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Solution of Some Fractional Order Telegraph Equations

Solución de algunas ecuaciones telegráficas de orden fraccional

Leda Galué

Universidad del Zulia, Maracaibo, Venezuela

Abstract. In recent years, there has been a great interest in fractional differential equations due to their frequent appearance in various fields, and their more accurate models of systems under consideration provided by fractional derivatives. In particular, fractional order telegraph equations have been considered and solved for many researchers, using different methods. In this paper we derived the solution of two homogeneous space-time fractional telegraph equations using the generalized differential transform method. The derivatives are considered in Caputo sense and the solutions are given in terms of generalized Mittag-Leffler function and the generalized Wright function. Further, various graphics are included which show the behavior of the solution obtained, and results given earlier by Momani, Odibat and Momani, Yildrim, Garg and Sharma, and Garg et al. are obtained as particular cases of ones our.

Key words and phrases. Fractional order telegraph equation, Generalized differential transform method, Caputo fractional derivative, Generalized Mittag-Leffler function, Generalized Wright function.

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Resumen. En los últimos años, ha habido un gran interés en las ecuaciones diferenciales fraccionales debido a su frecuente aparición en diversos campos, y a sus modelos más precisos de los sistemas en estudio proporcionados por las derivadas fraccionales. En particular, las ecuaciones telegráficas fraccionales han sido consideradas y resueltas por muchos investigadores, utilizando diferentes métodos. En este trabajo se derivó la solución de dos ecuaciones telegráficas homogéneas, con espacio-tiempo fraccionales, utilizando el método de la transformada diferencial generalizada. Las derivadas se consideran en el sentido Caputo y las soluciones se dan en términos de la función generalizada de Mittag-Leffler y la función de Wright generalizada. Además, se incluyen varias gráficas que muestran el comportamiento de la solución obtenida, y los resultados dados anteriormente por Momani, Odibat y Momani, Yildrim, Garg y Sharma, y Garg et al. se obtienen como casos particulares de los nuestros.

Palabras y frases clave. Ecuación telegráfica de orden fraccional, método de la transformada diferencial generalizada, derivada fraccional de Caputo, función generalizada de Mittag-Leffler, función de Wright generalizada.

1. Introduction

The telegraph equation is a partial differential equation with constant coefficients given by

$$u_{tt} - c^2 u_{xx} + a u_t + b u = 0, \tag{1}$$

where a, b and c are constants [5].

This equation is used in: signal analysis for transmission and propagation of electrical signals, modeling reaction diffusion in various branches of engineering sciences [5, 22], in the propagation of pressure waves in the study of pulsatile blood flow in arteries and in one-dimensional random motion of bugs along a hedge [5]. Also, the experimental results described in [7, 8, 20] seem to be better modelled by the telegraph equation than by the heat equation [3].

In recent years, there has been a great interest in fractional differential equations due to their frequent appearance in various fields and their more accurate models of systems under consideration provided by fractional derivatives. For example, fractional derivatives have been used successfully to model frequency dependent damping behavior of many viscoelastic materials [17] and the so called anomalous phenomena [13]. They are also used in modeling of many chemical processed, mathematical biology and many other problems in engineering [16, 20, 21]. In particular, fractional order telegraph equation models the transport of thermal neutrons inside a nuclear reactor with slab geometry [33].

Many authors have solved the classical and fractional order telegraph equations, among them we have: Biazar et al. [1], Cascaval et al. [3], Kaya [19], Momani [24], Odibat and Momani [26], Orsingher and Zhao [29], Orsingher and Beghin [28], Sevimlican [31], Yildirim [37], Huang [17], Garg et al. [13], Garg and Sharma [14], Hayat and Mohyud-Din [16], Hariharan et al. [15], El-Azab and El-Gamel [9], Gao and Chi [12], Das et al. [4], Dehghan et al. [6], Ford et al. [11], Karimi et al. [18], Figueiredo et al. [10], Xindong et al. [35], Yakubovich and Rodrigues [36], etc.

On the other hand, different methods such as, variational iteration method, transform method, Adomian decomposition method, juxtaposition of transforms, generalized differential transform method, homotopy perturbation method (HPM), wavelet method, matrix method, homotopy analysis method (HAM), are been used in order to solve fractional telegraph equations.

In this paper we derived the solution of two homogeneous space-time fractional telegraph equations using the generalized differential transform method. The derivatives are considered in Caputo sense and the solutions are given in terms of generalized Mittag-Leffler function and the generalized Wright function. Further, various graphics are included which show the behavior of the solutions obtained, and results given earlier by Momani [24], Odibat and Momani [26], Yildrim [37], Garg and Sharma [14], and Garg et al. [13] are obtained as particular cases of ones our.

Following we present some necessary definitions for the development for the next section.

Caputo Fractional Derivative Caputo fractional derivative of order α is defined as [2, 30]

$$D_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha-m+1}}\, d\xi, \qquad m-1 < \alpha \leq m,\ m \in \mathbb{N}. \quad (2)$$

The generalized Wright function An interesting generalization of the series ${}_pF_q$ is given by [32]

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1, A_1),\ \ldots\ ,(\alpha_p, A_p)\\(\beta_1, B_1),\ \ldots\ ,(\beta_q, B_q)\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n)\ldots\Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n)\ldots\Gamma(\beta_q + B_q n)} \frac{z^n}{n!}, \quad (3)$$

where $A_j > 0$ $(j = 1, \ldots, p)$; $B_j > 0$ $(j = 1, \ldots, q)$; $1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$, for suitably bounded values of $|z|$.

Mittag-Leffler function [23]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} = {}_1\Psi_1\left[\begin{smallmatrix}(1,1)\\(1,\alpha)\end{smallmatrix}; z\right],\ z \in \mathbb{C},\ \Re(\alpha) > 0. \quad (4)$$

For $\alpha = 1$, $E_\alpha(z)$ reduces to e^z.

Generalized Mittag-Leffler function Given by Wiman [34]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = {}_1\Psi_1\left[\begin{smallmatrix}(1,1)\\(\beta,\alpha)\end{smallmatrix}; z\right],\ z \in \mathbb{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0. \quad (5)$$

2. Generalized Two-Dimensional Differential Transform

Consider a function of two variables $u(x,y)$ and suppose that it can be represented as a product of two single variable functions, i.e. $u(x,y) = f(x)g(y)$. If $u(x,y)$ is analytic and differentiated continuously with respect to x and y in the domain of interest, then the generalized two-dimensional differential transform of the function $u(x,y)$ is given by ([25, 26, 27])

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\Big[\left(D^{\alpha}_{x_0}\right)^k\left(D^{\beta}_{y_0}\right)^h u(x,y)\Big]_{(x_0,y_0)}, \qquad (6)$$

where $0 < \alpha,\, \beta \leq 1$, $\left(D^{\alpha}_{x_0}\right)^k = D^{\alpha}_{x_0}D^{\alpha}_{x_0}\ldots D^{\alpha}_{x_0}$ (k times), $D^{\alpha}_{x_0}$ is defined by (2) and $U_{\alpha,\beta}(k,h)$ is the transformed function.

The generalized differential transform inverse of $U_{\alpha,\beta}(k,h)$ is given by

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{\alpha k}(y-y_0)^{\beta h}. \qquad (7)$$

Basic properties of the generalized two-dimensional differential transform:

Let $U_{\alpha,\beta}(k,h)$, $V_{\alpha,\beta}(k,h)$, $W_{\alpha,\beta}(k,h)$ be generalized two-dimensional differential transform of the functions $u(x,y)$, $v(x,y)$ and $w(x,y)$, respectively. Then

i) If $u(x,y) = v(x,y) \pm w(x,y)$ then $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$.

ii) If $u(x,y) = av(x,y)$, a is constant, then $U_{\alpha,\beta}(k,h) = aV_{\alpha,\beta}(k,h)$.

iii) If $u(x,y) = D^{\gamma}_{x_0}v(x,y)$, where $m-1 < \gamma \leq m$, $m \in \mathbb{N}$, then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}\big(k+\frac{\gamma}{\alpha},h\big)$.

iv) If $u(x,y) = D^{\mu}_{y_0}v(x,y)$, where $n-1 < \mu \leq n$, $n \in \mathbb{N}$, then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\beta h+\mu+1)}{\Gamma(\beta h+1)}V_{\alpha,\beta}\big(k,h+\frac{\mu}{\beta}\big)$.

3. Main Results

We consider the homogeneous space-time fractional telegraph equation

$$D^{(l+j)/l}_x u(x,t) = D^{p\beta}_t u(x,t) + D^{r\beta}_t u(x,t) + u(x,t), \quad 0 < x < 1, \quad t > 0 \quad (8)$$

where $l, j \in \mathbb{N}$, $j \leq l$, $\beta = 1/q$, $p,q,r \in \mathbb{N}$, $1 < p\beta \leq 2$, $0 < r\beta \leq 1$, $D^{(l+j)/l}_x \equiv D^{1/l}_x D^{1/l}_x \cdots D^{1/l}_x$ $\big((l+j)$ times$\big)$, $D^{p\beta}_t \equiv D^{\beta}_t D^{\beta}_t \cdots D^{\beta}_t$ (p times), $D^{r\beta}_t \equiv D^{\beta}_t D^{\beta}_t \cdots D^{\beta}_t$ (r times), $D^{1/l}_x$ and D^{β}_t are Caputo fractional derivatives, $p+r$ is odd, with the initial conditions:

$$u(0,t) = E_{\beta}(-t^{\beta}), \qquad (9)$$

$$D_x u(0,t) = E_{\beta}(-t^{\beta}). \qquad (10)$$

Applying generalized two dimensional differential transform with $\alpha = 1/l$, $x_0 = t_0 = 0$, to both side of (8), we obtain

$$U_{1/l,\beta}(k+l+j,h) = \frac{\Gamma\left(\frac{k}{l}+1\right)}{\Gamma\left(\frac{k}{l}+\frac{l+j}{l}+1\right)}\left[\frac{\Gamma(\beta h+\beta p+1)}{\Gamma(\beta h+1)}U_{1/l,\beta}(k,h+p) + \frac{\Gamma(\beta h+\beta r+1)}{\Gamma(\beta h+1)}U_{1/l,\beta}(k,h+r) + U_{1/l,\beta}(k,h)\right], \quad (11)$$

where we have used properties iii) and iv).

The application of (6) (with $\alpha = 1/l$, $x_0 = t_0 = 0$) to (9), and the use of the Caputo fractional derivative given in (2) joint with (4), lead us to

$$U_{1/l,\beta}(0,h) = \frac{(-1)^h}{\Gamma(\beta h+1)}. \quad (12)$$

From (6), (10), the property iii), (2) and (4), we get

$$U_{1/l,\beta}(l,h) = \frac{(-1)^h}{\Gamma(\beta h+1)}. \quad (13)$$

Now, making in (11) $k = 0$ we have

$$U_{1/l,\beta}(l+j,h) = \frac{1}{\Gamma\left(\frac{l+j}{l}+1\right)}\left[\frac{\Gamma(\beta h+\beta p+1)}{\Gamma(\beta h+1)}U_{1/l,\beta}(0,h+p) + \frac{\Gamma(\beta h+\beta r+1)}{\Gamma(\beta h+1)}U_{1/l,\beta}(0,h+r) + U_{1/l,\beta}(0,h)\right], \quad (14)$$

and from (12)

$$U_{1/l,\beta}(l+j,h) = \frac{(-1)^h}{\Gamma\left(\frac{l+j}{l}+1\right)\Gamma(\beta h+1)}, \quad (15)$$

where we have used the condition $p+r$ odd.

From (11) with $k = l$ and using (13)

$$U_{1/l,\beta}(2l+j,h) = \frac{(-1)^h}{\Gamma\left(\frac{l+j}{l}\times 1+2\right)\Gamma(\beta h+1)}, \quad (16)$$

and for $k = l+j$ in (11), by means of (15), we get

$$U_{1/l,\beta}(2l+2j,h) = \frac{(-1)^h}{\Gamma\left(\frac{l+j}{l}\times 2+1\right)\Gamma(\beta h+1)}. \quad (17)$$

Continuing with this procedure

$$U_{1/l,\beta}(3l+2j,h)=\frac{(-1)^h}{\Gamma\big(\frac{l+j}{l}\times 2+2\big)\Gamma(\beta h+1)}. \tag{18}$$

$$U_{1/l,\beta}(3l+3j,h)=\frac{(-1)^h}{\Gamma\big(\frac{l+j}{l}\times 3+1\big)\Gamma(\beta h+1)}. \tag{19}$$

$$U_{1/l,\beta}(4l+3j,h)=\frac{(-1)^h}{\Gamma\big(\frac{l+j}{l}\times 3+2\big)\Gamma(\beta h+1)}. \tag{20}$$

$$U_{1/l,\beta}(4l+4j,h)=\frac{(-1)^h}{\Gamma\big(\frac{l+j}{l}\times 4+1\big)\Gamma(\beta h+1)}, \tag{21}$$

and so successively.

Now, from the result (7) with $\alpha=1/l$, $x_0=t_0=0$

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{1/l,\beta}(k,h)\ x^{k/l}\ t^{\beta h} \tag{22}$$

and, making use of the equations (12), (13), (15)-(21) we can write

$$\begin{aligned}u(x,t)=\sum_{h=0}^{\infty}\frac{(-1)^h\ t^{\beta h}}{\Gamma(\beta h+1)}\Bigg[1+\frac{x}{\Gamma\big(\frac{l+j}{l}\times 0+2\big)}+\frac{x^{(l+j)/l}}{\Gamma\big(\frac{l+j}{l}\times 1+1\big)}+\\ \frac{x^{(2l+j)/l}}{\Gamma\big(\frac{l+j}{l}\times 1+2\big)}+\frac{x^{(2l+2j)/l}}{\Gamma\big(\frac{l+j}{l}\times 2+1\big)}+\frac{x^{(3l+2j)/l}}{\Gamma\big(\frac{l+j}{l}\times 2+2\big)}+\\ \frac{x^{(3l+3j)/l}}{\Gamma\big(\frac{l+j}{l}\times 3+1\big)}+\frac{x^{(4l+3j)/l}}{\Gamma\big(\frac{l+j}{l}\times 3+2\big)}+\frac{x^{(4l+4j)/l}}{\Gamma\big(\frac{l+j}{l}\times 4+1\big)}+\cdots\Bigg],\end{aligned} \tag{23}$$

that is,

$$\begin{aligned}u(x,t)=\sum_{h=0}^{\infty}\frac{(-1)^h\ t^{\beta h}}{\Gamma(\beta h+1)}\Bigg\{\Bigg[1+\frac{x^{(l+j)/l}}{\Gamma\big(\frac{l+j}{l}\times 1+1\big)}+\frac{\big(x^2\big)^{(l+j)/l}}{\Gamma\big(\frac{l+j}{l}\times 2+1\big)}+\\ \frac{\big(x^3\big)^{(l+j)/l}}{\Gamma\big(\frac{l+j}{l}\times 3+1\big)}+\frac{\big(x^4\big)^{(l+j)/l}}{\Gamma\big(\frac{l+j}{l}\times 4+1\big)}+\cdots\Bigg]+x\Bigg[\frac{1}{\Gamma\big(\frac{l+1}{l}\times 0+2\big)}+\\ \frac{x^{(l+j)/l}}{\Gamma\big(\frac{l+j}{l}\times 1+2\big)}+\frac{\big(x^2\big)^{(l+j)/l}}{\Gamma\big(\frac{l+j}{l}\times 2+2\big)}+\frac{\big(x^3\big)^{(l+j)/l}}{\Gamma\big(\frac{l+j}{l}\times 3+2\big)}+\cdots\Bigg]\Bigg\}\end{aligned} \tag{24}$$

which from the definitions given in (4) and (5) can be written as

$$u(x,t)=\Big\{E_{\frac{l+j}{l}}\big(x^{(l+j)/l}\big)+xE_{\frac{l+j}{l},2}\big(x^{(l+j)/l}\big)\Big\}E_{\beta}\big(-t^{\beta}\big). \tag{25}$$

Particular cases From (8)-(10) and (25):

i) If we put $l = j = 1$, the problem

$$\begin{aligned} D_x^2 u(x,t) &= D_t^{p\beta} u(x,t) + D_t^{r\beta} u(x,t) + u(x,t), \quad 0 < x < 1, \quad t > 0 \quad (26) \\ u(0,t) &= E_\beta(-t^\beta), \\ D_x u(0,t) &= E_\beta(-t^\beta) \end{aligned}$$

has the solution

$$u(x,t) = \Big\{E_2\big(x^2\big) + x\, E_{2,2}\big(x^2\big)\Big\} E_\beta\big(-t^\beta\big) \\ = E_1(x) E_\beta\big(-t^\beta\big) = e^x E_\beta\big(-t^\beta\big). \quad (27)$$

This is same as obtained by Garg et al. [13].

ii) If $l = 2$, $j = 1$, the problem

$$\begin{aligned} D_x^{3/2} u(x,t) &= D_t^{p\beta} u(x,t) + D_t^{r\beta} u(x,t) + u(x,t), \quad 0 < x < 1, \quad t > 0 \quad (28) \\ u(0,t) &= E_\beta(-t^\beta), \\ D_x u(0,t) &= E_\beta(-t^\beta) \end{aligned}$$

has as solution

$$u(x,t) = \Big\{E_{3/2}\big(x^{3/2}\big) + x E_{3/2,2}\big(x^{3/2}\big)\Big\} E_\beta\big(-t^\beta\big). \quad (29)$$

So, (29) is the same solution obtained by Garg et al. [13] using the generalized differential transform method, and by Garg and Sharma [14] employing the Adomian decomposition method.

Now, if we make $p = 2$, $q = r = 1$ in (28) the space-time fractional telegraph equation reduces to

$$D_x^{3/2} u(x,t) = D_t^2 u(x,t) + D_t u(x,t) + u(x,t), \quad 0 < x < 1, \quad t > 0 \quad (30)$$

and the solution is the same obtained by Momani [24], Odibat and Momani [26], and Yildrim [37].

Next we consider the homogeneous space-time fractional telegraph equation

$$D_x^{2\alpha} u(x,t) = D_t^{p\beta} u(x,t) + D_t^{r\beta} u(x,t) + u(x,t), \quad 0 < x < 1, \quad t > 0 \quad (31)$$

where $1 < 2\alpha \le 2$, $\beta = 1/q$, $p, q, r \in \mathbb{N}$, $1 < p\beta \le 2$, $0 < r\beta \le 1$, $D_x^{2\alpha} \equiv D_x^\alpha D_x^\alpha$, $D_t^{p\beta} \equiv D_t^\beta D_t^\beta \cdots D_t^\beta$ (p times), $D_t^{r\beta} \equiv D_t^\beta D_t^\beta \cdots D_t^\beta$ (r times), D_x^α and D_t^β are Caputo fractional derivatives, with the initial conditions

$$u(0,t) = E_{\beta,\gamma}\big(-t^{\beta}\big). \tag{32}$$

$$D_x^{\alpha} u(0,t) = E_{\beta,\gamma}\big(-t^{\beta}\big), \tag{33}$$

being $\Re(\gamma) > 0$.

Applying generalized two dimensional differential transform to both side of (31) and using properties iii) and iv), we obtain

$$U_{\alpha,\beta}(k+2,h) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+2\alpha+1)}\Bigg[\frac{\Gamma(\beta h+\beta p+1)}{\Gamma(\beta h+1)}U_{\alpha,\beta}(k,h+p) + \frac{\Gamma(\beta h+\beta r+1)}{\Gamma(\beta h+1)}U_{\alpha,\beta}(k,h+r) + U_{\alpha,\beta}(k,h)\Bigg]. \tag{34}$$

Following an analogue procedure to that used in order to establish (12) we have,

$$U_{\alpha,\beta}(0,h) = \frac{1}{\Gamma(\beta h+1)}\Bigg[\Big(D_{t_0}^{\beta}\Big)^{h} E_{\beta,\gamma}\big(-t^{\beta}\big)\Bigg]_{(x_0,t_0)=(0,0)} = \frac{(-1)^h}{\Gamma(\beta h+\gamma)}. \tag{35}$$

On the other hand, the application of (6) to the initial condition (33), using the property iii), yields

$$U_{\alpha,\beta}(1,h) = \frac{(-1)^h}{\Gamma(\alpha+1)\Gamma(\beta h+\gamma)}. \tag{36}$$

From (34) with $k = 0$

$$U_{\alpha,\beta}(2,h) = \frac{1}{\Gamma(2\alpha+1)}\Bigg[\frac{\Gamma(\beta h+\beta p+1)}{\Gamma(\beta h+1)}U_{\alpha,\beta}(0,h+p) + \frac{\Gamma(\beta h+\beta r+1)}{\Gamma(\beta h+1)}U_{\alpha,\beta}(0,h+r) + U_{\alpha,\beta}(0,h)\Bigg],$$

now using (35)

$$U_{\alpha,\beta}(2,h) = \frac{(-1)^h}{\Gamma(2\alpha+1)}\Bigg[\frac{\Gamma(\beta h+\beta p+1)}{\Gamma(\beta h+\beta p+\gamma)}\frac{(-1)^p}{\Gamma(\beta h+1)} + \frac{\Gamma(\beta h+\beta r+1)}{\Gamma(\beta h+\beta r+\gamma)}\frac{(-1)^r}{\Gamma(\beta h+1)} + \frac{1}{\Gamma(\beta h+\gamma)}\Bigg]. \tag{37}$$

For convenience we will use the following notation:

$$T_{i,j}(h) = \frac{\Gamma(\beta h+\beta(ip+jr)+1)}{\Gamma(\beta h+\beta(ip+jr)+\gamma)}\frac{(-1)^{h+ip+jr}}{\Gamma(\beta h+1)}, \tag{38}$$

so,

$$U_{\alpha,\beta}(0,h) = \frac{(-1)^h}{\Gamma(\beta h+\gamma)} = T_{0,0}(h), \tag{39}$$

$$U_{\alpha,\beta}(1,h) = \frac{1}{\Gamma(\alpha+1)} T_{0,0}(h). \tag{40}$$

It is easy to prove that

$$\frac{\Gamma(\beta h+\beta p+1)}{\Gamma(\beta h+1)} T_{i,j}(h+p) = T_{i+1,j}(h). \tag{41}$$

$$\frac{\Gamma(\beta h+\beta r+1)}{\Gamma(\beta h+1)} T_{i,j}(h+r) = T_{i,j+1}(h). \tag{42}$$

From (37) and (38) we can write

$$U_{\alpha,\beta}(2,h) = \frac{1}{\Gamma(2\alpha+1)} \big[T_{1,0}(h) + T_{0,1}(h) + T_{0,0}(h)\big]. \tag{43}$$

Making in (34) $k=1$, and taking into account (40), we have

$$U_{\alpha,\beta}(3,h) = \frac{1}{\Gamma(3\alpha+1)} \left[\frac{\Gamma(\beta h+\beta p+1)}{\Gamma(\beta h+\beta p+\gamma)} T_{0,0}(h+p) + \frac{\Gamma(\beta h+\beta r+1)}{\Gamma(\beta h+\beta r+\gamma)} T_{0,0}(h+r) + T_{0,0}(h)\right],$$

this, according to (41)-(42), can be written as

$$U_{\alpha,\beta}(3,h) = \frac{1}{\Gamma(3\alpha+1)} \big[T_{1,0}(h) + T_{0,1}(h) + T_{0,0}(h)\big]. \tag{44}$$

If $k=2$ in (34)

$$U_{\alpha,\beta}(4,h) = \frac{\Gamma(2\alpha+1)}{\Gamma(4\alpha+1)} \left[\frac{\Gamma(\beta h+\beta p+1)}{\Gamma(\beta h+1)} U_{\alpha,\beta}(2,h+p) + \frac{\Gamma(\beta h+\beta r+1)}{\Gamma(\beta h+1)} U_{\alpha,\beta}(2,h+r) + U_{\alpha,\beta}(2,h)\right],$$

then from (43), (41) and (42) we get

$$U_{\alpha,\beta}(4,h) = \frac{1}{\Gamma(4\alpha+1)} \big[T_{2,0}(h) + 2T_{1,1}(h) + 2T_{1,0}(h) + 2T_{0,1}(h) + T_{0,2}(h) + T_{0,0}(h)\big]. \tag{45}$$

For $k=3$ in (34) we have

$$U_{\alpha,\beta}(5,h) = \frac{\Gamma(3\alpha+1)}{\Gamma(5\alpha+1)}\left[\frac{\Gamma(\beta h+\beta p+1)}{\Gamma(\beta h+1)}U_{\alpha,\beta}(3,h+p) + \frac{\Gamma(\beta h+\beta r+1)}{\Gamma(\beta h+1)}U_{\alpha,\beta}(3,h+r) + U_{\alpha,\beta}(3,h)\right]$$

and, from (44), (41) and (42), we obtain

$$U_{\alpha,\beta}(5,h) = \frac{1}{\Gamma(5\alpha+1)}\big[T_{2,0}(h) + 2T_{1,1}(h) + 2T_{1,0}(h) + 2T_{0,1}(h) + T_{0,2}(h) + T_{0,0}(h)\big]. \quad (46)$$

Analogously we obtain the following expressions:

$$U_{\alpha,\beta}(6,h) = \frac{1}{\Gamma(6\alpha+1)}\big[T_{3,0}(h) + 3T_{2,1}(h) + 3T_{2,0}(h) + 6T_{1,1}(h) + 3T_{1,2}(h) + 3T_{1,0}(h) + 3T_{0,2}(h) + T_{0,3}(h) + 3T_{0,1}(h) + T_{0,0}(h)\big]. \quad (47)$$

$$U_{\alpha,\beta}(7,h) = \frac{1}{\Gamma(7\alpha+1)}\big[T_{3,0}(h) + 3T_{2,1}(h) + 3T_{2,0}(h) + 6T_{1,1}(h) + 3T_{1,2}(h) + 3T_{1,0}(h) + 3T_{0,2}(h) + T_{0,3}(h) + 3T_{0,1}(h) + T_{0,0}(h)\big]. \quad (48)$$

$$U_{\alpha,\beta}(8,h) = \frac{1}{\Gamma(8\alpha+1)}\big[T_{4,0}(h) + 4T_{3,1}(h) + 4T_{3,0}(h) + 12T_{2,1}(h) + 6T_{2,2}(h) + 6T_{2,0}(h) + 12T_{1,2}(h) + 4T_{1,3}(h) + 12T_{1,1}(h) + 4T_{1,0}(h) + 4T_{0,3}(h) + T_{0,4}(h) + 6T_{0,2}(h) + 4T_{0,1}(h) + T_{0,0}(h)\big]. \quad (49)$$

$$U_{\alpha,\beta}(9,h) = \frac{1}{\Gamma(9\alpha+1)}\big[T_{4,0}(h) + 4T_{3,1}(h) + 4T_{3,0}(h) + 12T_{2,1}(h) + 6T_{2,2}(h) + 6T_{2,0}(h) + 12T_{1,2}(h) + 4T_{1,3}(h) + 12T_{1,1}(h) + 4T_{1,0}(h) + 4T_{0,3}(h) + T_{0,4}(h) + 6T_{0,2}(h) + 4T_{0,1}(h) + T_{0,0}(h)\big]. \quad (50)$$

Substituting the earlier results (39), (40), (43)-(50)) in (7), we have

$$u(x,t)=\sum_{h=0}^{\infty}t^{\beta h}\Bigg[\left(1+\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right)T_{0,0}(h)+\left(\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}\right)\times$$
$$\big[T_{1,0}(h)+T_{0,1}(h)+T_{0,0}(h)\big]+\left(\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{5\alpha}}{\Gamma(5\alpha+1)}\right)\times$$
$$\big[T_{2,0}(h)+2T_{1,1}(h)+2T_{1,0}(h)+2T_{0,1}(h)+T_{0,2}(h)+T_{0,0}(h)\big]+$$
$$\left(\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}+\frac{x^{7\alpha}}{\Gamma(7\alpha+1)}\right)\big[T_{3,0}(h)+3T_{2,1}(h)+3T_{2,0}(h)+6T_{1,1}(h)+$$
$$3T_{1,2}(h)+3T_{1,0}(h)+3T_{0,2}(h)+T_{0,3}(h)+3T_{0,1}(h)+T_{0,0}(h)\big]+$$
$$\left(\frac{x^{8\alpha}}{\Gamma(8\alpha+1)}+\frac{x^{9\alpha}}{\Gamma(9\alpha+1)}\right)\big[T_{4,0}(h)+4T_{3,1}(h)+4T_{3,0}(h)+12T_{2,1}(h)+$$
$$6T_{2,2}(h)+6T_{2,0}(h)+12T_{1,2}(h)+4T_{1,3}(h)+12T_{1,1}(h)+4T_{1,0}(h)+$$
$$4T_{0,3}(h)+T_{0,4}(h)+6T_{0,2}(h)+4T_{0,1}(h)+T_{0,0}(h)\big]+\cdots\Bigg],\quad(51)$$

that is,

$$u(x,t)=\sum_{h=0}^{\infty}t^{\beta h}\Bigg[\left(1+\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\cdots+\frac{x^{8\alpha}}{\Gamma(8\alpha+1)}+\frac{x^{9\alpha}}{\Gamma(9\alpha+1)}+\cdots\right)\times$$
$$T_{0,0}(h)+\left(\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+\cdots+\frac{x^{8\alpha}}{\Gamma(8\alpha+1)}+\frac{x^{9\alpha}}{\Gamma(9\alpha+1)}+\cdots\right)\times$$
$$\big[T_{1,0}(h)+T_{0,1}(h)\big]+$$
$$\left(\frac{x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{x^{5\alpha}}{\Gamma(5\alpha+1)}+\cdots+\frac{x^{8\alpha}}{\Gamma(8\alpha+1)}+\frac{x^{9\alpha}}{\Gamma(9\alpha+1)}+\cdots\right)\times$$
$$\big[T_{2,0}(h)+2T_{1,1}(h)+T_{1,0}(h)+T_{0,1}(h)+T_{0,2}(h)\big]+$$
$$\left(\frac{x^{6\alpha}}{\Gamma(6\alpha+1)}+\frac{x^{7\alpha}}{\Gamma(7\alpha+1)}+\frac{x^{8\alpha}}{\Gamma(8\alpha+1)}+\frac{x^{9\alpha}}{\Gamma(9\alpha+1)}+\cdots\right)\times$$
$$\big[T_{3,0}(h)+3T_{2,1}(h)+2T_{2,0}(h)+4T_{1,1}(h)+$$
$$3T_{1,2}(h)+T_{1,0}(h)+2T_{0,2}(h)+T_{0,3}(h)+T_{0,1}(h)\big]+$$
$$\left(\frac{x^{8\alpha}}{\Gamma(8\alpha+1)}+\frac{x^{9\alpha}}{\Gamma(9\alpha+1)}+\cdots\right)\big[T_{4,0}(h)+4T_{3,1}(h)+3T_{3,0}(h)+$$
$$9T_{2,1}(h)+6T_{2,2}(h)+3T_{2,0}(h)+9T_{1,2}(h)+4T_{1,3}(h)+6T_{1,1}(h)+$$
$$T_{1,0}(h)+3T_{0,3}(h)+T_{0,4}(h)+3T_{0,2}(h)+T_{0,1}(h)\big]+\cdots\Bigg].\quad(52)$$

This can be written as

$$u(x,t) = \sum_{h=0}^{\infty} T_{0,0}(h)\ t^{\beta h} \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)}\ +$$
$$\sum_{h=0}^{\infty} \big[T_{1,0}(h) + T_{0,1}(h)\big]\ t^{\beta h} \sum_{k=2}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)}\ +$$
$$\sum_{h=0}^{\infty} \big[T_{2,0}(h) + 2T_{1,1}(h) + T_{1,0}(h) + T_{0,1}(h) + T_{0,2}(h)\big]\ t^{\beta h} \sum_{k=4}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)}\ +$$
$$\sum_{h=0}^{\infty} \big[T_{3,0}(h) + 3T_{2,1}(h) + 2T_{2,0}(h) + 4T_{1,1}(h) + 3T_{1,2}(h)\ +$$
$$T_{1,0}(h) + 2T_{0,2}(h) + T_{0,3}(h) + T_{0,1}(h)\big]\ t^{\beta h} \sum_{k=6}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} + \cdots. \quad (53)$$

Making changes of indexes we have

$$u(x,t) = \sum_{h=0}^{\infty} T_{0,0}(h)\ t^{\beta h} \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} + x^{2\alpha} \sum_{h=0}^{\infty} \big[T_{1,0}(h) + T_{0,1}(h)\big]\ t^{\beta h}\ \times$$
$$\sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+2\alpha+1)} + x^{4\alpha} \sum_{h=0}^{\infty} \big[T_{2,0}(h) + 2T_{1,1}(h) + T_{1,0}(h)\ +$$
$$T_{0,1}(h) + T_{0,2}(h)\big]\ t^{\beta h} \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+4\alpha+1)}\ +$$
$$x^{6\alpha} \sum_{h=0}^{\infty} \big[T_{3,0}(h) + 3T_{2,1}(h) + 2T_{2,0}(h) + 4T_{1,1}(h) + 3T_{1,2}(h) + T_{1,0}(h)\ +$$
$$2T_{0,2}(h) + T_{0,3}(h) + T_{0,1}(h)\big]\ t^{\beta h} \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+6\alpha+1)} + \cdots. \quad (54)$$

From (54), (4) and (5) we have

$$u(x,t) = \sum_{h=0}^{\infty} T_{0,0}(h)\ t^{\beta h}\ E_{\alpha}(x^{\alpha})\ +$$
$$x^{2\alpha} \sum_{h=0}^{\infty} \big[T_{1,0}(h) + T_{0,1}(h)\big]\ t^{\beta h} E_{\alpha,2\alpha+1}(x^{\alpha})\ +$$
$$x^{4\alpha} \sum_{h=0}^{\infty} \big[T_{2,0}(h) + 2T_{1,1}(h) + T_{1,0}(h) + T_{0,1}(h) + T_{0,2}(h)\big] t^{\beta h} E_{\alpha,4\alpha+1}(x^{\alpha})\ +$$

$$x^{6\alpha}\sum_{h=0}^{\infty}\big[T_{3,0}(h)+3T_{2,1}(h)+2T_{2,0}(h)+4T_{1,1}(h)+3T_{1,2}(h)+ T_{1,0}(h)+2T_{0,2}(h)+T_{0,3}(h)+T_{0,1}(h)\big]\, t^{\beta h}E_{\alpha,6\alpha+1}(x^{\alpha})+\cdots. \quad (55)$$

Observe that from (38) and (3) we have

$$\sum_{h=0}^{\infty}T_{i,j}(h)\, t^{\beta h}=(-1)^{ip}(-1)^{jr}\,{}_2\Psi_2\left[{(\beta ip+\beta jr+1,\beta),(1,1) \atop (\beta ip+\beta jr+\gamma,\beta),(1,\beta)};-t^{\beta}\right].$$

Therefore,

$$\begin{aligned}u(x,t)=\;&E_{\alpha}(x^{\alpha})\,E_{\beta,\gamma}\big(-t^{\beta}\big)+x^{2\alpha}E_{\alpha,2\alpha+1}(x^{\alpha})\times\\
&\Big\{(-1)^p\,{}_2\Psi_2\left[{(\beta p+1,\beta),(1,1)\atop(\beta p+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+(-1)^r\,{}_2\Psi_2\left[{(\beta r+1,\beta),(1,1)\atop(\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]\Big\}+\\
&x^{4\alpha}E_{\alpha,4\alpha+1}(x^{\alpha})\Big\{{}_2\Psi_2\left[{(2\beta p+1,\beta),(1,1)\atop(2\beta p+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+\\
&2(-1)^{p+r}\,{}_2\Psi_2\left[{(\beta p+\beta r+1,\beta),(1,1)\atop(\beta p+\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+\\
&(-1)^p\,{}_2\Psi_2\left[{(\beta p+1,\beta),(1,1)\atop(\beta p+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+(-1)^r\,{}_2\Psi_2\left[{(\beta r+1,\beta),(1,1)\atop(\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+\\
&{}_2\Psi_2\left[{(2\beta r+1,\beta),(1,1)\atop(2\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]\Big\}+x^{6\alpha}E_{\alpha,6\alpha+1}(x^{\alpha})\times\\
&\Big\{(-1)^p\,{}_2\Psi_2\left[{(3\beta p+1,\beta),(1,1)\atop(3\beta p+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+\\
&3(-1)^r\,{}_2\Psi_2\left[{(2\beta p+\beta r+1,\beta),(1,1)\atop(2\beta p+\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+2\,{}_2\Psi_2\left[{(2\beta p+1,\beta),(1,1)\atop(2\beta p+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+\\
&4(-1)^{p+r}\,{}_2\Psi_2\left[{(\beta p+\beta r+1,\beta),(1,1)\atop(\beta p+\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+\\
&3(-1)^p\,{}_2\Psi_2\left[{(\beta p+2\beta r+1,\beta),(1,1)\atop(\beta p+2\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+\\
&(-1)^p\,{}_2\Psi_2\left[{(\beta p+1,\beta),(1,1)\atop(\beta p+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+2\,{}_2\Psi_2\left[{(2\beta r+1,\beta),(1,1)\atop(2\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+\\
&(-1)^r\,{}_2\Psi_2\left[{(3\beta r+1,\beta),(1,1)\atop(3\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]+(-1)^r\,{}_2\Psi_2\left[{(\beta r+1,\beta),(1,1)\atop(\beta r+\gamma,\beta),(1,\beta)};-t^{\beta}\right]\Big\}+\cdots. \quad (56)\end{aligned}$$

As a particular case, if $\gamma=1$ and we use (4)-(5), then

$$\begin{aligned}u(x,t)=E_{\beta}\big(-t^{\beta}\big)\big\{&E_{\alpha}(x^{\alpha})+\big[(-1)^p+(-1)^r\big]\,x^{2\alpha}E_{\alpha,2\alpha+1}(x^{\alpha})+\\
&\big[2+2(-1)^{p+r}+(-1)^p+(-1)^r\big]\,x^{4\alpha}E_{\alpha,4\alpha+1}(x^{\alpha})+\\
&\big[5(-1)^p+5(-1)^r+4(-1)^{p+r}+4\big]\,x^{6\alpha}E_{\alpha,6\alpha+1}(x^{\alpha})+\cdots\big\}. \quad (57)\end{aligned}$$

Now, from (57) we get

i) If $p+r$ is odd: $u(x,t) = E_\beta\big(-t^\beta\big)\ E_\alpha(x^\alpha)$, which was given by Garg et al. [13].

ii) If $p+r$ is even, being p and r even:

$$u(x,t) = E_\beta\big(-t^\beta\big)\ \big\{E_\alpha(x^\alpha) + 2x^{2\alpha}E_{\alpha,2\alpha+1}(x^\alpha) + 6x^{4\alpha}E_{\alpha,4\alpha+1}(x^\alpha) + 18x^{6\alpha}E_{\alpha,6\alpha+1}(x^\alpha) + \cdots\big\}. \quad (58)$$

iii) If $p+r$ is even, being p and r odd, we have

$$u(x,t) = E_\beta\big(-t^\beta\big)\ \big\{E_\alpha(x^\alpha) - 2x^{2\alpha}E_{\alpha,2\alpha+1}(x^\alpha) + 2x^{4\alpha}E_{\alpha,4\alpha+1}(x^\alpha) - 2x^{6\alpha}E_{\alpha,6\alpha+1}(x^\alpha) + \cdots\big\} \quad (59)$$

4. Graphic Representation

In this section some figures are presented which show the behavior of the solution $u(x,t)$ for different values of the parameters, with $0 < x < 1$, $0 < t < 1$.

i) Figures of $u(x,t)$ as defined by (25) (Figure 1 and Figure 2):

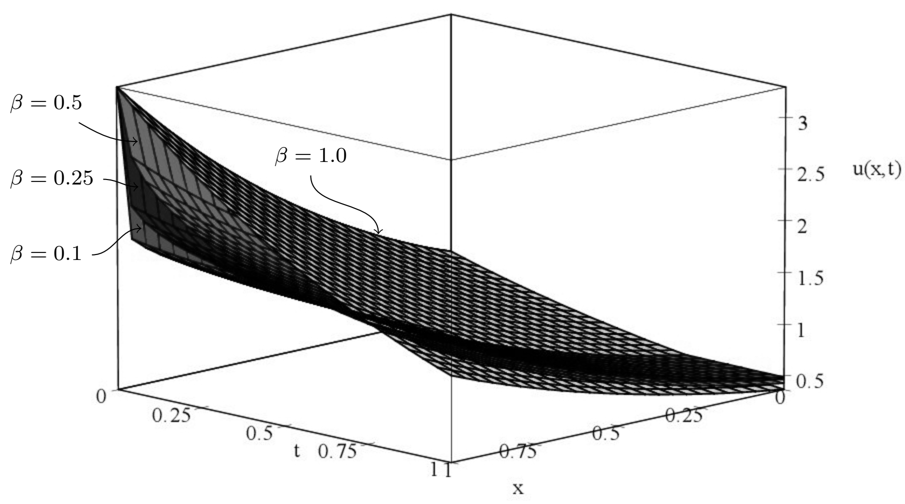

FIGURE 1. $u(x,t)$ for $l = 10$, $j = 5$ and diferent values of β.

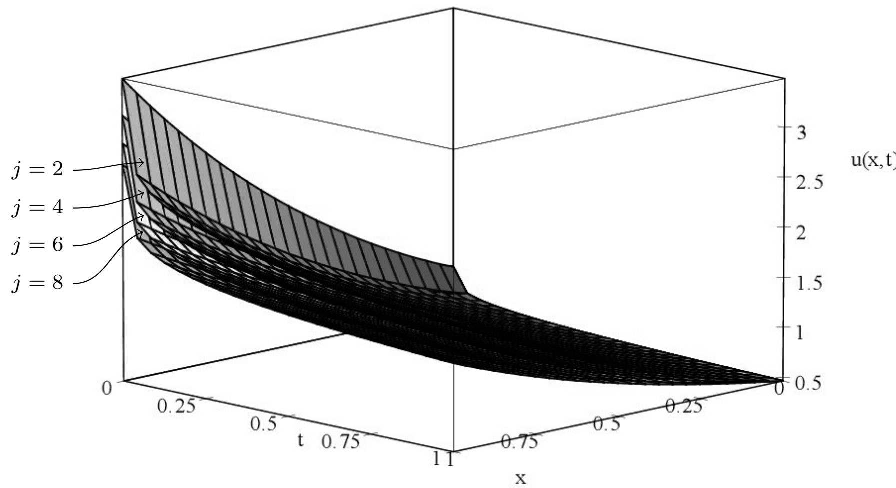

FIGURE 2. $u(x,t)$ for $l = 10$, $\beta = 0.25$ and diferent values of j.

ii) Figures of $u(x,t)$ as defined by (56) (Figure 3 and Figure 4):

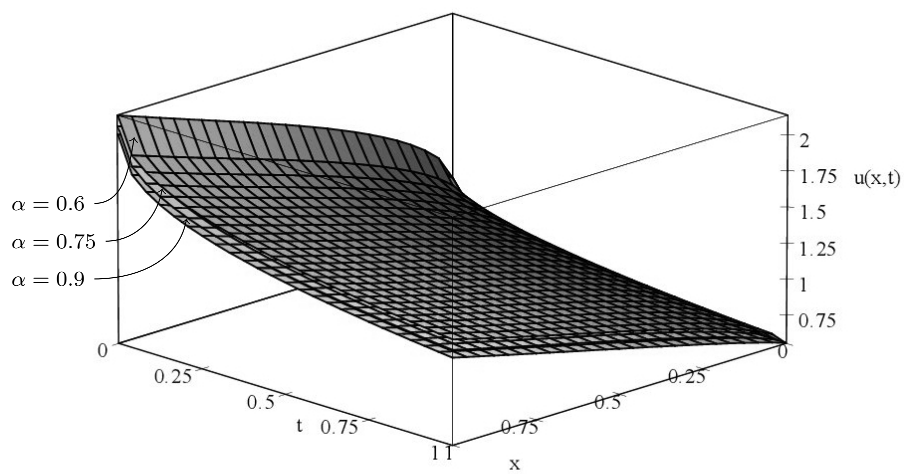

FIGURE 3. $u(x,t)$ for $\beta = 0.5$, $\gamma = 2$, $p = 3$, $r = 1$ and different values of α.

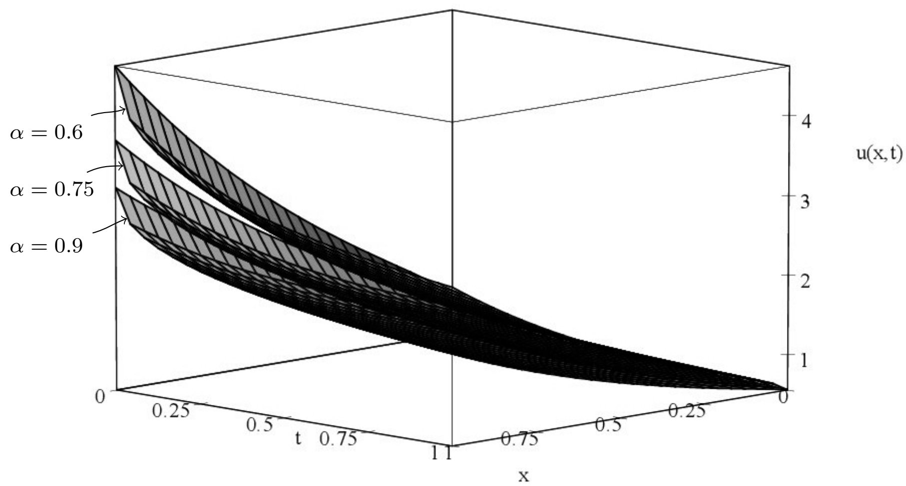

FIGURE 4. $u(x, t)$ for $\beta = 0.5$, $\gamma = 2$, $p = 3$, $r = 2$ and different values of α.

iii) Figures of $u(x, t)$ as defined by (58) (Figure 5 and Figure 6):

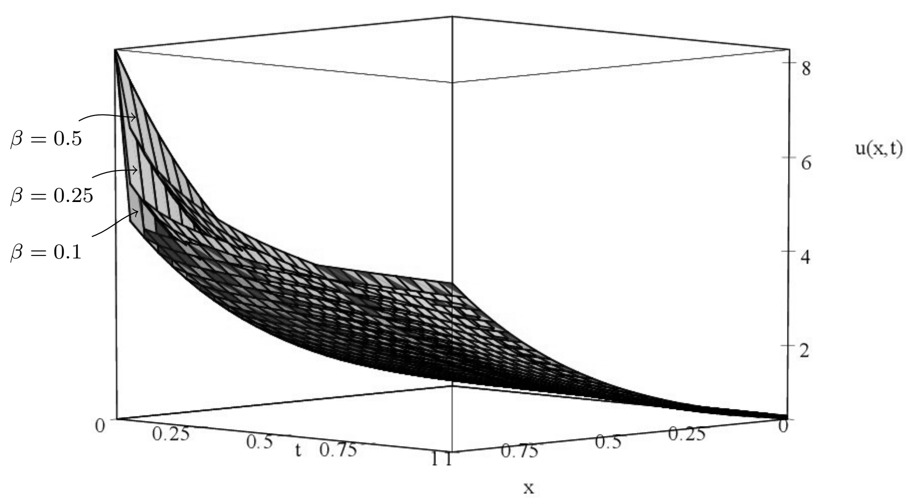

FIGURE 5. $u(x, t)$ for $\alpha = 0.75$ and different values of β.

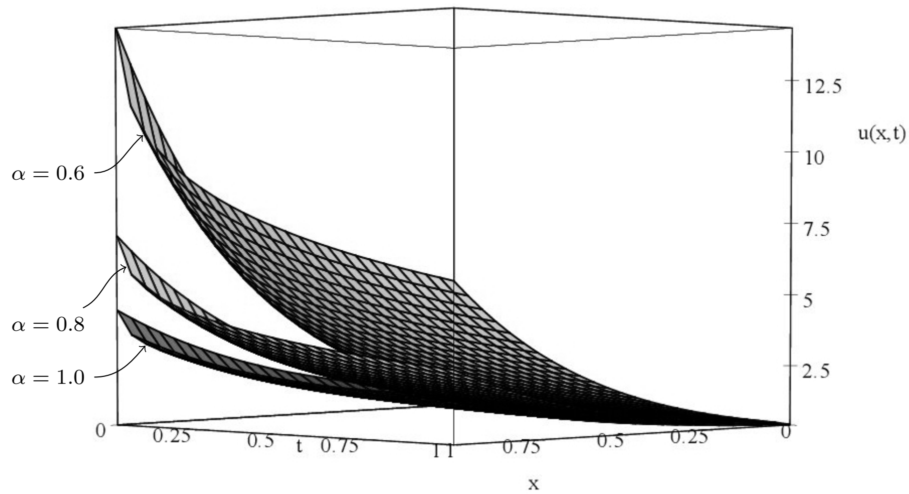

FIGURE 6. $u(x,t)$ for $\beta = 0.5$ and different values of α.

iv) Figures of $u(x,t) = E_\beta(-t^\beta)E_\alpha(x^\alpha)$ (Figure 7 and Figure 8):

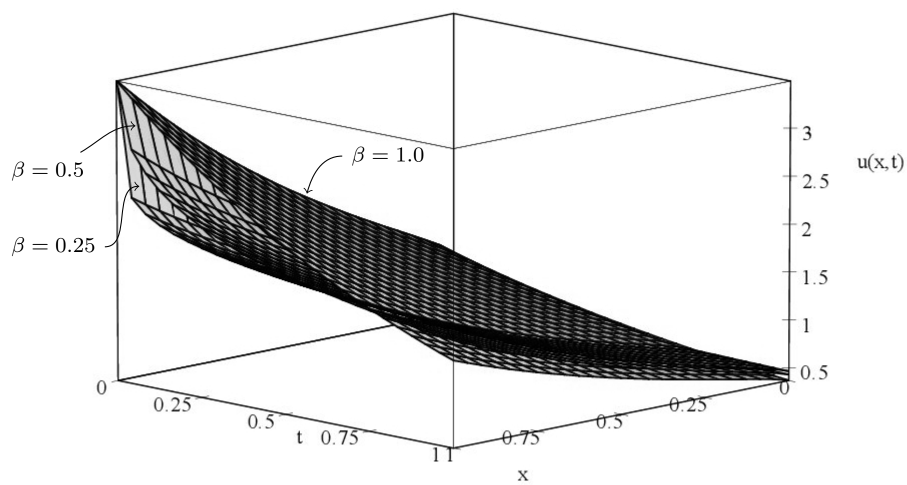

FIGURE 7. $u(x,t)$ for $\alpha = 0.75$ and different values of β.

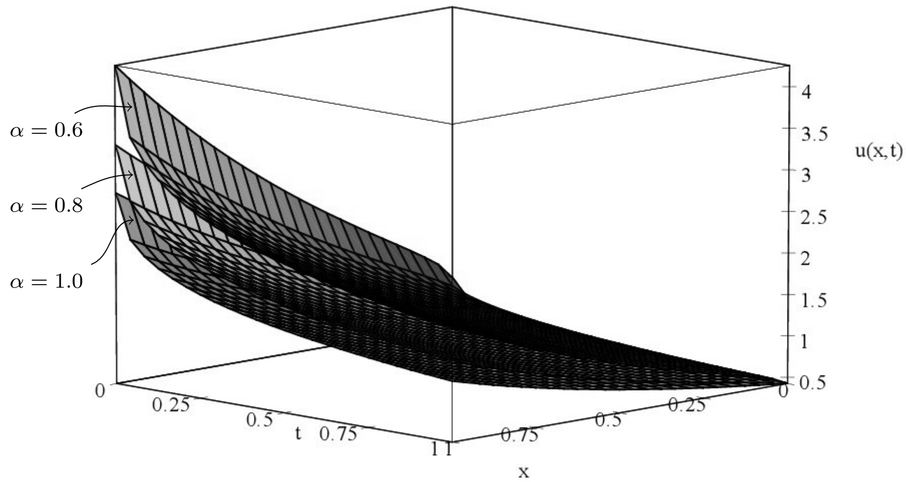

FIGURE 8. $u(x, t)$ for $\beta = 0.5$ and different values of α.

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Centro de Investigación de Matemática Aplicada
Universidad del Zulia
Facultad de Ingeniería
Maracaibo, Venezuela
e-mail: lgalue@hotmail.com

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