Introduction to Representations of Braid Groups

Introducción a las representaciones de los grupos de trenzas

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Abstract. These are lecture notes prepared for a minicourse given at the Cimpa Research School *Algebraic and geometric aspects of representation theory*, held in Curitiba, Brazil in March 2013. The purpose of the course is to provide an introduction to the study of representations of braid groups. Three general classes of representations of braid groups are considered: homological representations via *mapping class groups*, monodromy representations via the Knizhnik-Zamolodchikov connection, and solutions of the Yang-Baxter equation via *quasi-triangular bialgebras*. Some of the remarkable relations between these three different constructions are described.

Key words and phrases. Braid groups, Representation theory.

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Resumen. Estas notas fueron preparadas para un minicurso enseñado en la escuela Cimpa *Algebraic and geometric aspects of representation theory*, en Curitiba, Brazil en Marzo de 2013. El propósito del curso es presentar una introducción al estudio de las representaciones de los grupos de trenzas. Tres clases generales de representaciones son consideradas: representaciones homológicas de *mapping class groups*, representaciones de monodromía de la connección de Knizhnik-Zamolodchikov, y soluciones de la ecuación de Yang-Baxter en términos de *quasi-triangular bialgebras*. Algunas de las notables relaciones entre estas construcciones son descritas.

Palabras y frases clave. Grupos de trenzas, teoría de representaciones.

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1. Introduction

Braid groups were introduced by Emil Artin in 1925, and by now play a role in various parts of mathematics including knot theory, low dimensional topology, public key cryptography and deformation quantization. The braid group B_n admits different equivalent definitions: the fundamental group of the configuration space $C_n(\mathbb{C})$ of n unordered points in the complex plane, the mapping class group of a disk with n marked points, and Artin's presentation. Each of these descriptions provides a method for constructing representations of B_n: fundamental groups act via monodromy, mapping class groups act on the homology groups of spaces, and solutions to the Yang-Baxter equations produce solutions to Artin's relations. The aim of these notes is to introduce these methods, and describe some of the relations between these seemingly unrelated constructions. The notes are organized as follows:

In Section §2 we present three different ways in which the braid group B_n can be defined, explain how these definitions are equivalent, and introduce some general facts regarding the braid groups. Section §3 regards the group B_n as the mapping class group of a disk with n marked points. This description produces representations of B_n via the action on the homology of spaces which are functorially associated to the punctured disk. The Burau representation is discussed and the relationship with the Alexander-Conway polynomial is explained. The Lawrence-Krammer-Bigelow representation is also constructed. Section §4 begins with the introduction of the Knizhnik-Zamolodchikov connection and explains how representations of complex semisimple Lie algebras produce, via the Knizhnik-Zamolodchivok connection, monodromy representations of the braid group. A theorem of Kohno is explained, which describes the relationship between monodromy representations associated to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and the Lawrence-Krammer-Bigelow representation. In Section §5 we introduce the Yang-Baxter equation, and explain the way in which solutions of this equation produce representations of B_n. We define the notion of a quasi-triangular bialgebra, and show that modules over quasi-triangular bialgebras come equipped with solutions to the Yang-Baxter equation. Finally, the Drinfeld-Kohno theorem is discussed. This remarkable theorem explains the precise relationship between the monodromy representations constructed via the Knizhnik-Zamolodchikov connection, and the representations constructed as solutions of the Yang-Baxter equation provided by quantum enveloping algebras. We have included some proofs of the results we discussed, when they are enlightening and sufficiently simple. However, no attempt has been made to present all proofs of the theorems we mentioned, as this would go far beyond the scope of these notes. When proofs are missing, we have tried to provide references to the many excellent sources that were used in preparing this course.

2. Braid Groups

Artin's braid groups admit several equivalent descriptions, which we will review in this section.

2.1. Artin's Presentation

We denote by D_2 the unit disk in the complex plane and fix n marked points

$$-1 < p_1 < p_2 < \cdots < p_n < 1 \in D_2 \cap \mathbb{R}.$$

Definition 2.1. A braid is a collection of n paths $f_i : I = [0, 1] \mapsto D_2$, called strands, such that:

(1) $f_i(0) = p_i$.

(2) $f_i(1) = p_{\tau(i)}$, for some permutation τ in the symmetric group Σ_n.

(3) For each $t \in I$, $f_i(t) \neq f_j(t)$ provided that $i \neq j$.

Braids are pictured geometrically as a collection of n strands in three dimensional space. One is usually interested in braids only up to isotopies fixing the endpoints of each strand. We will abuse the notation and take the word braid to mean an isotopy class. Some examples of braids are the following:
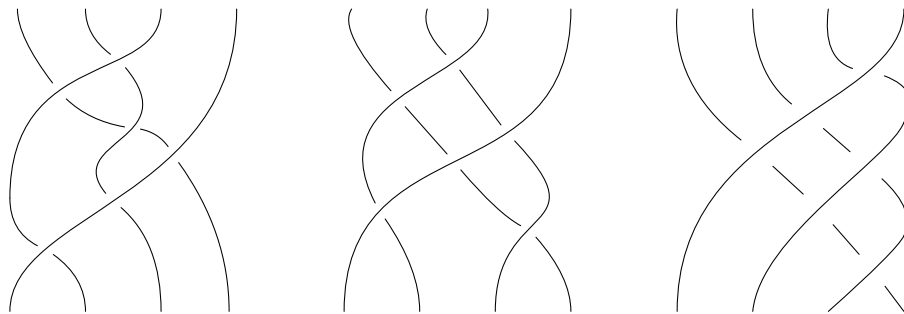

FIGURE 1. Examples of braids.

Definition 2.2. The braid group in n strands, denoted B_n, consists of the set of all braids with n strands with multiplication given by glueing.

It is a good exercise to become convinced that this operation indeed gives B_n the structure of a group. The following result, due to Emil Artin, gives a presentation for the group B_n.

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Theorem 2.3 (Artin [3])**.** *Denote by* $\mathcal{B}_n$ *the group generated by the symbols* $\sigma_1, \ldots, \sigma_{n-1}$ *modulo the relations:*

(1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$,   *for* $i = 1, \ldots, n-2$.

(2) $\sigma_i \sigma_j = \sigma_j \sigma_i$,   *for* $|i - j| \geq 2$.

There is an isomorphism of groups $\phi : \mathcal{B}_n \to B_n$ *determined by:*

$$\sigma_i \quad \mapsto \quad || \bowtie ||.$$

Proof. The fact that the first relation is satisfied follows from the equality

$$\text{(diagram)} = \text{(diagram)}.$$

The fact that the second relation is satisfied follows from

$$\text{(diagram)} = \text{(diagram)}.$$

We conclude that ϕ defines a homomorphism.

Next, one needs to prove that ϕ is actually an isomorphism. Surjectivity is easy since the elements $\phi(\sigma_i)$ clearly generate B_n. For a proof of injectivity see [15, Theorem 1.12].   ☑

From now on we will identify the groups B_n and $\mathcal{B}_n$ using the isomorphism above.

2.2. Configuration Spaces

The group B_n can also be described as the fundamental group of the configuration space of points in the plane. We will now briefly recall some definitions regarding configuration spaces. Let M be a smooth manifold. The configuration space of n ordered points in M, denoted $\widehat{C}_n(M)$ is the manifold

$$\widehat{C}_n(M) := \big\{ (x_1, \ldots, x_n) \in M^n : x_i \neq x_j, \quad \text{if} \quad i \neq j \big\}.$$

There is a natural action of the symmetric group Σ_n on the space $\widehat{C}_n(M)$, given by permuting the coordinates. The configuration space of n unordered points in M is the quotient space

$$C_n(M) := \widehat{C}_n(M)/\Sigma_n.$$

Theorem 2.4 (Fox-Neuwirth [11], Fadell-van Buskirk [9])**.** *There is a natural isomorphism between the braid group in n strands and the fundamental group of the configuration space of n unordered points in the plane*

$$B_n \cong \pi_1\big(C_n(\mathbb{C}), \mathbf{p}\big).$$

Here $\mathbf{p} = \{1, 2, \dots, n\} \in C_n(\mathbb{C})$.

The isomorphism above is defined as follows: a braid can be thought of as a path in the configuration space $\widehat{C}_n(\mathbb{C})$. However, seen as a path in this configuration space it is not closed, since the endpoint of a strand may be different from its starting point. By composing with the quotient map to the unordered configuration space, this path becomes a closed path, and therefore defines an element of the fundamental group.

2.3. Mapping Class Groups

The braid groups B_n can also be realized as mapping class groups, which exposes the very interesting relation with the topology of surfaces. Let M, N be compact manifolds, possibly with boundary. We denote by $C(M, N)$ the space of all continuous functions from M to N. There is a natural topology on $C(M, N)$, the compact open topology, defined as follows: for any $K \subset M$ compact and $U \subset N$ open, set:

$$V(K, U) \coloneqq \{f \in C(M, N) : f(K) \subset U\}.$$

The compact open topology is the smallest topology for which all $V(K, U)$ are open. We denote by $\mathsf{Homeo}(M)$ the group of homeomorphisms of M. This is a subset of $C(M, M)$ and in this way inherits a topology which gives it the structure of a topological group.

Let us now specialize the discussion to the case of an orientable compact surface S. The group of orientation preserving homeomorphism of S that fix the boundary pointwise will be denoted by $\mathsf{Homeo}^+(S, \partial S)$. The connected component of the identity in $\mathsf{Homeo}^+(S, \partial S)$ is denoted by $\mathsf{Homeo}_0(S, \partial S)$ and consists of the set of homeomorphism that are isotopic to the identity via an isotopy that fixes the boundary pointwise.

Definition 2.5. The mapping class group of a surface S denoted $\mathsf{Mod}(S)$ is defined as

$$\mathsf{Mod}(S) = \mathsf{Homeo}^+(S, \partial S)/\mathsf{Homeo}_0(S, \partial S).$$

***Remark* 2.6.** The mapping class group can alternatively be defined by replacing homeomorphisms by diffeomorphism and isotopies by homotopies. The following is a list of possible definitions of $\mathsf{Mod}(S)$

$$\mathsf{Mod}(S) \cong \mathsf{Homeo}^+(S, \partial S)/\mathsf{Homeo}_0(S, \partial S)$$
$$\cong \mathsf{Diff}^+(S, \partial S)/\mathsf{Diff}_0(S, \partial S)$$
$$\cong \pi_0\big(\mathsf{Homeo}^+(S, \partial S)\big)$$
$$\cong \pi_0\big(\mathsf{Diff}^+(S, \partial S)\big).$$

Example 2.7. The notation $\mathsf{Mod}(S)$ for the mapping class group is motivated by the example of the torus $\mathbb{T}$, where

$$\mathsf{Mod}(\mathbb{T}) \cong SL_2(\mathbb{Z}).$$

The isomorphism is defined by the action of $\mathsf{Mod}(\mathbb{T})$ on homology.

Definition 2.8. Let S be a surface and $Q \subset S$ a finite set of marked points. We denote by $\mathsf{Homeo}(\mathsf{S}, \mathsf{Q})$ the group of homeomorphisms of S that fix Q as a set and fix the boundary pointwise. The mapping class group of S, seen as a surface with marked points is

$$\mathsf{Mod}(S, Q) := \pi_0\big(\mathsf{Homeo}^+(S, \partial S) \cap \mathsf{Homeo}(S, Q)\big).$$

Example 2.9 (Alexander-Tietze Theorem)**.** The mapping class group of the disk is trivial

$$\mathsf{Mod}(D) = \{0\}.$$

In order to prove this fact we consider an automorphism $f : D \to D$ that fixes the boundary. Then we define the isotopy

$$h_t(z) = \begin{cases} z, & \text{if } \quad t \le |z| \le 1; \\ th\big(\frac{z}{t}\big), & \text{if } \quad |z| \le t. \end{cases}$$

Example 2.10. An example of a nontrivial mapping class is that of the half twist. Here S is the disk with two marked points, that we may take to be $p = \big(-\frac{1}{2}, 0\big)$ and $q = \big(\frac{1}{2}, 0\big)$. The half twist H is the class of the diffeormorphism described by Figure 2.

Theorem 2.11. *Let D be a disk with n marked points $Q = \{p_1, \ldots, p_n\}$. There is an isomorphism*

$$\psi : B_n \to \mathsf{Mod}(D, Q),$$

characterized by the property that

$$\sigma_i \mapsto H_i.$$

Here H_i denotes the class of the homeomorphism which is supported on a disk containing only the i-th and $i + 1$-th punctures, and which acts as a half twist on that disk.

The reader interested in the proof of this theorem may consult [5]. In this section we have closely followed the book by Farb and Margalit [10], which is an excellent reference for learning about mapping class groups.
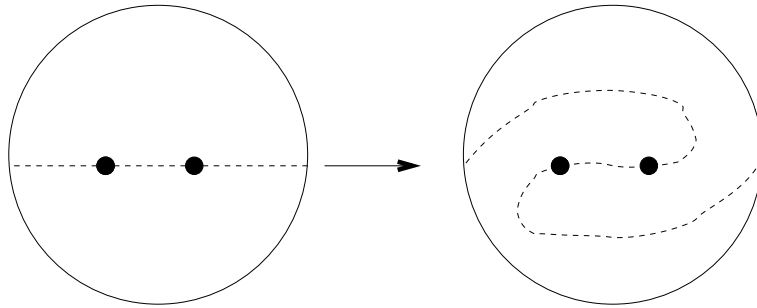
FIGURE 2. Halftwist of a two pointed disk.

2.4. Pure Braids

There is a natural group homomorphism $\pi : B_n \to \Sigma_n$ which sends a braid to the corresponding permutation of the marked points. Alternatively, the homomorphism is characterized by the property that

$$\sigma_i \mapsto (i, i+1).$$

Definition 2.12. The pure braid group on n strands, denoted by P_n, is the kernel of the homomorphism $\pi : B_n \to \Sigma_n$.

The group P_n consists of those braids such that each strand starts and ends at the same point. These are closed paths in the configuration space $\widehat{C}_n(\mathbb{C})$, and in this way one obtains an identification

$$P_n \cong \pi_1\big(\widehat{C}_n(\mathbb{C}), \mathbf{p}\big).$$

Remark 2.13. Let G be an abelian group. An Eilenberg-Maclane space $K(G, n)$ is a topological space such that:

- $\pi_n\big(K(G, n)\big) \cong G$.

- $\pi_m\big(K(G, n)\big) \cong \{0\}, \quad$ for $m \neq n$.

Given G and n, the spaces $K(G, n)$ exist and are well defined up to homotopy equivalence. The configuration spaces of points in the plane are important examples of Eilenberg-Maclane spaces,

$$C_n(\mathbb{C}) \cong K(B_n, 1)$$
$$\widehat{C}_n(\mathbb{C}) \cong K(P_n, 1)$$

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3. Homological Representations

In this section we discuss the Burau representations of braid groups, their geometric interpretation as action of mapping classes on homology, and their relation with the Alexander-Conway polynomial of links. We also introduce the Lawrence-Krammer-Bigelow representation, which has the remarkable property of being faithful. Our exposition follows closely the book by Kassel and Turaev [15].

3.1. The Burau Representation

Let us denote by Λ the ring of Laurent polynomials with integer coefficients

$$\Lambda := \mathbb{Z}\big[t, t^{-1}\big].$$

In [6], W. Burau constructed representations of the braid group B_n on the space $\mathsf{Aut}(\Lambda^n)$ of $n \times n$ matrices with coefficients in Λ. Let us define the $n \times n$ matrix U_i as follows:

$$U_i = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix}$$

Proposition 3.1. *There is a unique group homomorphism* $\psi_n : B_n \to \mathsf{Aut}(\Lambda^n)$ *characterized by the property*

$$\sigma_i \mapsto U_i.$$

This homomorphism is called the Burau representation of B_n.

Proof. We begin by proving that the matrices U_i are invertible. Let us set

$$U = \begin{bmatrix} -t & t \\ 1 & 0 \end{bmatrix}.$$

Since the matrices U_i are block diagonal matrices whose blocks are either identity matrices of U it is enough to prove that U is an invertible matrix. For this we exhibit an explicit inverse

$$U^{-1} = \begin{bmatrix} 0 & 1 \\ t^{-1} & 1-t^{-1} \end{bmatrix}$$

One can directly check that $UU^{-1} = U^{-1}U = \mathrm{id}$. The block form of the matrices implies that $U_i U_j = U_j U_i$ provided that $|i-j| \geq 2$. Finally, it remains to show that

$$U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1}.$$

This can be checked in the case $n = 3$ and becomes the following exercise in matrix multiplication

$$\begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix}. \quad ☑$$

Remark 3.2. Setting $t = 1$, each of the matrices U_i becomes a permutation matrix, and one obtains representations of B_n that factor through the symmetric groups. For this reason, the Burau representations may be seen as deformation of the usual permutation representation.

The Burau representation admits a one dimensional invariant subspace. By taking the quotient of Λ^n by this invariant subspace one obtains the reduced Burau representations.

Proposition 3.3. *Let n be a natural number greater that 2 and $V_1, \ldots, V_{n-1}$ be the $(n-1) \times (n-1)$ matrices defined as follows:*

$$V_1 = \begin{bmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{bmatrix} \qquad V_{n-1} = \begin{bmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{bmatrix},$$

and for $1 < i < n-1$

$$V_i = \begin{bmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{bmatrix}.$$

Also, Let C be the $n \times n$ matrix over Λ

$$C = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then

$$C^{-1} U_i C = \begin{bmatrix} V_i & 0 \\ X_i & 1 \end{bmatrix},$$

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where X_i is the row of length $n-1$ which is $(0,\ldots,0)$ in case $i \neq n-1$ and is $(0,\ldots,0,1)$ for $i = n-1$.

Proof. We set

$$W_i := \begin{bmatrix} V_i & 0 \\ X_i & 1 \end{bmatrix}.$$

It suffices to prove that $U_i C = C W_i$ for all $i = 1,\ldots,n-1$. A direct computation shows that

$$U_i C = \begin{bmatrix} 1 & 1 & 1 & \cdots & & 1 & 1 \\ 0 & 1 & 1 & \cdots & & 1 & 1 \\ 0 & 0 & 1 & \cdots & & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 1-t & \cdots & & 1 \\ 0 & 0 & 0 & 1 & & \cdots & 1 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & 0 & 1 \end{bmatrix}.$$

Similarly, a simple calculation gives

$$C W_i = \begin{bmatrix} 1 & 1 & 1 & \cdots & & 1 & 1 \\ 0 & 1 & 1 & \cdots & & 1 & 1 \\ 0 & 0 & 1 & \cdots & & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 1-t & \cdots & & 1 \\ 0 & 0 & 0 & 1 & & \cdots & 1 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & 0 & 1 \end{bmatrix}.$$

Here, the matrices on the right hand side of the equations are those obtained from C by replacing the entry (i,i) by $1-t$ and the entry $(i+1,i)$ by 1. This completes the proof. ☑

Since conjugation with the matrix C is an automorphism, we conclude that the matrices W_i satisfy the braid relations. Moreover, since $\det(W_i) = \det(V_i)$, the matrices V_i are invertible as elements of $\mathrm{End}(\Lambda^{n-1})$. The fact that the last column of the matrices W_i is nonzero only in the last entry also implies that the matrices V_i satisfy the braid relations. These observations allow us to make the following definition

Definition 3.4. For $n \geq 2$ the reduced Burau representation of the braid group B_n is the representation $\psi_n^r : B_n \to \mathsf{Aut}\big(\Lambda^{n-1}\big)$ characterized by the property that

$$\sigma_i \mapsto V_i.$$

For $n = 2$ the reduced Burau representation is given by $\psi_2^r(\sigma_1) = -t$.

Remark 3.5. The Burau representation is known to be faithful for $n \leq 3$ and not faithful for $n \geq 5$. It is not known whether it is faithful for $n = 4$. A theorem of Bigelow asserts that ψ_4 is faithful if and only if the Jones polynomial detects the unknot.

3.2. Homological Interpretation

Let us denote by D the closed disk in the plane with n distinguished interior points $p_1 < \cdots < p_n \in D \cap \mathbb{R}$ and set $Q := \{p_1, \ldots, p_n\}$. Observe that for any point p in the interior of D,

$$H_1\big(D - \{p\}, \mathbb{Z}\big) \cong \mathbb{Z},$$

is generated by the homology class of a small circle around p oriented counterclockwise. Set $\Sigma := D - Q$ and fix a basepoint $d \in \partial D$. We define the group homomorphism $\phi : \pi_1(\Sigma, d) \to \mathbb{Z}$ by

$$[\gamma] \mapsto \sum_{i=1}^{n} w_i(\gamma),$$

where $w_i(\gamma)$ is the winding number of γ around p_i. The kernel of ϕ determines a covering space $\widetilde{\Sigma} \to \Sigma$, whose group of covering transformation is the infinite cyclic group $\mathbb{Z}$. We choose a point $\widetilde{d} \in \widetilde{\Sigma}$ over d and set

$$\widetilde{H} := H\big(\widetilde{\Sigma}, \mathbb{Z}\widetilde{d}, \mathbb{Z}\big).$$

Let F be a self homeomorphism of D that permutes the elements of Q. The restriction f of F to Σ preserves the total winding number i.e.,

$$\phi \circ f_* = \phi,$$

and therefore f can be lifted to a homeomorphism $\widetilde{F}$ of $\widetilde{\Sigma}$. This construction defines a representation

$$\Psi_n : \mathsf{Mod}(D, Q) \to \mathsf{Aut}\big(\widetilde{H}\big).$$

Since the group of deck transformations of $\widetilde{\Sigma} \to \Sigma$ is the infinite cyclic group, the homology group $\widetilde{H}$ has an action of $\mathbb{Z}$, and therefore it is a module over the ring Λ. The following theorem shows that the homological representation constructed above coincides with the Burau representation.

Theorem 3.6. *There exists an isomorphism of Λ-modules $\mu : \Lambda^n \to \widetilde{H}$ such that the following diagram commutes*

$$
\begin{array}{ccc}
B_n & \xrightarrow{\ \cong\ } & \mathsf{Mod}(D, Q) \\
\downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \Psi_n} \\
\mathsf{Aut}(\Lambda^n) & \xrightarrow{\ \underline{\mu}\ } & \mathsf{Aut}\big(\widetilde{H}\big),
\end{array}
$$

*Here $\underline{\mu}$ is the isomorphism induced by μ.*

Proof. For a proof the reader may consult [15, Theorem 3.7]. ☑

3.3. Braids, Knots and the Alexander Conway Polynomial

A knot is an isotopy class of an embedding of a circle in three dimensional euclidian space. More generally, a link with k components is an isotopy class of an embedding of k circles in $\mathbb{R}^3$. The fundamental problem of knot theory is the classification of knots and links. A typical example of a knot is
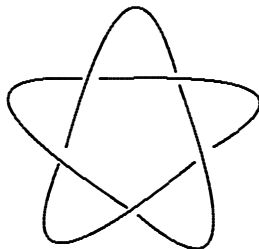
The relationship between links and braids is given by the operation of closure of a braid, which produces a link by connecting the strands of the braid, as explained in the diagram
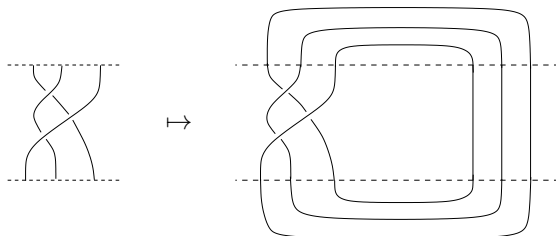

A link obtained as the closure of a braid comes with a natural orientation, given by declaring the strands of the braid to be flowing downwards.

Theorem 3.7 (Alexander [1])**.** *Any oriented link can be obtained as the closure of a braid.*

Proof. For a proof of this theorem the reader may consult [15, Theorem 2.3].
☑

The theorem above immediately raises the question of deciding when two braids produce the same oriented link upon closure. This question is answered by Markov's theorem. We say that two braids $\beta, \beta' \in B_n$ are related by the Markov move M_1 if their are conjugate, i.e. if there exists a $\gamma \in B_n$ such that $\gamma \beta \gamma^{-1} = \beta'$. The following picture describes the action of the operation M_1.
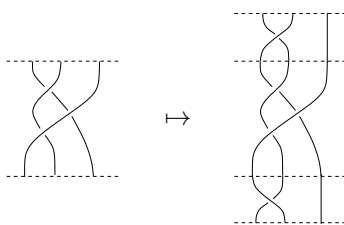
Let $\iota_n : B_n \to B_{n+1}$ be the natural inclusion of groups characterized by $\iota(\sigma_i) = \sigma_i$. Two braids $\beta \in B_n, \beta' \in B_{n+1}$ are related by the second Markov move M_2 if $\sigma_n^{\pm} \iota(\beta) = \beta'$. The following picture describes the action of the second Markov move on a braid
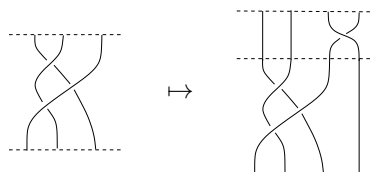
Definition 3.8. The Markov equivalence relation M is the smallest equivalence relation on the set of all braids such that two braids are equivalent if they are related by the Markov moves M_1 and M_2. We say that two braids are Markov equivalent if they are equivalent with respect to M.

Theorem 3.9 (Markov [21])**.** *Two braids are Markov equivalent if and only if their closures represent the same oriented link.*

Proof. The interested reader may consult [15, Theorem 2.8] .   ☑

Definition 3.10. A Markov function f with values in a set X is a sequence of functions $f_n : B_n \to X$ with the properties:

- $f_n\big(\gamma^{-1}\beta\gamma\big) = f_n(\beta)$.

- $f_n(\beta) = f_{n+1}\big(\sigma_n \iota_n(\beta)\big) = f_{n+1}\big(\sigma_n^{-1} \iota_n(\beta)\big).$

Clearly, Markov's theorem implies that any Markov function produces a link invariant with values in the set X. Let us now describe a Markov function provided by the Burau representation. Let $g : \Lambda = \mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[s, s^{-1}]$ be the ring homomorphism characterized by $t \mapsto s^2$. We define the group homomorphsim

$$\langle \;\; \rangle : B_n \to \mathbb{Z},$$

by setting $\langle \sigma_i \rangle = 1$.

For $n \geq 2$ we define the function $f_n : B_n \to X = \mathbb{Z}[s, s^{-1}]$ by the formula

$$f_n(\beta) = (-1)^{n+1} \frac{s^{-\langle \beta \rangle}(s - s^{-1})}{s^n - s^{-n}} g\big(\det\big(\psi_n^r(\beta) - \mathrm{id}\big)\big).$$

Theorem 3.11. *The sequence of functions f_n defined above is a Markov function with values in the set $X = \mathbb{Z}[s, s^{-1}]$.*

Proof. The interested reader may consult [15, Lemma 3.12].   ☑

The Markov function f_n defined above in terms of the Burau representation defines an invariant of oriented links known as the Alexander-Conway polynomial.

Definition 3.12. The Alexander-Conway polynomial of an oriented link L, denoted $\nabla(L)$, is the polynomial $f_n(\beta) \in \mathbb{Z}[s, s^{-1}]$, where β is any braid whose closure is L.

For the purpose of computation it is often useful to describe the skein relations for the Alexander Conway polynomials. These relations express the change in the value of the polynomial that occurs when modifying the crossing of a link diagram. A Conway triple L_+, L_-, L_0 is a triple of link diagrams that differ only locally at one crossing, which looks, respectively, as follows:
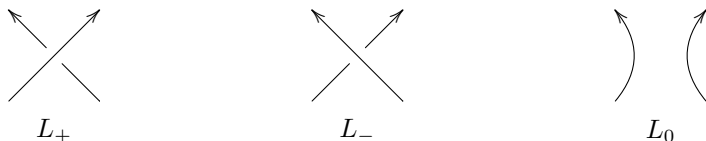
$$L_+ \qquad\qquad L_- \qquad\qquad L_0$$

Theorem 3.13. *The Alexander-Conway polynomial is the unique invariant of oriented links ∇ with values in $\mathbb{Z}[s, s^{-1}]$ which satisfies the following properties:*

- $\nabla(L) = 1$, *if L is the unknot.*

- *For any Conway triple L_+, L_-, L_0 :*

$$\nabla(L_+) - \nabla(L_-) = \big(s^{-1} - s\big)\nabla(L_0).$$

Proof. The proof can be found in [15, Theorem 3.13].   ☑

It is fairly easy to show that there is at most one invariant satisfying the skein relations above. The existence is the more interesting part of the theorem, and that is where the explicit construction of the invariants, provided by the Burau representations, is needed.

3.4. The Lawrence-Krammer-Bigelow Representation

We will now discuss another homological representation introduced by R. Lawrence [20] and studied by D. Krammer [19] and S. Bigelow [4]. Let us fix a natural number $n \geq 1$ and denote by D the unit disk with n distinguished points $Q = \{p_1, \ldots, p_n\}$. We also set

$$\Sigma := D - Q, \qquad \mathcal{F} := \widehat{C}_2(\Sigma), \qquad \mathcal{C} := C_2(\Sigma).$$

Here we use the notation introduced above for configuration spaces. That is, $\mathcal{F}$ is the configuration space of two ordered points in Σ, and $\mathcal{C}$ is the configuration space of two unordered points in Σ. There is a natural double sheeted covering $\mathcal{F} \to \mathcal{C}$. A point in $\mathcal{C}$ is an unordered set of distinct points of Σ and we use the notation $\{x, y\}$ for points of $\mathcal{C}$. A path in $\mathcal{C}$ can be represented as a pair of paths $\{\zeta_1, \zeta_2\}$ where $\zeta_i : I \to \Sigma$ such that for all $s \in I, \zeta_1(s) \neq \zeta_2(s)$. A loop in $\mathcal{C}$ is a path such that $\{\zeta_1(0), \zeta_2(0)\} = \{\zeta_1(1), \zeta_2(1)\}$. That is, a pair of maps (ζ_1, ζ_2) defines a loop in $\mathcal{C}$ if either they define a loop in $\mathcal{F}$ or define a path in $\mathcal{F}$ that permutes the starting points.

Given a loop $\zeta = \{\zeta_1, \zeta_2\}$ in $\mathcal{C}$ we define the number $w(\zeta) \in \mathbb{Z}$ as follows. If the pair (ζ_1, ζ_2) defines a loop in $\mathcal{F}$, then each of the ζ_i are loops in Σ and we set

$$w(\zeta) := \phi(\zeta_1) + \phi(\zeta_2),$$

where, as above, ϕ is the group homomorphism that counts the total winding number with respect to Q. If the pair (ζ_1, ζ_2) does not define a loop in $\mathcal{F}$, then it defines a path that permutes the endpoints. Therefore $\zeta_1 \circ \zeta_2$ defines a loop is Σ and we set

$$w(\zeta) := \phi(\zeta_1 \circ \zeta_2).$$

We define a second numerical invariant $u(\zeta)$ as follows. Consider the map $\gamma_\zeta : I \to S^1$ given by the formula

$$\gamma_\zeta(s) := \left(\frac{\zeta_1(s) - \zeta_2(s)}{|\zeta_1(s) - \zeta_2(s)|} \right)^2.$$

Observe that this map is well defined for ζ, since ζ_1 and ζ_2 play a symmetric role. Since ζ is a loop, we also conclude that $\gamma_\zeta(0) = \gamma_\zeta(1)$ and therefore γ_ζ defines a map $S^1 \to S^1$ whose degree is the value of u at ζ. It is a simple

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exercise to prove that the maps w and u are well defined on homotopy classes and are group homomorphisms $\pi_1(\mathcal{C}) \to \mathbb{Z}$.

We fix two distinct points $d_1, d_2 \in \partial(\Sigma)$ and take $\{d_1, d_2\}$ as the basepoint of $\mathcal{C}$. We denote by

$$\rho : \pi_1\big(\mathcal{C}, \{d_1, d_2\}\big) \to \mathbb{Z} \oplus \mathbb{Z},$$

the group homomorphism given by the formula

$$\rho(\zeta) = q^{w(\zeta)} t^{u(\zeta)},$$

where q and t are the canonical generators of $\mathbb{Z} \oplus \mathbb{Z}$.

Lemma 3.14. *The group homomorphism* $\rho : \pi_1\big(\mathcal{C}, \{d_1, d_2\}\big) \to \mathbb{Z} \oplus \mathbb{Z}$ *is surjective.*

Proof. It suffices to prove that both q and t are in the image of ρ. Let us first prove that q is in the image. Let $\zeta = \{\zeta_1, \zeta_2\}$ be the loop in $\mathcal{C}$ such that ζ_1 is constant and ζ_2 is a small counterclockwise loop around p_1. Then

$$w(\zeta) = \phi(\zeta_1) + \phi(\zeta_2) = 1.$$

Since ζ_2 stays close to p_1 and ζ_1 is constant, we conclude that $u(\zeta) = 0$. Thus $\rho(\zeta) = q$.

Let us now prove that t is in the image of ρ. Consider a small disk $B \subset \Sigma$ and define the loop $\varphi = \{\varphi_1, \varphi_2\}$ as follows. Fix two distinct points $x, y \in \partial(B)$. Set φ_1 the path that goes along ∂B from x to y in counterclockwise direction and φ_2 the path that goes along ∂B from y to x in counterclockwise direction. Then

$$w(\varphi) = \phi(\varphi_1 \circ \varphi_2) = 0$$

and

$$u(\varphi) = \deg(\mathrm{id}) = 1.$$

We conclude that $\rho(\varphi) = t$.   ☑

Let $\widetilde{\mathcal{C}} \to \mathcal{C}$ be the covering space that corresponds to the subgroup $\ker(\rho)$ of $\pi_1\big(C, \{d_1, d_2\}\big)$. The group $\mathbb{Z} \oplus \mathbb{Z}$ acts as the group of deck transformations of $\widetilde{\mathcal{C}}$. Therefore, the space

$$\mathcal{H} := H_2\big(\widetilde{\mathcal{C}}, \mathbb{Z}\big),$$

has the structure of a module over the ring

$$R := \mathbb{Z}\big[q, q^{-1}, t, t^{-1}\big].$$

We will now describe an action of the braid group B_n on the R-module $\mathcal{H}$. We will again use the identification

$$B_n \cong \mathsf{Mod}(D, Q).$$

Take a homeomorphism f of D that fixes the boundary pointwise and permutes the elements of Q. The map f induces a homeomorphism $\widehat{f} : \mathcal{C} \to \mathcal{C}$ given by

$$\widehat{f}(\{x, y\}) := \{f(x), f(y)\}.$$

Since f fixes the boundary pointwise, we know that $\widehat{f}(\{d_1, d_2\}) = \{d_1, d_2\}$ and therefore $\widehat{f}$ induces an isomorphism $\widehat{f}_* : \pi_1(\mathcal{C}, \{d_1, d_2\}) \to \pi_1(\mathcal{C}, \{d_1, d_2\})$.

Lemma 3.15. *The homomorphism $\widehat{f}_*$ preserves the invariant ρ, i.e. $\rho \circ \widehat{f}_* = \rho$.*

Proof. We need to prove that $w \circ \widehat{f}_* = w$ and $u \circ \widehat{f}_* = u$. The first statement reduces to the fact that the total winding number is preserved by $\widehat{f}_*$, which we have already seen. Let us now prove the second statement. Consider the inclusion of configuration spaces

$$\iota : \mathcal{C} = C_2(\Sigma) \hookrightarrow C_2(D),$$

induced by the inclusion $\Sigma \hookrightarrow D$. The definition of the invariant u can be extended word by word to an invariant $\widehat{u}$ of loops in $C_2(D)$ in such a way that the diagram

$$\pi_1(\mathcal{C}, \{x, y\}) \xrightarrow{\ \iota\ } \pi_1(C_2(D), \{x, y\})$$

$$u \searrow \qquad \downarrow \widehat{u}$$

$$\mathbb{Z}$$

commutes. On the other hand, $\widehat{f}$ extends to a homeomorphism $\overline{f}$ of $C_2(D)$ which, by the Alexander-Tietze Theorem proved in Example 2.9, is isotopic to the identity. Thus $\overline{f}$ acts trivially on the fundamental group of $C_2(D)$. Then, for any element $\gamma \in \pi_1(\mathcal{C}, \{x, y\})$,

$$u \circ \widehat{f}_*(\gamma) = \widehat{u} \circ \iota \circ \widehat{f}_*(\gamma) = \widehat{u} \circ \overline{f}_* \circ \iota \circ \gamma = \widehat{u} \circ \iota(\gamma) = u(\gamma). \qquad \checkmark$$

The previous lemma implies that the homeomorphism $\widehat{f}$ lifts naturally to a homeomorphism $\widetilde{f} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{C}}$ that commutes with the deck transformations of the covering. This homeomorphism induces a homomorphism in homology $\widetilde{f}_* : H_2(\widetilde{\mathcal{C}}, \mathbb{Z}) \to H_2(\widetilde{\mathcal{C}}, \mathbb{Z})$.

Definition 3.16. The Lawrence-Krammer-Bigelow representation of the braid group B_n is the homomorphism

$$B_n \to \mathsf{Aut}_R\big(H_2(\widetilde{\mathcal{C}}, \mathbb{Z})\big),$$

given by the formula

$$f \mapsto \widetilde{f}_*.$$

Theorem 3.17 (Krammer [19], Bigelow [4])**.** *The following statements hold for all $n \geq 2$:*

(1) *There is a natural isomorphism of R-modules*

$$\mathcal{H} \cong R^{\frac{n(n-1)}{2}}.$$

(2) *The Lawrence-Krammer-Bigelow representation $B_n \to \mathsf{Aut}_R(\mathcal{H})$ is faithful.*

Proof. The interested reader may consult [15, Theorem 3.15].   ☑

The second statement of the theorem implies that B_n is a linear group, i.e. it is isomorphic to a group of matrices with real coefficients. This can be obtained by setting q, t to be algebraically independent real numbers.

4. The Knizhnik-Zamolodchikov Connection

We will now describe representations of braid groups that arise by monodromy of certain flat connections, known as Knizhnik-Zamolodchivov connections, on configuration spaces of points in the plane. More about this connections and the relationship with the Jones polynomial can be found in Kohno's book [16].

4.1. Cohomology of Configuration Spaces and the KZ Connection

As before, we denote by $\widehat{C}_n(\mathbb{C})$ the configuration space of n ordered points in the plane. The cohomology of $\widehat{C}_n(\mathbb{C})$ has been computed explicitly by Arnold in [2]. Since the original article is in russian, some readers may prefer to read [22] for a detailed explanation of the computation. Let us now describe Arnold's result.

Definition 4.1. The algebra A_n is the graded commutative algebra over $\mathbb{C}$ generated by degree one elements $a_{ij} = a_{ji}$ for $i \neq j$, $1 \leq i, j \leq n$ modulo the Arnold relation

$$a_{ij}a_{jk} + a_{ki}a_{ij} + a_{jk}a_{ki} = 0, \quad \text{for} \quad i < j < k.$$

We also define the differential forms w_{ij} for $1 \leq i, j \leq n$ in $\widehat{C}_n(\mathbb{C})$ given by

$$w_{ij} = d\big(\log(z_i - z_j)\big) = \frac{dz_i - dz_j}{z_i - z_j}.$$

Theorem 4.2 (Arnold)**.** *There is a homomorphism of differential graded algebras $A_n \to \Omega\big(\widehat{C}_n(\mathbb{C})\big)$ given by*

$$a_{ij} \mapsto w_{ij}.$$

Moreover, this homomorphism induces an isomorphism in cohomology

$$A_n \cong H\big(\widehat{C}_n(\mathbb{C}), \mathbb{C}\big).$$

Proof. The fact that the formula above gives a homomorphism is an explicit computation with the differential forms w_{ij}. For the proof that the map is an isomorphism we recommend [22]. ☑

Arnold's computation can be used to define natural flat connections on configuration spaces.

Definition 4.3. For each $n \geq 2$, the Kohno-Drinfeld Lie algebra is the Lie algebra $\mathfrak{t_n}$ generated by the symbols $t_{ij} = t_{ji}$ for $1 \leq i, j \leq n$ modulo the relations:

$$[t_{ij}, t_{kl}] = 0, \quad \text{if} \quad \#\{i,j,k,l\} = 4,$$
$$[t_{ij}, t_{ik} + t_{jk}] = 0, \quad \text{if} \quad \#\{i,j,k\} = 3.$$

Definition 4.4. The Knizhnik-Zamolodchikov connection is the connection on the configuration space $\widehat{C}_n(\mathbb{C})$ with values in the Kohno-Drinfeld Lie algebra, given by the formula

$$\theta_n := \sum_{i<j} t_{ij} w_{ij}.$$

Lemma 4.5. *The Knizhnik-Zamolodchikov connection θ_n is flat, i.e.*

$$d\theta_n + \frac{1}{2}\big[\theta_n, \theta_n\big] = 0.$$

Proof. Since each of the forms $w_{ij} = d\big(\log(z_i - z_j)\big)$ is closed, it is enough to prove that

$$[\theta_n, \theta_n] = 0.$$

For this we use the Arnold relations as follows

$$[\theta_n, \theta_n] = \sum_{i<j,k<l} [t_{ij}\omega_{ij}, t_{kl}w_{kl}] = \sum_{i<j,k<l} [t_{ij}, t_{kl}]w_{ij}w_{kl}.$$

In view of the first relation in the Drinfeld-Kohno Lie algebra, it suffices to sum over sets of indices so that $\#\{i,j,k,l\} = 3$. Thus, the expression above is equal to

$$\sum_{i<j,k<l,\#\{i,j,k,l\}=3} [t_{ij}, t_{kl}]w_{ij}w_{kl}.$$

We now rewrite this expression by considering all the possible ways in which a pair of indices can be equal, and obtain

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$$\sum_{i<j<k} \big[t_{ij}, t_{ik}\big]w_{ij}w_{ik} + \big[t_{ik}, t_{ij}\big]w_{ik}w_{ij} + \big[t_{jk}, t_{ij}\big]w_{jk}w_{ij} +$$

$$\big[t_{ij}, t_{jk}\big]w_{ij}w_{jk} + \big[t_{ik}, t_{jk}\big]w_{ik}w_{jk} + \big[t_{jk}, t_{ik}\big]w_{jk}w_{ik}.$$

Using now the second relation in the Drinfeld-Kohno Lie algebra, we can rewrite the expression as

$$2\sum_{i<j<k} \big[t_{ij}, t_{jk}\big]\big(w_{ij}w_{jk} + w_{ki}w_{ij} + w_{jk}w_{ki}\big),$$

which vanishes in view of the Arnold relation. ☑

The KZ connection is a flat connection on the configuration space with values in the Drinfeld-Kohno Lie algebra. We will see how this connection can be used to construct flat connections on vector bundles on configurations spaces. Recall that given a finite dimensional complex Lie algebra $\mathfrak{g}$ the Killing form is the symmetric bilinear form κ on $\mathfrak{g}$ defined by

$$\kappa(x, y) := \mathsf{tr}\big(\mathsf{ad}(x) \circ \mathsf{ad}(y)\big),$$

where ad denotes the adjoint representation of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is called semisimple if the Killing form κ is nondegenerate. We denote by $\mathsf{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple then the killing form defines an isomorphism

$$\kappa^\sharp : \mathfrak{g} \to \mathfrak{g}^*,$$

which in turn induces identifications

$$\mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \mathrm{End}(\mathfrak{g}).$$

We will denote by Ω the element of $\mathfrak{g} \otimes \mathfrak{g}$ which corresponds to id $\in \mathrm{End}(\mathfrak{g})$ under the identification above. Explicitly, one can choose an orthonormal basis $\{I_\mu\}$ for $\mathfrak{g}$ with respect to the Killing form and then

$$\Omega = \sum_\mu I_\mu \otimes I_\mu.$$

The Casimir element of $\mathfrak{g}$, denoted by C, is the image of $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ in the universal enveloping algebra. Since the Killing form is ad invariant i.e. $\kappa\big(\mathsf{ad}(x)(y), z\big) + \kappa\big(y, \mathsf{ad}(x)(z)\big) = 0$, the map $\kappa^\sharp : \mathfrak{g} \to \mathfrak{g}^*$ is a morphism of representations of $\mathfrak{g}$. Since id $\in \mathrm{End}(\mathfrak{g})$ is an invariant element for the action of $\mathfrak{g}$, so is Ω. We conclude that C is a central element of $\mathsf{U}(\mathfrak{g})$.

Recall that $\mathsf{U}(\mathfrak{g})$ admits a coproduct

$$\Delta : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g}),$$

which is the unique algebra homomorphism with the property that

$$\Delta(x) = 1 \otimes x + x \otimes 1,$$

for all $x \in \mathfrak{g}$.

Lemma 4.6. *We regard $\mathfrak{g}$ as a subspace of $\mathsf{U}(\mathfrak{g})$ via the obvious inclusion. Then*

$$\Omega = \frac{1}{2}\big(\Delta(C) - 1 \otimes C - C \otimes 1\big).$$

Proof. This is a direct computation:

$$\begin{aligned}
\Delta(C) &= \Delta\left(\sum_\mu I_\mu I_\mu \right) \\
&= \sum_\mu \Delta(I_\mu)\Delta(I_\mu) \\
&= \sum_\mu (1 \otimes I_\mu + I_\mu \otimes 1)(1 \otimes I_\mu + I_\mu \otimes 1) \\
&= 1 \otimes C + C \otimes 1 + 2\Omega.
\end{aligned}$$ $\checkmark$

Let $\iota^{12} : \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g})$ be the map

$$x \otimes y \mapsto x \otimes y \otimes 1,$$

and define ι^{23}, ι^{13} analogously. Then, for $1 \leq i < j \leq 3$ we set

$$\Omega^{ij} := \iota^{ij}(\Omega).$$

Lemma 4.7. *The following relation is satisfied*

$$\big[\Omega^{12}, \Omega^{23} + \Omega^{13}\big] = 0.$$

Proof. First, we observe that since C is a central element in $\mathsf{U}(\mathfrak{g})$, $1 \otimes 1 \otimes C$, $1 \otimes C \otimes 1$, $C \otimes 1 \otimes 1$ are central elements in $\mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g})$. In view of Lemma 4.6, we know that for each pair $1 \leq i < j \leq 3$

$$\Omega^{ij} = \frac{1}{2}\iota^{ij}\big(\Delta(C)\big) + X^{ij},$$

where X^{ij} is central. Therefore it suffices to prove that

$$\Big[\iota^{12}\big(\Delta(C)\big), \iota^{23}\big(\Delta(C)\big) + \iota^{13}\big(\Delta(C)\big)\Big] = 0.$$

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In order to prove this we compute

$$\iota^{23}\big(\Delta(C)\big) = \iota^{23}\Big(1 \otimes C + C \otimes 1 + 2\sum_\mu I_\mu \otimes I_\mu\Big)$$
$$= 1 \otimes 1 \otimes C + 1 \otimes C \otimes 1 + 2\sum_\mu 1 \otimes I_\mu \otimes I_\mu.$$

Similarly

$$\iota^{13}\big(\Delta(C)\big) = \iota^{13}\Big(1 \otimes C + C \otimes 1 + 2\sum_\mu I_\mu \otimes I_\mu\Big)$$
$$= 1 \otimes 1 \otimes C + C \otimes 1 \otimes 1 + 2\sum_\mu I_\mu \otimes 1 \otimes I_\mu.$$

Therefore
$$\iota^{13}\big(\Delta(C)\big) + \iota^{23}\big(\Delta(C)\big) = 2\sum_\mu \Delta(I_\mu) \otimes I_\mu + X,$$

where X is central. Finally we compute

$$\frac{1}{2}\Big[\iota^{12}\big(\Delta(C)\big), \iota^{23}\big(\Delta(C)\big) + \iota^{13}\big(\Delta(C)\big)\Big]$$
$$= \Big[\Delta(C) \otimes 1, \sum_\mu \Delta(I_\mu) \otimes I_\mu\Big] = \sum_\mu \big[\Delta(C), \Delta(I_\mu)\big] \otimes I_\mu = 0. \quad \text{☑}$$

The previous lemma will be the key to constructing an action of the pure braid group P_n on tensor products of representations of a complex semisimple Lie algebra. Given a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ of a Lie algebra $\mathfrak{g}$, we will also denote by ρ the corresponding homomorphism of associative algebras $\rho : \mathsf{U}(\mathfrak{g}) \to \mathrm{End}(V)$.

Lemma 4.8. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\rho_1 : \mathfrak{g} \to \mathrm{End}(V_1), \ldots, \rho_n : \mathfrak{g} \to \mathrm{End}(V_n)$ be representations of $\mathfrak{g}$. Then there is a homomorphism of Lie algebras*

$$\Upsilon_n : \mathfrak{t}_n \to \mathrm{End}\big(V_1 \otimes \cdots \otimes V_n\big)$$

given by the formula

$$t_{ij} \mapsto (\rho_1 \otimes \cdots \otimes \rho_n) \circ \lambda^{ij}(\Omega) \in \mathrm{End}\big(V_1\big) \otimes \cdots \otimes \mathrm{End}\big(V_n\big)$$
$$\subset \mathrm{End}\big(V_1 \otimes \cdots \otimes V_n\big),$$

where $\lambda^{ij} : \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})^{\otimes n}$ is the morphism of algebras given by

$$x \otimes y \mapsto 1 \otimes \cdots \otimes 1 \otimes \underbrace{x}_{i} \otimes 1 \otimes \cdots \otimes 1 \otimes \underbrace{y}_{j} \otimes 1 \otimes \cdots \otimes 1.$$

Proof. We need to prove that the endomorphisms $\Upsilon(t_{ij})$ satisfy the Khono-Drinfeld relations. It is clear from the definition that

$$\big[\Upsilon(t_{ij}), \Upsilon(t_{kl})\big] = 0,$$

if $\#\{i, j, k, l\} = 4$. It remains to prove that

$$\big[\Upsilon(t_{ij}), \Upsilon(t_{ik}) + \Upsilon(t_{ik})\big] = 0,$$

if $\#\{i, j, k\} = 3$. Clearly, it is enough to consider the case $n = 3$. Since $\rho_1 \otimes \cdots \otimes \rho_n$ is a morphism of algebras, it suffices to prove that

$$\big[\Omega^{12}, \Omega^{23} + \Omega^{13}\big] = 0,$$

which is precisely the claim of Lemma 4.7.   ☑

Lemma 4.8 together with the flatness of the KZ connection imply the following result

Theorem 4.9. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\rho_1 : \mathfrak{g} \to \mathrm{End}\big(V_1\big), \ldots, \rho_n : \mathfrak{g} \to \mathrm{End}\big(V_n\big)$ be representations of $\mathfrak{g}$. The vector space $\big(V_1 \otimes \cdots \otimes V_n\big)$ has the structure of a representation of the pure braid group P_n, given by holonomy of the connection $\Upsilon_n(\theta_n)$ on the trivial vector bundle over $\widehat{C}_n(\mathbb{C})$ with fiber $\big(V_1 \otimes \cdots \otimes V_n\big)$.*

Proof. Since the KZ connection θ_n is flat and Υ_n is a morphism of Lie algebras, the connection $\Upsilon_n(\theta_n)$ is a flat connection on the trivial vector bundle over $\widehat{C}_n(\mathbb{C})$. Using the identification

$$P_n \cong \pi_1\Big(\widehat{C}_n(\mathbb{C}), \mathbf{p}\Big),$$

one obtains the desired representation.   ☑

The construction above can be symmetrized to obtain representation of the braid groups B_n.

Theorem 4.10. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be representation of $\mathfrak{g}$. The vector space $V^{\otimes n}$ has the structure of a representation of the braid group B_n.*

Proof. By applying the construction in Theorem 4.9 to the case $\rho_i = \rho$, one obtains a flat connection $\Upsilon_n(\theta_n)$ on the trivial vector bundle over $\widehat{C}_n(\mathbb{C})$ with fiber $V^{\otimes n}$. The symmetric group Σ_n acts on $V^{\otimes n}$ and also on $\widehat{C}_n(\mathbb{C})$ by permuting the coordinates. Therefore it also acts diagonally on $\Omega\big(\widehat{C}(\mathbb{C})\big) \otimes \mathrm{End}\big(V^{\otimes n}\big)$. One easily checks that $\Upsilon_n(\theta_n)$ is invariant under this diagonal action, and therefore descends to a flat connection form on the vector bundle $\big(\widehat{C}_n(\mathbb{C}) \times V^{\otimes n}\big)/\Sigma_n$ over $\widehat{C}_n(\mathbb{C})/\Sigma_n \cong C_n(\mathbb{C})$. The holonomy of this connection gives a representation of $\pi_1\big(C_n(\mathbb{C}), \mathbf{p}\big) \cong B_n$ on the vector space $V^{\otimes n}$.   ☑

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4.2. Representations of $\mathfrak{sl}_2(\mathbb{C})$ and Lawrence-Krammer-Bigelow

We will consider the KZ connection for the semisimple Lie algebras $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ of 2×2 complex matrices of trace zero. Let us fix the following basis for $\mathfrak{sl}_2(\mathbb{C})$:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

In terms of the basis, the bracket is given by:

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

Definition 4.11. Let $\lambda \in \mathbb{C}$ be a complex number. The Verma module M_λ with highest weight λ is the representation of $\mathfrak{sl}_2(\mathbb{C})$ defined as follows. As a vector space it is generated by the elements $F^j(v_\lambda)$ for $j \geq 0$. The action of $\mathfrak{sl}_2(\mathbb{C})$ on M_λ is given by:

$$\begin{aligned} F\big(F^j(v_\lambda)\big) &= F^{j+1}(v_\lambda), \\ E\big(F^j(v_\lambda)\big) &= j(\lambda - j + 1)F^{j-1}(v_\lambda), \\ H\big(F^j(v_\lambda)\big) &= (\lambda - 2j)F^j(v_\lambda). \end{aligned}$$

Given $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ we set $|\Lambda| = \lambda_1 + \cdots + \lambda_n$, and we define the space of weight vectors with weight $|\Lambda| - 2m$ to be the vector space

$$W\big[|\Lambda| - 2m\big] := \big\{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} : H(x) = \big(|\Lambda| - 2m\big)x\big\}.$$

The space of null vectors $N[|\Lambda| - 2m]$ is

$$N\big[|\Lambda| - 2m\big] := \big\{x \in W\big[|\Lambda| - 2m\big] : E(x) = 0\big\}.$$

Lemma 4.12. *The vector space $W\big[|\Lambda| - 2m\big]$ is finite dimensional, of dimension equal to the number of ordered partitions of length n of the number m.*

Proof. Since H acts on M_λ in a diagonal manner with respect to the given basis, a sum of basis elements is in $W\big[|\Lambda| - 2m\big]$ if and only if each of the basis elements is. Therefore, it suffices to count the number of basis elements in $W\big[|\Lambda| - 2m\big]$. A basis element $x = F^{j_i} \otimes \cdots \otimes F^{j_n}$ belongs to $W\big[|\Lambda| - 2m\big]$ precisely when

$$\big(|\Lambda| - 2m\big)x = H(x) = \sum_i (\lambda - 2j_i)x,$$

that is, precisely when $\sum_i j_i = m$. ☑

Given a nonzero complex number τ we define the KZ connection with parameter τ,

$$\theta_{n,\tau} := \frac{1}{\tau}\theta_n.$$

Note that since $d\theta_n = 0 = [\theta_n, \theta_n]$, the connection $\theta_{n,\tau}$ is flat. We now fix another complex number $\lambda \in \mathbb{C}$ and consider the representation $M_\lambda^{\otimes n}$. Using the notation in Theorem 4.9, the differential form $\Upsilon_n(\theta_{n,\tau})$ is a flat connection on $M_\lambda^{\otimes n}$.

Proposition 4.13. *The flat connection $\Upsilon_n\theta_{n,\tau}$ commutes with the action ρ : $\mathfrak{sl}_2(\mathbb{C}) \to \mathrm{End}\big(M_\lambda^{\otimes n}\big)$, i.e. for any vector field Y on the configuration space $\widehat{C}_n(\mathbb{C})$*

$$\rho(v)\big(\Upsilon_n\theta_{n,\tau}(Y)(x)\big) = \Upsilon_n\theta_{n,\tau}(Y)\big(\rho(v)(x)\big),$$

for any $v \in \mathfrak{sl}_2(\mathbb{C})$ and $x \in M_\lambda^{\otimes n}$. The vector space $N[n\lambda - 2m]$ is a representation

$$\mu_{n,m,\lambda,\tau} : B_n \to \mathsf{Aut}\big(N[n\lambda - 2m]\big)$$

of the braid group B_n.

Proof. It suffices to prove that each of the endomorphisms $\Upsilon_n(t_{ij})$ commutes with ρ. The action of $\mathfrak{sl}_2(\mathbb{C})$ on $M_\lambda^{\otimes n}$ is given by $\rho^{\otimes n} \circ \Delta^n(x) : \mathfrak{g} \subset \mathsf{U}(\mathfrak{g}) \to \mathrm{End}\big(M_\lambda^{\otimes n}\big)$. Therefore, it suffices to prove that each element $\Omega^{ij} = \lambda^{ij}(\Omega)$ commutes with $\Delta^n(v)$. By Lemma 4.6 we know that Ω differs from $\Delta(C)$ by a central element, so it suffices to prove that

$$\big[\lambda^{ij}\big(\Delta(C)\big), \Delta^n(v)\big] = 0,$$

which is a consequence of the fact that C is a central element of $\mathsf{U}(\mathfrak{g})$. This completes the proof of the first claim. Since the connection $\Upsilon_n\theta_{n,\tau}$ commutes with the action of ρ, then it preserves the finite dimensional vector space $N[n\lambda - 2m]$, and therefore it restricts to a flat connection on the trivial vector bundle over $\widehat{C}_n(\mathbb{C})$ with fiber $N[n\lambda - 2m]$. As before, the connection is invariant under the action of Σ_n and therefore it descends to a flat connection on the quotient vector bundle over $C_n(\mathbb{C})$. The holonomy of this connection gives $N[n\lambda - 2m]$ the structure of a representation of B_n. ☑

We are now ready to state the following remarkable theorem of Khono which describes the relation between the Lawrence-Krammer-Bigelow representations and the representations on the null spaces of Verma modules of $\mathfrak{sl}_2(\mathbb{C})$ given by holonomy of the KZ connection.

Theorem 4.14 (Kohno [18]). *There exists an open dense subset $U \subset \mathbb{C}^2$, such that for $(\tau, \lambda) \in U$ the representation*

$$\mu_{n,2,\lambda,\tau} : B_n \to \mathsf{Aut}\big(N[n\lambda - 4]\big)$$

is equivalent to the representation obtained from the Lawrence-Krammer-Bigelow representation

$$B_n \to \mathsf{Aut}_R(\mathcal{H}) \cong \mathsf{Aut}\Big(R^{\frac{n(n-1)}{2}}\Big),$$

by setting

$$q = \exp^{-2\pi i \frac{\lambda}{\tau}}, \; t = \exp^{\frac{2\pi i}{\tau}}.$$

Here, as before, R is the ring $R = \mathbb{Z}\big[q, q^{-1}, t, t^{-1}\big]$.

5. The Yang-Baxter Equation and the Drinfeld-Kohno Theorem

5.1. The Yang-Baxter Equation and Quasi-Triangular Bialgebras

The Yang-Baxter equation was originally introduced in the field of statistical mechanics. Solutions to this equation provide a systematic way to construct representations of the braid groups B_n. We will introduce the notion of quasi-triangular bialgebra, and explain that these are algebraic structures with the property that their modules are naturally endowed with solutions to the Yang-Baxter equation.

Definition 5.1. Let V be a complex vector space. An automorphism R of $V \otimes V$ is called an r-matrix if it satisfies the Yang-Baxter equation

$$(R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) = (\mathrm{id} \otimes R) \circ (R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes R) \in \mathrm{End}\big(V^{\otimes 3}\big).$$

Lemma 5.2. *Given an r-matrix $R \in \mathsf{Aut}(V \otimes V)$, there is a representation*

$$\rho_R : B_n \to \mathsf{Aut}\big(V^{\otimes n}\big),$$

defined by

$$\sigma_i \mapsto \mathrm{id}^{i-1} \otimes R \otimes \mathrm{id}^{n-i-1}.$$

Proof. The identity

$$\rho(\sigma_i)\rho(\sigma_{i+1})\rho(\sigma_i) = \rho(\sigma_{i+1})\rho(\sigma_i)\rho(\sigma_{i+1}),$$

follows from the fact that R satisfies the Yang-Baxter equation. The relation $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 2$ is clear from the form of the map ρ_R.   ☑

Let us now introduce some algebraic definitions.

Definition 5.3. A bialgebra is a tuple $(A, \mu, \eta, \Delta, \epsilon)$ where:

- A is a vector space over $\mathbb{C}$.

- μ is map $A \otimes A \to A$, called the product.

- η is a map $\mathbb{C} \to A$, called the unit.

- Δ is a map $\Delta : A \to A \otimes A$, called the coproduct.

- ϵ is a map $A \to \mathbb{C}$, called the counit.

Satisfying the conditions:

- (A, μ, η) is a unital associative algebra.

- (A, Δ, ϵ) is a counital coassociative coalgebra.

- $\Delta : A \to A \otimes A$ is a morphism of associative algebras.

Of course, we will abuse the notation and simply say that A is a bialgebra.

Example 5.4. We have already encountered an important example of a bialgebra, the universal enveloping algebra $\mathsf{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. Recall that the coproduct is the unique algebra homomorphism $\Delta : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g}) \otimes \mathsf{U}(\mathfrak{g})$, with the property that $\Delta(x) = 1 \otimes x + x \otimes 1$, for any $x \in \mathfrak{g}$. The bialgebra $\mathsf{U}(\mathfrak{g})$ is cocommutative but in general it is not commutative.

Given modules V, W of a bialgebra A, the vector space $V \otimes W$, is also a module of A with action given by

$$a(v \otimes w) := \Delta(a)(v \otimes w).$$

Thus, bialgebras are algebras for which the category of modules admits a tensor product. In general, the modules $V \otimes W$ and $W \otimes V$ need not be isomorphic. In case the coproduct Δ is cocommutative, the linear map

$$\tau_{V,W} : V \otimes W \to W \otimes V,$$
$$v \otimes w \mapsto w \otimes v,$$

is an isomorphism of A-modules. The condition that A is cocommutative is sufficient, but not necessary in order to have consistent isomorphism of A-modules $V \otimes W \cong W \otimes V$. This consideration leads to the following definition:

Definition 5.5. An almost-cocommutative bialgebra is a bialgebra $(A, \mu, \eta, \Delta, \epsilon)$ together with an invertible element $R \in A \otimes A$, called the universal r-matrix, such that for any $a \in A$

$$\tau_{A,A} \circ \Delta(a) = R\Delta(a)R^{-1},$$

where $\tau_{A.A} : A \otimes A \to A \otimes A$ is given by $a \otimes b \mapsto b \otimes a$.

Given an almost-cocommutative bialgebra, and modules V, W the map

$$\tau_{R,V,W} := \tau_{V,W} \circ R : V \otimes W \to W \otimes V,$$

is an isomorphism of A-modules. Thus, an almost-cocommutative bialgebra is a bialgebra for which there are natural isomorphisms of A-modules $V \otimes W \cong W \otimes V$.

Let us now consider A-modules V, W, Z. There are several ways to obtain isomorphisms of A-modules between $V \otimes W \otimes Z$ and $W \otimes V \otimes Z$, and it is natural to expect that these isomorphisms should coincide. Namely, one would expect the following diagrams to commute:

$$
\begin{array}{ccc}
V \otimes W \otimes Z & \xrightarrow{\tau_{R,V,W \otimes Z}} & W \otimes Z \otimes V \\
{\scriptstyle \tau_{R,V,W} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}} \\
W \otimes V \otimes Z & \xrightarrow{\mathrm{id} \otimes \tau_{R,V,Z}} & W \otimes Z \otimes V \ ,
\end{array}
\qquad (1)
$$

$$
\begin{array}{ccc}
V \otimes W \otimes Z & \xrightarrow{\tau_{R,V \otimes W,Z}} & Z \otimes V \otimes W \\
{\scriptstyle \mathrm{id} \otimes \tau_{R,W,Z}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}} \\
V \otimes Z \otimes W & \xrightarrow{\tau_{R,V,Z} \otimes \mathrm{id}} & Z \otimes V \otimes W \ .
\end{array}
\qquad (2)
$$

The commutativity of these diagrams can be assured by imposing conditions on the universal r-matrix R. This leads to the following definition.

Definition 5.6. A quasi-triangular bialgebra is an almost-cocommutative bialgebra A such that the universal r-matrix $R \in A \otimes A$ satisfies the equations:

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23},$$
$$(\mathrm{id} \otimes \Delta)R = R_{13}R_{12}.$$

Here $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$ and $R_{13} = (\tau_{A,A} \otimes \mathrm{id})(R_{23}) = (\mathrm{id} \otimes \tau_{A,A})(R_{12})$.

Example 5.7. The simplest example of a quasi-triangular bialgebra is a co-commutative bialgebra, with r-matrix $R = 1 \otimes 1$. Thus, the universal enveloping algebra $\mathsf{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a quasi-triangular bialgebra.

Remark 5.8. The definition of a quasi-triangular bialgebra is designed to guarantee that the category of modules has a well behaved tensor product. Namely, a tensor product with natural isomorphisms

$$C_{VW} : V \otimes W \cong W \otimes V,$$

for which the diagrams (1) and (2) commute. This kind of category is known as a braided monoidal category. Observe that in diagrams (1) and (2) expressions of the type $V \otimes W \otimes Z$ appear. This notation is using implicitly that the natural isomorphisms of vector spaces

$$(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$$

are isomorphism of representations. In more generality, one may consider the case where there are natural isomorphisms of representations

$$(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z),$$

which are not the obvious isomorphisms of vector spaces.

This corresponds to the general definition of braided monoidal category. The algebraic structure on A that guarantees that the category of modules is a braided monoidal category is that of a quasi-triangular quasi-bialgebra. Note that the fact that for a bialgebra the obvious linear isomorphism $(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$ is a morphism of modules corresponds to the fact that the coproduct is coassociative. Thus, in a quasi-triangular quasi-bialgebra one does not require Δ to be strictly coassociative. Instead, there is an element $\phi \in A \otimes A \otimes A$, called the Drinfeld associator, which controls the lack of associativity much in the same way in which R controls the lack of commutativity. We will not discuss the details of these elegant structures here. The interested reader may consult Kasssel's book [14].

Proposition 5.9. *Let A be a quasi-triangular bialgebra with universal r-matrix $R \in A \otimes A$. Given A-modules V, W, Z there are isomorphisms of A-modules*

$$\tau_{R,V,W} : V \otimes W \to W \otimes V,$$

defined by

$$\tau_{R,V,W} \coloneqq \tau_{V,W} \circ R,$$

satisfying the following equations:

$$(\tau_{R,V \otimes W,Z}) = (\tau_{R,V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{R,W,Z}), \tag{3}$$

$$(\tau_{R,V,W \otimes Z}) = (\mathrm{id} \otimes \tau_{R,V,Z}) \circ (\tau_{R,V,W} \otimes \mathrm{id}), \tag{4}$$

$$(\tau_{R,W,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{R,V,Z}) \circ (\tau_{R,V,W} \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes \tau_{R,V,W}). \circ (\tau_{R,V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{R,W,Z}). \tag{5}$$

Proof. Let us first check that $\tau_{R,V,W}$ is an isomorphism of A-modules. Since R is invertible, $\tau_{R,V,W}$ is an invertible linear map. Let us check that it is A-linear

$$\tau_{R,V,W}\big(a(v \otimes w)\big) = \tau_{R,V,W}\big(\Delta(a)(v \otimes w)\big) = \tau_{V,W} \circ R \circ \Delta(a)(v \otimes w)$$
$$= \tau_{V,W} \circ \tau_{A,A}\big(\Delta(a)\big) \circ R(v \otimes w) = \Delta(a) \circ \tau_{V,W} \circ R(v \otimes w)$$
$$= a \circ \tau_{R,V,W}(v \otimes w).$$

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Equations (3) and (4) are analogous. Let us prove equation (3). For this we use the fact that $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$, and compute

$$(\tau_{R,V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{R,W,Z})$$
$$= (\tau_{V,Z} \circ R \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{W,Z} \circ R)$$
$$= (\tau_{V,Z} \otimes \mathrm{id}) \circ R_{12} \circ (\mathrm{id} \otimes \tau_{W,Z}) \circ R_{23}$$
$$= (\tau_{V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{W,Z})(\mathrm{id} \otimes \tau_{A,A})(R_{12}) \circ R_{23}$$
$$= (\tau_{V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{W,Z})R_{13} \circ R_{23}$$
$$= (\tau_{V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{W,Z})(\Delta \otimes \mathrm{id})R$$
$$= (\tau_{V \otimes W,Z}) \circ (\Delta \otimes \mathrm{id})R$$
$$= \tau_{R,V \otimes W,Z}.$$

Let us now prove equation (5). We compute:

$$(\tau_{R,W,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{R,V,Z}) \circ (\tau_{R,V,W} \otimes \mathrm{id})$$
$$= (\tau_{W,Z} \otimes \mathrm{id}) \circ R_{12} \circ (\mathrm{id} \otimes \tau_{V,Z}) \circ R_{23} \circ (\tau_{V,W} \otimes \mathrm{id}) \circ R_{12}$$
$$= (\tau_{W,Z} \otimes \mathrm{id}) \circ R_{12} \circ (\mathrm{id} \otimes \tau_{V,Z}) \circ (\tau_{V,W} \otimes \mathrm{id}) \circ R_{13} \circ R_{12}$$
$$= (\tau_{W,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{V,Z}) \circ R_{13} \circ (\tau_{V,W} \otimes \mathrm{id}) \circ R_{13} \circ R_{12}$$
$$= (\tau_{W,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{V,Z}) \circ (\tau_{V,W} \otimes \mathrm{id}) \circ R_{23} \circ R_{13} \circ R_{12}$$
$$= (\tau_{W,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{V,Z}) \circ (\tau_{V,W} \otimes \mathrm{id}) \circ R_{23} \circ R_{13} \circ R_{12}.$$

On the other hand

$$(\mathrm{id} \otimes \tau_{R,V,W}) \circ (\tau_{R,V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{R,W,Z})$$
$$= (\mathrm{id} \otimes \tau_{V,W}) \circ R_{23} \circ (\tau_{V,Z} \otimes \mathrm{id}) \circ R_{12} \circ (\mathrm{id} \otimes \tau_{W,Z}) \circ R_{23}$$
$$= (\mathrm{id} \otimes \tau_{V,W}) \circ R_{23} \circ (\tau_{V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{W,Z}) \circ R_{13} \circ R_{23}$$
$$= (\mathrm{id} \otimes \tau_{V,W}) \circ (\tau_{V,Z} \otimes \mathrm{id}) \circ R_{13} \circ (\mathrm{id} \otimes \tau_{W,Z}) \circ R_{13} \circ R_{23}$$
$$= (\mathrm{id} \otimes \tau_{V,W}) \circ (\tau_{V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{W,Z}) \circ R_{12} \circ R_{13} \circ R_{23}.$$

Clearly,

$$(\tau_{W,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{V,Z}) \circ (\tau_{V,W} \otimes \mathrm{id}) =$$
$$(\mathrm{id} \otimes \tau_{V,W}) \circ (\tau_{V,Z} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{W,Z}).$$

Therefore, it suffices to prove that

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}.$$

For this we compute

$$\begin{aligned}
R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes \mathrm{id})R \\
&= (R \otimes 1)(\Delta \otimes \mathrm{id})R \\
&= (\tau_{A,A} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R)R_{12} \\
&= (\tau_{A,A} \otimes \mathrm{id})(R_{13}R_{23})R_{12} \\
&= R_{23}R_{13}R_{12}.
\end{aligned}$$

This completes the proof.   ☑

Corollary 5.10. *Let A be a quasi-triangular bialgebra with universal r-matrix R, and V a module over A. Then $\tau_{R,V,V} \in \mathsf{Aut}_A(V)$ is a solution of the Yang-Baxter equation. In particular, the vector space $V^{\otimes n}$ is a representation of the braid group B_n.*

5.2. The Drinfeld-Kohno Theorem

We have seen that modules over quasi-triangular bialgebras produce representations of the braid groups. However, so far we have not encountered nontrivial examples of quasi-triangular bialgebras. A rich source of examples comes from Lie theory: The universal enveloping algebra $\mathsf{U}(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$ can be deformed to obtain a quantum enveloping algebra. These provide interesting examples of quasi-triangular bialgebras and of representations of the braid groups.

In order to describe the Drinfeld-Jimbo bialgebras, it will be necessary to review some facts regarding the classification of semisimple Lie algebras. A good reference for this subject is Humphreys' book [12]. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an abelian subalgebra with the property that $N_\mathfrak{g}(\mathfrak{h}) = \mathfrak{h}$. Where $N_\mathfrak{g}(\mathfrak{h})$ is the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$, i.e.,

$$N_\mathfrak{g}(\mathfrak{h}) := \big\{x \in \mathfrak{g} : [\mathfrak{h},x] \subset \mathfrak{h}\big\}.$$

Cartan subalgebras always exist and moreover, any two Cartan subalgebras are conjugated to one another. Let us now on fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For any element $x \in \mathfrak{h}$, the linear map $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable. Since $\mathfrak{h}$ is abelian, all the maps $\mathrm{ad}(x)$ commute and therefore $\mathfrak{g}$ decomposes as a direct sum of eigenspaces

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

for some functionals $\alpha \in \mathfrak{h}^*$ so that

$$[h,y] = \alpha(h)y, \qquad \text{if } y \in \mathfrak{g}_\alpha, h \in \mathfrak{h}.$$

The set of roots, denoted by Δ, is the set of nonzero $\alpha \in \mathfrak{h}^*$ in the decomposition above.

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Lemma 5.11. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. For any $\alpha \in \mathfrak{h}^*$ set*

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [h, x] = \alpha(h)x, \quad \text{for all } h \in \mathfrak{h}\}.$$

Then

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}.$$

If $x \in \mathfrak{g}_\alpha$ for $\alpha \neq 0$, $\mathrm{ad}(x)$ is nilpotent. If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ with respect to the Killing form. In particular, the restriction of κ to $\mathfrak{h} = \mathfrak{g}_0$ is nondegenerate.

Proof. The first claim follows from direct computation. Take $h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$

$$[h[x, y]] = [[h, x], y] + [x, [h, y]] = \big(\alpha(x) + \beta(x)\big)[x, y].$$

For the second claim, consider $y \in \mathfrak{g}_\beta$, then

$$\mathrm{ad}(x)^n(y) \in \mathfrak{g}_{n\alpha+\beta},$$

since $\mathfrak{g}_\gamma$ is nonzero only for finitely many γ, we conclude that $\mathrm{ad}(x)^n = 0$ for n sufficiently large. Let us now prove the last statement. We choose a basis for $\mathfrak{g}$ compatible with the decomposition $g = \bigoplus_\gamma \mathfrak{g}_\gamma$ then if $x \in \mathfrak{g}_\alpha$ and $y \in g_\beta$ then

$$\mathrm{ad}(x) \circ \mathrm{ad}(y) : \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+\alpha+\beta},$$

so if $\alpha + \beta \neq 0$ the matrix associated to $\mathrm{ad}(x) \circ \mathrm{ad}(y)$ has zeros in the diagonal and therefore

$$\kappa(x, y) = 0. \qquad \qquad \checkmark$$

The set $\Delta \subset \mathfrak{h}^*$ of roots of $\mathfrak{g}$ has a beautiful combinatorial structure which makes it an abstract root system. Let us recall this definition

Definition 5.12. Let E be a finite dimensional real vector space with a symmetric positive definite bilinear form $(,)$. A root system in E is a finite subset $\Delta \subset E$ which spans E and does not containing zero, such that:

(1) If $\alpha \in \Delta$ then $-\alpha \in \Delta$, and no other multiple of α is in Δ.

(2) For each $\alpha \in \Delta$ the reflection σ_α with respect to the plane orthogonal to α fixes the set Δ.

(3) If $\alpha, \beta \in \Delta$ then

$$\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}.$$

Coming back to the roots of the Lie algebra $\mathfrak{g}$, since the Killing form κ is nondegenerate when restricted to $\mathfrak{h}$, it induces a bilinear form in $\mathfrak{h}^*$ which we still denote by κ.

Proposition 5.13. *The set Δ spans a real vector space $E \subset \mathfrak{h}^*$ of dimension equal to the complex dimension of $\mathfrak{h}^*$. Moreover, the bilinear form κ is positive definite when restricted to E, and the set Δ is an abstract root system in E.*

Proof. The interested reader may find the proof of this proposition in Humphreys' book [12]. ☑

The complete structure of an abstract root system $\Delta \subset E$ can be described by its Cartan matrix, which is defined as follows. Given $v \in E$ denote by T_v the space of vectors orthogonal to v. We say that $v \in E$ is singular if $v \in T_\alpha$ for some $\alpha \in \Delta$. We say that $v \in E$ is regular if it is not singular. Clearly, regular vectors exist since E is not the union of finitely many hyperplanes. Given a regular element $v \in E$, the set Δ decomposes into positive and negative roots

$$\Delta = \Delta^+ \coprod \Delta^-,$$

where

$$\Delta^+ := \{\alpha \in \Delta : (v, \alpha) > 0\} \quad \text{and} \quad \Delta^- := \{\alpha \in \Delta : (v, \alpha) < 0\}.$$

We say that a positive root α is simple if it cannot be written in the form $\alpha = \beta_1 + \beta_2$ where β_1, β_2 are positive roots.

Definition 5.14. Let Δ be an abstract root system in E and $v \in E$ a regular element. Fix an ordering $\alpha_1, \ldots, \alpha_l$ of the simple roots. The Cartan matrix of Δ with respect to v is the matrix

$$C_{i,j} := \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

Proposition 5.15. *The Cartan matrix C of the root system Δ is well defined, up to conjugation by a permutation matrix. Moreover, C has the following properties:*

(1) *The entries C_{ij} are non-positive integers if $i \neq j$ and $C_{ii} = 2$.*

(2) *There exists a unique diagonal matrix $D = \mathsf{Diag}(d_1, \ldots, d_l)$ with $d_i \in \{1, 2, 3\}$, such that DC is symmetric and positive definite.*

Proof. The interested reader can find the proof in Humphreys' book [12]. ☑

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We are ready to define the quantum enveloping algebra of a complex semisimple Lie algebra. These algebras are topological in the sense that all tensor products appearing in the definition are topological tensor products, which we now define. We denote by $\mathbb{C}[[h]]$ the algebra of formal power series in the variable h with complex coefficients. A module M over $\mathbb{C}[[h]]$ has the structure of a topological vector space: A basis of neighborhoods for zero is $\{h^n M\}_{n \geq 0}$.

Definition 5.16. Let M and N be $\mathbb{C}[[h]]$-modules. The topological tensor product of M and N, denoted $M \widehat{\otimes} N$ is the $\mathbb{C}[[h]]$-module

$$M \widehat{\otimes} N := \varprojlim \frac{M \otimes_{\mathbb{C}[[h]]} N}{h^n(M \otimes_{\mathbb{C}[[h]]} N)}.$$

There is a natural map $M \otimes_{\mathbb{C}[[h]]} N \to M \widehat{\otimes} N$.

Definition 5.17. A topological quasi-triangular bialgebra is a $\mathbb{C}[[h]]$-module A together with structure maps and universal r-matrix as in the definition of quasi-triangular bialgebra, where all tensor products are replaced by topological tensor products and structure maps are required to be continuous.

The following notation will be useful in describing the Drinfeld-Jimbo construction. Given an invertible element q in an algebra A and a positive integer n, we set:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}},$$
$$[n]_q! := [n]_q[n-1]_q \cdots [1]_q,$$
$$\binom{n}{k}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

Definition 5.18. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra with Cartan matrix C of size $l \times l$. The Drinfeld-Jimbo algebra $\mathsf{U}_h(\mathfrak{g})$ is the quotient of the free algebra over $\mathbb{C}[[h]]$ on the set $\{H_i, E_i, F_i\}_{i \leq l}$ by the closure of the two sided ideal generated by the following relations:

$$[H_i, H_j] = 0,$$
$$[E_i, F_j] - \delta_{ij}\left(\frac{\sinh\left(\frac{hd_i H_i}{2}\right)}{\sinh\left(\frac{hd_i}{2}\right)}\right) = 0,$$
$$[H_i, E_j] - C_{ij}E_j = 0,$$
$$[H_i, F_j] + C_{ij}F_j = 0,$$

$$\sum_{k=0}^{1-C_{ij}} (-1)^k \binom{1-C_{ij}}{k}_{q_i} E_i{}^k E_j E_i^{1-C_{ij}-k} = 0, \quad \text{for } i \neq j,$$

$$\sum_{k=0}^{1-C_{ij}} (-1)^k \binom{1-C_{ij}}{k}_{q_i} F_i{}^k F_j F_i^{1-C_{ij}-k} = 0, \quad \text{for } i \neq j.$$

Here $q_i := \exp\left(\frac{hd_i}{2}\right)$.

Theorem 5.19 (Drinfeld [7]-Jimbo [13]). *The algebra $\mathsf{U}_h(\mathfrak{g})$ is a topological bialgebra with structure maps characterized by the following properties:*

$$\Delta_h(H_i) = H_i \otimes 1 + 1 \otimes H_i,$$

$$\Delta_h(E_i) = E_i \otimes \exp\left(\frac{hd_iH_i}{4}\right) + \exp\left(\frac{-hd_iH_i}{4}\right) \otimes E_i,$$

$$\Delta_h(F_i) = E_i \otimes \exp\left(\frac{hd_iH_i}{4}\right) + \exp\left(\frac{-hd_iH_i}{4}\right) \otimes F_i,$$

$$\epsilon_h(H_i) = \epsilon_h(E_i) = \epsilon_h(F_i) = 0.$$

There exists a universal r-matrix R_h, which can be written explicitly, making $\mathsf{U}_h(\mathfrak{g})$ a topological quasi-triangular bialgebra. Moreover, there is a canonical isomorphism of algebras

$$\frac{\mathsf{U}_h(\mathfrak{g})}{h\big(\mathsf{U}_h(\mathfrak{g})\big)} \cong \mathsf{U}(\mathfrak{g}).$$

Proof. The proof of this theorem can be found in [7].   ☑

Given a finite dimensional representation V of the complex semisimple Lie algebra $\mathfrak{g}$, there exists a corresponding representation of the Drinfeld-Jimbo algebra $\mathsf{U}_h(\mathfrak{g})$. This fact can be deduced from the following result

Theorem 5.20. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra with Drinfeld-Jimbo algebra $\mathsf{U}_q(\mathfrak{g})$. There exists an isomorphism of topological algebras*

$$\varphi : \mathsf{U}_h(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})[[h]],$$

which is the identity modulo h. Moreover, given any other such isomorphism φ' there exists an invertible element $F \in \mathsf{U}(\mathfrak{g})[[h]]$, congruent to 1 modulo h, such that for any $x \in \mathsf{U}_h(\mathfrak{g})$

$$\varphi'(x) = F\varphi(x)F^{-1}.$$

Proof. The proof of this theorem can be found in Kassel's book [14].   ☑

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The theorem above implies that given a complex finite dimensional representation V of $\mathfrak{g}$, the $\mathbb{C}[[h]]$-module $V[[h]]$ is a representation of $\mathsf{U}_h(\mathfrak{g})$, well defined up to isomorphism. Indeed, $V[[h]]$ is a module over $\mathsf{U}(\mathfrak{g})[[h]]$ and we can choose the isomorphism φ as above to make it a module over the Drinfeld-Jimbo algebra. Since φ is unique up to conjugation, the isomorphism class of the representation is well defined. Because $\mathsf{U}_h(\mathfrak{g})$ is a topological quasi-triangular bialgebra, the universal r-matrix $R_h \in \mathsf{U}_h(\mathfrak{g})\widehat{\otimes}\mathsf{U}_h(\mathfrak{g})$ defines an r-matrix on $V[[h]]\widehat{\otimes}V[[h]]$. We conclude that for each $n \geq 2$ the $\mathbb{C}[[h]]$-module $V[[h]]^{\widehat{\otimes}n} \cong V^{\otimes n}[[h]]$ is a representation of the braid group B_n by $\mathbb{C}[[h]]$-linear automorphisms. We denote this representation by

$$\rho_{DJ} : B_n \to \mathsf{Aut}_{\mathbb{C}[[h]]}\big(V^{\otimes n}[[h]]\big).$$

For a finite dimensional representation V of a complex semisimple Lie algebra $\mathfrak{g}$, there are two different constructions of representations of the braid groups. On the one hand, the monodromy of the Knizhnik-Zamolodchikov connections gives the vector space $V^{\otimes n}$ the structure of a representation of B_n. On the other hand, the universal r-matrix in the Drinfeld-Jimbo algebra $\mathsf{U}_h(\mathfrak{g})$ provides a representation of B_n on $V^{\otimes n}[[h]]$. The relationship between these two seemingly unrelated constructions is provided by the Drinfeld-Kohno theorem. For a complex number h, let $\theta_{n,h}$ be the Knizhnik-Zamolodchikov connection with parameter h defined by

$$\theta_{n,h} := \frac{h}{2\pi i}\theta_n.$$

Since $d\theta_n = 0 = [\theta_n, \theta_n]$, the connection $\theta_{n,h}$ is flat. As before, for each $h \in \mathbb{C}$ we obtain a representation of the braid group $\rho_h : B_n \to V^{\otimes n}$. The monodromy of the connection $\theta_{n,h}$ can be computed explicitly in terms of iterated integrals and in particular, depends on h in an analytic manner. Therefore, by taking Taylor series with respect to the parameter h, we obtain a representation

$$\rho_{KZ} : B_n \to \mathsf{Aut}_{\mathbb{C}[[h]]}\big(V^{\otimes n}[[h]]\big).$$

Theorem 5.21 (Drinfeld [8] - Kohno [17]). *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and V a finite dimensional complex representation. For each $n \geq 2$, the Knizhnik-Zamolodchikov representation of the braid group B_n*

$$\rho_{KZ} : B_n \to \mathsf{Aut}_{\mathbb{C}[[h]]}\big(V^{\otimes n}[[h]]\big),$$

obtained by taking Taylor series on the monodromy of the Knizhnik-Zamolodchikov connection is equivalent to the Drinfeld-Jimbo representation

$$\rho_{DJ} : B_n \to \mathsf{Aut}_{\mathbb{C}[[h]]}\big(V^{\otimes n}[[h]]\big),$$

obtained via the universal r-matrix of the Drinfeld-Jimbo algebra $\mathsf{U}_h(\mathfrak{g})$.

Proof. The interested reader may refer to Kassel's book [14].   ☑

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