big(\langle V_1, \ldots, V_s\rangle \cap F_n(X)\big) \neq \varnothing$. Similarly, we conclude the result if we assume that $s \leq l \leq n$. ☑

The following shows that neither (1) nor (3) implies (2).

Proposition 3.4. *There exists a transitive map $f : X \to X$ such that $F_n(f)$ is transitive, for each $n \in \mathbb{N}$, and $C_n(f)$ is not transitive, for any $n \in \mathbb{N}$.*

***Proof*.** Let $f : S^1 \to S^1$ be defined by $f(z) = z^2$, for each $z \in S^1$. It is not difficult to see that f is exact. Hence, $F_n(f)$ is transitive, for each $n \in \mathbb{N}$, by Proposition 3.3.

Let $n \in \mathbb{N}$ and let $B \in C_n(X)$. We prove that $\omega\big(B, C_n(f)\big) \neq C_n(X)$. Suppose first that $B \in F_n(X)$. Then $\omega\big(B, C_n(f)\big) = \omega\big(B, F_n(f)\big) \subset F_n(X)$. Thus, $\omega\big(B, C_n(f)\big) \neq C_n(X)$. Now, we assume that $B \in C_n(X) \setminus F_n(X)$. Hence, $\mathrm{Int}(B) \neq \varnothing$. Therefore, by Lemma 3.1, $\omega\big(B, C_n(f)\big) = \{X\}$. The proof now follows from Theorem 2.4. ☑

Proposition 3.5. *Let X be a continuum, let $n \in \mathbb{N}$ and let $f : X \to X$ be a map. If $B \in C_n(X)$ $\big($or $B \in F_n(X)\big)$ is such that $\omega\big(B, C_n(f)\big) = C_n(X)$ $\big($or $\omega\big(B, F_n(f)\big) = F_n(X)$, respectively$\big)$ and $p \in B$, then $\omega(p, f) = X$.*

Proof. Let U be an open subset of X. Since $\omega\big(B, C_n(f)\big) = C_n(X)$, there exists $k \in \mathbb{N}$ such that $f^k(B) \in \langle U \rangle$. Thus, $f^k(B) \subset U$ and $f^k(p) \in U$. Therefore, $\omega(p, f) = X$. ☑

Theorem 3.6. *Let X be a continuum and let $f : X \to X$ be a map. If either $C_n(f)$ or $F_n(f)$ is transitive, for some $n \in \mathbb{N}$, then f is transitive.*

Proof. It follows from Proposition 3.5 and Theorem 2.4. Another proof can be found in [2, Theorem 4]. ☑

Theorem 3.7 completes all possible relationships between (1), (2) and (3).

Theorem 3.7. *Let X be a continuum, let $n \in \mathbb{N}$ and let $f : X \to X$ be a map. If $C_n(f)$ is transitive then $F_n(f)$ is transitive.*

Proof. Suppose that $C_n(f)$ is transitive. Then, by Theorem 2.4, there exists $B \in C_n(X)$ such that $\omega\big(B, C_n(f)\big) = C_n(X)$.

We prove that $B \in C_n(X) \setminus C_{n-1}(X)$. Let $U_1, \ldots, U_n$ be pairwise disjoint, open subsets of X. Since $\omega\big(B, C_n(f)\big) = C_n(X)$, there is a positive integer k such that $f^k(B) \in \langle U_1, \ldots, U_n \rangle$. Thus, $f^k(B)$ has exactly n components and $B \in C_n(X) \setminus C_{n-1}(X)$.

Let $B_1, \ldots, B_n$ be the components of B. Let $x_i \in B_i$ for each $i \in \{1, \ldots, n\}$. Let $A = \{x_1, \ldots, x_n\} \in F_n(X)$. We prove that $\omega\big(A, F_n(f)\big) = F_n(X)$. Let $V_1, \ldots, V_s$ be pairwise disjoint open subsets of X, $s \leq n$. Since $\omega\big(B, C_n(f)\big) = C_n(X)$, $f^k(B) \in \langle V_1, \ldots, V_s \rangle$, for some $k \in \mathbb{N}$. Observe that each component of $f^k(B)$ intersects $f^k(A)$. Thus, $f^k(A) \in \langle V_1, \ldots, V_s \rangle$. Since $f^k(A) \in F_n(X)$, $f^k(A) \in \langle V_1, \ldots, V_s \rangle \cap F_n(X)$ and $\omega\big(A, F_n(f)\big) = F_n(X)$. Therefore, by Theorem 2.4, $F_n(f)$ is transitive. ☑

We finish this section with a question.

Question 3.8. Let X be a continuum and let $f : X \to X$ be a map. If $m \neq n$ and $C_n(f)$ is transitive, then does it follow that $C_m(f)$ is transitive?

4. Transitivity of $C_n(f)$

In [1, Teorema 7.18], it is shown a map $f : [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ such that the induced map $C_1(f)$ is transitive. It is not known, another map $f : X \to X$, where X is a continuum, such that $C_1(f)$ is transitive. In particular, we do not know if there exists a map f defined on a finite-dimensional continuum such that $C_1(f)$ is transitive [1, Question 7.20]. In this section, with similar arguments to those given in [1], we present some particular cases when the induced map $C_n(f)$ cannot be transitive, for any $n \in \mathbb{N}$.

The proof of the following lemma is the same as the one given in [1, Theorem 4.3] and will be omitted.

Lemma 4.1. *Let X be a continuum, let $n \in \mathbb{N}$ and let $f : X \to X$ be a map. If $B \in C_n(X)$ is such that $\omega\big(B, C_n(f)\big) = C_n(X)$, then $\operatorname{Int}\big(f^k(B)\big) = \varnothing$, for each $k \in \mathbb{N}$.*

Theorem 4.2. *Let X be a continuum and let $f : X \to X$ be a map. If X contains a free arc, then the induced map $C_n(f) : C_n(X) \to C_n(X)$ is not transitive, for any $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Suppose that $C_n(f)$ is transitive. Then, by Theorem 2.4, there exists $B \in C_n(X)$ such that $\omega\big(B, C_n(f)\big) = C_n(X)$. Let $\alpha \subset X$ be a free arc with end points p and q. Let U and V be nonempty, disjoint open subsets of $\alpha \setminus \{p,q\}$. Let $W_1, \dots, W_{n-1}$ be nonempty, pairwise disjoint and open subsets of $X \setminus \alpha$. Observe that $\mathcal{U} = \big\langle \alpha \setminus \{p,q\}, U, V, W_1, \dots, W_{n-1} \big\rangle$ is a nonempty open subset of $C_n(X)$. Since $\omega\big(B, C_n(f)\big) = C_n(X)$, $f^k(B) \in \mathcal{U}$ for some $k \in \mathbb{N}$. Notice that $W_1, \dots, W_{n-1}$ and $\alpha \setminus \{p,q\}$ are n nonempty pairwise disjoint open subsets of X. Thus, $f^k(B)$ has exactly n components. Let B_0 be the component of $f^k(B)$ such that $B_0 \subset \alpha \setminus \{p,q\}$. Since $B_0 \cap U \neq \varnothing$, $B_0 \cap V \neq \varnothing$ and $U \cap V = \varnothing$, we have that $\operatorname{Int}(B_0) \neq \varnothing$. Hence, $\operatorname{Int}\big(f^k(B)\big) \neq \varnothing$, contradicting Lemma 4.1. Therefore, $C_n(f)$ is not transitive. ☑

The next lemma is a simple result that follows from the definition of continuum of type λ.

Lemma 4.3. *Let X be a continuum of type λ, where X is irreducble between p and q and let $m : X \to [0,1]$ be a monotone map such that $m(p) = 0, m(q) = 1$ and $\operatorname{Int}\big(m^{-1}(t)\big) = \varnothing$ for each $t \in [0,1]$. If K is a subcontinuum of X such that $m(K) = [a,b]$, where $a < b$, then $\operatorname{Int}(K) \neq \varnothing$.*

Proof. Since m is monotone, both $m^{-1}\big([0,a]\big)$ and $m^{-1}\big([b,1]\big)$ are proper subcontinua of X. Notice that, $m^{-1}\big([0,a]\big) \cap m^{-1}\big([b,1]\big) = \varnothing$, $m^{-1}(a) \cap K \neq \varnothing$ and $m^{-1}(b) \cap K \neq \varnothing$. Thus, $m^{-1}\big([0,a]\big) \cup K \cup m^{-1}\big([b,1]\big)$ is a continuum such that $\{p,q\} \subset m^{-1}\big([0,a]\big) \cup K \cup m^{-1}\big([b,1]\big)$. Since X is irreducible between p and

q, we have that $X = m^{-1}\big([0,a]\big) \cup K \cup m^{-1}\big([b,1]\big)$. Therefore, $m^{-1}(a,b) \subset K$ and $\operatorname{Int}(K) \neq \varnothing$. ☑

Corollary 4.4. *Let X be a continuum of type λ, where X is irreducble between p and q, let $m : X \to [0,1]$ be a monotone map such that $m(p) = 0$, $m(q) = 1$ and $\operatorname{Int}\left(m^{-1}(t)\right) = \varnothing$ for each $t \in [0,1]$, and let $n \in \mathbb{N}$. If $A \in C_n(X)$ is such that $m(A)$ has a nondegenerate component, then $\operatorname{Int}(A) \neq \varnothing$.*

Proof. Let $A \in C_n(X)$ be such that $m(A)$ has a nondegenerate component. Then A has a component A_0 such that $m(A_0)$ is nondegenerate. Now, the corollary follows from Lemma 4.3. ☑

Theorem 4.5. *Let X be a continuum and let $f : X \to X$ be a map. If X is a continuum of type λ, then $C_n(f)$ is not transitive, for any $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Suppose that $C_n(f)$ is transitive. Then, by Theorem 2.4, there exists $B \in C_n(X)$ such that $\omega\big(B, C_n(f)\big) = C_n(X)$. We show that $\operatorname{Int}\left(f^k(B)\right) \neq \varnothing$, for some $k \in \mathbb{N}$.

Let $m : X \to [0,1]$ be a monotone map such that $m(p) = 0, m(q) = 1$ and $\operatorname{Int}\left(m^{-1}(t)\right) = \varnothing$ for each $t \in [0,1]$, where X is irreducible between p and q. Let $0 = t_0 < t_1 < \cdots < t_{n-1} < a < b < t_n = 1$. Let $U_i = m^{-1}\big((t_{i-1}, t_i)\big)$, for each $i \in \{1, \ldots, n\}$. Notice that $U_1, \ldots, U_n$ are nonempty, pairwise disjoint open subsets of X. Let $V = m^{-1}\big((t_{n-1}, a)\big)$ and $W = m^{-1}\big((b, t_n)\big)$ be disjoint open subsets of U_n. Observe that $\mathcal{U} = \langle U_1, \ldots, U_n, V, W\rangle$ is a nonempty open subset of $C_n(X)$. Since $\omega\big(B, C_n(f)\big) = C_n(X)$, $f^k(B) \in \mathcal{U}$ for some $k \in \mathbb{N}$. Thus, $f^k(B)$ has exactly n components. Let B_0 be the component of $f^k(B)$ such that $B_0 \subset U_n$. Since $B_0 \cap V \neq \varnothing$, $B_0 \cap W \neq \varnothing$ and $V \cap W = \varnothing$, we have that $[a,b] \subset m(B_0)$ is nondegenerate. Therefore, by Corollary 4.4, $\operatorname{Int}\left(f^k(B)\right) \neq \varnothing$, which contradicts Lemma 4.1. Therefore, $C_n(f)$ is not transitive. ☑

In the remainder of this paper, we will focus on maps which are defined on dendrites. A *cut point* of X is a point p such that $X \setminus \{p\}$ is not connected. We write $\operatorname{Cut}(X)$ to represent the family of cut points of a dendrite X.

Proposition 4.6. *Let X be a dendrite, let $n \in \mathbb{N}$ and let $f : X \to X$ be a map. If $p \in \operatorname{Cut}(X)$ and $B \in C_n(X)$ is such that $\omega\big(B, C_n(f)\big) = C_n(X)$, then $p \in f^k(B)$, for some $k \in \mathbb{N}$.*

Proof. Let U and V be nonempty, disjoint and open subsets of X such that $X \setminus \{p\} = U \cup V$. The sets $U \cup \{p\}$ and $V \cup \{p\}$ are subcontinua of X, by [12, Theorem 4, p. 133]. Also, by [12, Theorem 5, p. 173], there exist subcontinua M and N of X, such that $\{p\} \subsetneq M \subsetneq U \cup \{p\}$ and $\{p\} \subsetneq N \subsetneq V \cup \{p\}$. Hence, $M \cup N$ is a continuum, $(M \cup N) \cap U \neq \varnothing$, $(M \cup N) \cap V \neq \varnothing$ and $M \cup N \neq X$. It is not difficult to show that there are nonempty open subsets $W_1, \ldots, W_{n-1}$ and W of X such that:

(1) $M \cup N \subset W$ and $X \setminus \overline{W} \neq \varnothing$;

(2) $\cup_{i=1}^{n-1} W_i \subset X \setminus \overline{W}$;

(3) $W_i \cap W_j = \varnothing$ for $i \neq j$.

Let $\mathcal{U} = \langle W, U \cap W, V \cap W, W_1, \ldots, W_{n-1} \rangle$ be an open subset of $C_n(X)$. If $x_i \in W_i$, for each $i \in \{1, \ldots, n-1\}$, then $M \cup N \cup \{x_1, \ldots, x_{n-1}\} \in \mathcal{U}$. Thus, $\mathcal{U} \neq \varnothing$.

Since $\omega(B, C_n(f)) = C_n(X)$, $f^k(B) \in \mathcal{U}$, for some $k \in \mathbb{N}$. Thus, $f^k(B)$ has exactly n components. Let B_0 be the component of $f^k(B)$ such that $B_0 \subset W$. Since $B_0 \cap (U \cap W) \neq \varnothing$, $B_0 \cap (V \cap W) \neq \varnothing$, we have that $p \in B_0$. Therefore, $p \in f^k(B)$ and our proof is complete. ☑

If $a, b \in X$ and X is a dendrite, then there exists a unique arc α joining a and b in X. We will denote that α by ab. The idea of the following proof is similar to [1, Theorem 6.2].

Theorem 4.7. *Let X be a continuum and let $f : X \to X$ be a map. If X is a dendrite, then the induced map $C_n(f) : C_n(X) \to C_n(X)$ is not transitive, for any $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. Suppose that $C_n(f)$ is transitive. Then, by Theorem 2.4, there exists $B \in C_n(X)$ such that $\omega(B, C_n(f)) = C_n(X)$. The family $\mathrm{Cut}(X)$ has uncountably infinitely many points, by [14, Theorem 10.8]. Let $p \in \mathrm{Cut}(X)$. Then, there exists $k \in \mathbb{N}$ such that $p \in f^k(B)$, by Proposition 4.6. Thus, since $\omega(f^k(B), C_n(f)) = C_n(X)$, by Proposition 3.5, $\omega(p, f) = X$. Therefore, $f^l(p) \neq f^s(p)$ for each $l \neq s$. Since $p \in \mathrm{Cut}(X)$ and $f(p) \neq p$, we have that there exists a component W of $X \setminus \{p\}$ such that $f(p) \in W$. Observe that $L = W \cup \{p\}$ is a proper subcontinuum of X, by [12, Theorem 4, p. 133].

Claim 4.8. There exists $q_1 \in pf(p)$, $q_1 \neq p$ such that $q_1 \in \mathrm{Cut}(X)$ and $q_1 \in pf(q_1) \cap f(pq_1)$.

Let $r : X \to pf(p)$ be the first point map defined in [14, Lemma 10.24]; i. e., $r(x) \in pf(p)$ is such that $r(x)$ is a point of any arc in X from x to any point of $pf(p)$. Thus, $(r \circ f|_{pf(p)}) : pf(p) \to pf(p)$ is a map and $C_1 = \{z \in pf(p) : (r \circ f|_{pf(p)})(z) = z\}$ is a nonempty closed subset of X. Let $q_1 \in C_1$ be the closest point to p in $pf(p)$. Since $p \neq f(p) = r(f(p))$, $q_1 \neq p$.

Since $r(f(q_1)) = q_1$, it is clear that $q_1 \in pf(q_1) \cap f(q_1)f(p)$. Also, $f(q_1)f(p) \subset f(pq_1)$. Thus, $q_1 \in pf(q_1) \cap f(pq_1)$. We show that $q_1 \in \mathrm{Cut}(X)$. Suppose that $q_1 \neq f(p)$. Hence, $q_1 \in pf(p) \setminus \{p, f(p)\}$ and $q_1 \in \mathrm{Cut}(X)$, by [14, Theorem 10.7]. Similarly, assume that $q_1 = f(p)$. Since $\omega(p, f) = X$, $f(f(p)) \neq f(p)$ and $f(f(p)) \notin pf(p)$. Therefore, $q_1 \in pf(f(p)) \setminus \{p, f(f(p))\}$ and, by [14, Theorem 10.7], $q_1 \in \mathrm{Cut}(X)$.

Claim 4.9. The arc $pq_1 \subset f^m(B)$, for some $m \in \mathbb{N}$.

Since $p, q_1 \in \mathrm{Cut}(X)$, there are $a, b \in X$ such that $pq_1 \subset ab \setminus \{a, b\}$ [14, Theorem 10.7]. Let $U_1, \dots, U_n, V_1$ and V_2 be nonempty, connected and open subsets of X such that:

(1) $U_1, \dots, U_n$ are pairwise disjoint subsets of X.

(2) $ab \subset U_1$.

(3) $V_1 \cup V_2 \subset U_1 \setminus pq_1$, $a \in V_1$, $b \in V_2$ and $V_1 \cap V_2 = \varnothing$.

Since X is a dendrite, it is not difficult to check that $U_1, \dots, U_n, V_1$ and V_2 do indeed exist. Also, $\mathcal{U} = \langle V_1, V_2, U_1, \dots, U_n \rangle$ is nonempty open subset of $C_n(X)$. Since $\omega\big(B, C_n(f)\big) = C_n(X)$, $f^m(B) \in \mathcal{U}$ for some $m \in \mathbb{N}$. Thus, since $U_1, \dots, U_n$ are nonempty, pairwise disjoint subsets of X, we have that there exists a component B_0 of $f^m(B)$ such that $B_0 \subset U_1$. Furthermore, $B_0 \cap V_i \neq \varnothing$ for each $i \in \{1, 2\}$. Therefore, since X is hereditarily unicohenret and (3), $pq_1 \subset B_0$ and $pq_1 \subset f^m(B)$.

Observe that $q_1 \in f(pq_1)$, by Claim 4.8. Hence, $\big\{q_1, f(q_1)\big\} \subset f(pq_1) \subset f^{m+1}(B)$, by Claim 4.9. Since $f(pq_1)$ is connected, q_1 and $f(q_1)$ belong to the same component of $f^{m+1}(B)$. Therefore, $q_1 f(q_1) \subset f^{m+1}(B)$.

The proof of the following claim is similar to the proof of Claim 4.8; we have to use the cut point q_1 instead of p.

Claim 4.10. There exists $q_2 \in q_1 f(q_1)$, $q_2 \neq q_1$ such that $q_2 \in \mathrm{Cut}(X)$ and $q_2 \in q_1 f(q_2) \cap f(q_1 q_2)$.

Notice that $q_2 f(q_2) \subset f^{m+2}(B)$, $q_1 f(q_1) \subset L$ and $q_2 f(q_2) \subset L$. Then we can inductively construct a sequence $(q_i)_{i=1}^{\infty} \subset \mathrm{Cut}(X)$ such that $q_i f(q_i) \subset L \cap f^{m+i}(B)$, for each $i \in \mathbb{N}$. Let $\mathcal{L} = \langle L, X \rangle \subset C_n(X)$. Since $C_n(X) \setminus \mathcal{L} = \langle X \setminus L \rangle$ is open, $\mathcal{L}$ is a proper nonempty, closed subset of $C_n(X)$. Furthermore, $f^{m+i}(B) \in \mathcal{L}$ for each $i \in \mathbb{N}$. Thus, $\omega\big(f^m(B), C_n(f)\big) \subset \mathcal{L}$. Since $\omega\big(B, C_n(f)\big) = \omega\big(f^m(B), C_n(f)\big)$, we contradict the fact that $\omega\big(B, C_n(f)\big) = C_n(X)$. Therefore, $C_n(f)$ is not transitive. ☑

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