Revista Colombiana de Matemáticas
Volumen 49(2015)2, páginas 235-259

Principal spin-bundles and triality

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Abstract. In this paper we construct a family of spin Lie groups G with an outer automorphism of order three (triality automorphism) and we describe the subgroups of fixed points for this kind of automorphisms. We will take advantage of this work to study the action of the group of outer automorphisms of G on the moduli space of principal G-bundles and describe the subvariety of fixed points in $M(G)$ for the action of the outer automorphism of order three of G. Finally, we further study the case of $\mathrm{Spin}(8, \mathbb{C})$.

Key words and phrases. triality, Spin-principal bundles, moduli space, fixed points, stability.

2010 Mathematics Subject Classification. 14D20.

Resumen. En este artículo construimos una familia de grupos de Lie espinoriales G dotados de un automorfismo externo de orden tres (trialidad) y describimos los subgrupos de puntos fijos para esta clase de automorfismos. Usaremos esto para estudiar la acción del grupo de automorfismos externos de G en el espacio de moduli de G-fibrados principales y describir la subvariedad de puntos fijos en $M(G)$ para la acción del automorfismo externo de orden tres de G. Finalmente, profundizaremos en el estudio del caso $\mathrm{Spin}(8, \mathbb{C})$.

Palabras y frases clave. Trialidad, Spin-fibrado principal, espacio de moduli, puntos fijos, estabilidad.

Introduction

Let X be a compact Riemann surface of genus $g \geq 2$ and let G be a complex reductive Lie group with Lie algebra $\mathfrak{g}$. The notions of stability, semistability and polystability for principal G-bundles over X were given by Ramanathan in [15], obtaining that the moduli space of polystable principal G-bundles, $M(G)$ is a complex variety whose open subvariety of non-singular points is $M_s(G)$, the subset of stable principal G-bundles. Stable principal bundles are of interest in many different areas like surface group representations. In [15, Theorem 7.1] it is proved that there exists an isomorphism between $M_s(G)$ and the quotient of the unitary representations of $\pi_1(X \setminus \{x_0\})$ in G modulo the action of G by conjugation. This kind of spaces have a very rich topology and geometry and have been intensively studied in relation to several types of moduli spaces of bundles, including unitary principal bundles ([13]) and Higgs bundles ([8]).

A way of studying the geometry of $M(G)$ is by the study of subvarieties of the moduli space. Given an automorphism of $M(G)$, the subset of fixed points in $M(G)$ for this automorphism is a natural subvariety. It is also natural to study automorphisms of finite order of $M(G)$. The case of involutions was developed by Garcia-Prada in [5] for the more general case of Higgs bundles. We are interested in automorphisms of order three. In [1], we studied the case in which $G = \mathrm{Spin}(8, \mathbb{C})$ and the automorphisms of order three come from the triality automorphism of G. Triality is a very special phenomenon which appears frequently in algebra and geometry and plays an important role in many areas of mathematics and physics. In this paper we will see how the group of outer automorphisms of G, $\mathrm{Out}(G)$, acts in $M(G)$ (it is well known that the action of inner automorphisms is trivial) inducing a subgroup of the group of automorphisms of $M(G)$ (in the spirit of [12]). When $G = \mathrm{Spin}(8, \mathbb{C})$, this says that triality induces an automorphism of order three of $M(G)$. Here, taking advantage of the Cayley-Dickson construction of Cayley algebras and the ideas of A. Elduque in [3], we will construct a family of orthogonal Lie algebras, including $\mathfrak{g} = \mathfrak{o}(8, \mathbb{C})$, having S_3 as a subgroup of automorphisms. This gives rise to a family of moduli spaces, all of them with structure group of Spin type having S_3 as a subgroup of the group of automorphisms. The main aim of this paper is to study the subvariety of fixed points in $M(G)$ for the automorphisms of order three listed above. We will see that this subvariety always contains the reductions of the structure group of the principal bundle to the subgroup of G of fixed points of the corresponding order three automorphism. Moreover, we will see that stable and simple fixed points are always of this form (and they do not exist for the cases in which there is only one automorphism of G lifting the outer automorphism of order three of G). We also give a family of Spin groups parametrized by the integers for which there are no stable fixed points for the action of triality. Finally, we fix our attention in the simple group $\mathrm{Spin}(8, \mathbb{C})$, for which we can give a complete description of the subvariety of fixed points in the moduli space.

This paper is organized as follows. In Section 1 we recall some properties of the Clifford algebra of a nonsingular quadratic form q, which plays an important role in our construction of orthogonal algebras with a triality automorphism. Section 2 deals with the construction of the group of outer automorphism of the Clifford algebra by taking advantage of the theory of Cayley algebras. Section 3 is intended to compute the subgroup of fixed points for all outer automorphisms of order three of the algebras constructed above. In Section 4 we establish appropriate stability conditions for $\mathrm{Spin}(n, \mathbb{C})$ and $\mathrm{SO}(n, \mathbb{C})$-bundles and we describe precisely the stability condition when applied to principal G_2-bundles, which will play a role in our study. Section 5 is devoted to describing how $\mathrm{Out}(G)$ acts nontrivially on $M(G)$, then $\mathrm{Out}(G)$ is a subgroup of $\mathrm{Aut}(M(G))$. In Section 6 we establish a geometric description of the subvariety of stable fixed points in $M(G)$ for the action of outer automorphisms of order three in terms of the moduli space of principal G_2-bundles. Finally, in Section 7 we further this study for the case of $\mathrm{Spin}(8, \mathbb{C})$.

1. Preliminaries in Clifford algebras

Let V be a complex vector space of even dimension $n = 2l$ and b a symmetric nonsingular bilinear form on V with associated quadratic form $q : V \to \mathbb{C}$, so $b(x, y) = q(x + y) - q(x) - q(y)$ for $x, y \in V$. The bilinear form b defines an involution $\sigma_b : \mathrm{End}(V) \to \mathrm{End}(V)$ through the identity

$$b(\sigma_b(f)(x), y) = b(x, f(y))$$

for $f \in \mathrm{End}(V)$ and $x, y \in V$, which is called the adjoint involution. The subspace of skew-symmetric endomorphisms of V, that is,

$$\mathfrak{o}(b) = \{f \in \mathrm{End}(V) : \sigma_b(f) = -f\} = \{f \in \mathrm{End}(V) : b(f(x), y) + b(x, f(y)) = 0\},$$

is a Lie subalgebra of $\mathrm{End}(V)$ for the canonical Lie bracket of $\mathrm{End}(V)$ given by $[f, g] = f \circ g - g \circ f$.

Recall that, if TV is the tensor algebra of V with the tensor product and I is the ideal of TV generated by the elements of the form $x \otimes x - q(x)$ for $x \in V$, the Clifford algebra of the quadratic space (V, q) is defined by TV/I. The Clifford algebra is central simple of dimension 2^n. We can see the vector space V as a vector subspace of $C(V, q)$ via the natural inclusion $V \hookrightarrow C(V, q)$. This inclusion allows us to identify V with a subspace of generators of the Clifford algebra. We will consider the even Clifford algebra, $C_0(V, q)$, the subalgebra of $C(V, q)$ generated by tensor products of an even number of elements of V. In the case in which $n = 2l$ is even, we have the following result (for a proof and details, see [19]):

Proposition 1.1. *There exists a unique involution τ of $C(V, q)$ which is the identity on V. If n is congruent to 0 modulo 4, then the involution τ restricts to an involution τ_0 of $C_0(V, q)$, which is the identity on $Z = Z(C_0(V, q))$. Further, if n is congruent to 0 modulo 8 then there exists a nonsingular symmetric bilinear form on $C_0(V, q)$ such that τ_0 is the associated adjoint involution.*

***Remark* 1.2.** The involution τ is defined by permuting the order in the elements of $C(V, b)$, that is, if $x, y \in V$,

$$\tau(xy) = yx. \tag{1}$$

From now on, we will suppose that n is congruent to 0 modulo 8. In this case, we will consider, as in the case of V, the Lie algebra of skew-symmetric endomorphisms of $C_0(V,q)$, $\mathrm{Skew}(C_0(V,q),\tau_0)$.

Let $[V,V]$ the subspace of $C(V,q)$ spanned by the elements of the form $[x,y] = x \otimes y - y \otimes x$ for $x, y \in V$. We define the adjoint linear map

$$\mathrm{ad} : [V,V] \to \mathrm{End}(V)$$

by $\mathrm{ad}(\alpha) = \mathrm{ad}_\alpha$ where $\mathrm{ad}_\alpha(z) = [\alpha, z]$. This linear map is well-defined because for each $x, y, z \in V$, we have

$$[[x,y],z] = 2(b(y,z)x - b(x,z)y) \in V. \tag{2}$$

To see this, just express $b(y,z)$ and $b(x,z)$ in terms of q and apply the definition of the ideal I which defines $C(V,q)$. It is not difficult to see that ad is injective and its image is a Lie subalgebra of $\mathrm{Skew}(C_0(V,q),\tau_0)$ (see [10]). Moreover, ad induces an isomorphism of Lie algebras

$$\mathrm{ad} : [V,V] \to \mathfrak{o}(q) \hookrightarrow \mathrm{End}(V).$$

(For a proof, see [10]).

Definition 1.3. A similitude on the quadratic space of dimension n, (V,b) is a linear automorphism of V such that for all $x, y \in V$,

$$b(f(x),y) = m(f)b(x, f^{-1}(y))$$

for some $m(f) \in \mathbb{C}^*$ called the multiplier of the similitude. A similitude f is proper if $\det(f) = m(f)^{n/2}$.

***Remark* 1.4.** In the preceding definition, f is a similitude if and only if for all $x \in V$ we have $q(f(x)) = m(f)q(x)$.

Similitudes of V form a group $\mathrm{GO}(V,b)$ or simply $\mathrm{GO}(b)$. Moreover, proper similitudes of V form a normal subgroup of $\mathrm{GO}(b)$ of index 2 which we will denote $\mathrm{GO}^+(b)$. A similitude f of (V,b) is an isometry if, and only if, $m(f) = 1$. In this case, the similitude is proper, so we have a natural injection $\mathrm{SO}(b) \hookrightarrow \mathrm{GO}^+(b)$.

The following result relates similitudes and automorphisms of $C(V,b)$ and $C_0(V,b)$.

Proposition 1.5. *Any isometry* $f \in \mathrm{SO}(b)$ *induces an automorphism* $C(f)$ *of* $C_0(V,b)$ *such that* $\mathrm{ad} \circ C(f) \circ \mathrm{ad}^{-1} = \mathrm{Int}(f)$*. If* $f \in \mathrm{Go}(b)$ *is a similitude with multiplier* $m(f)$*, then* f *induces an automorphism* $C(f)$ *of* $C_0(f)$ *such that*

$$C(f)(xy) = m(f)^{-1} f(x) f(y) \tag{3}$$

and $\mathrm{ad} \circ C(f) \circ \mathrm{ad}^{-1} = \mathrm{Int}(f)$.

Proof. The first claim follows from the second. Let $f \in \mathrm{GO}(b)$ be a similitude with multiplier $m(f)$. We define $C(f)$ as in (3). This is a good definition because if $x \in V$,

$$C(f)(xx) = m(f)^{-1} f(x) f(x) = m(f)^{-1} q(f(x)) = m(f)^{-1} m(f) q(x) = q(x).$$

Linearity and injectivity are immediate, so we consider $\mathrm{ad} \circ C(f) \circ \mathrm{ad}^{-1} : \mathfrak{o}(b) \to \mathfrak{o}(b)$. Let $A \in \mathfrak{o}(b)$. For simplicity, suppose that there exist $x, y \in V$ such that $[x, y] = \mathrm{ad}^{-1}(A)$. Then,

$$\mathrm{ad} \circ C(f) \circ \mathrm{ad}^{-1}(A) = \mathrm{ad} \circ C(f)([x, y]).$$

For $z \in V$, from the definition of similitude and (2),

$$\begin{aligned} \mathrm{ad} \circ C(f)([x, y])(z) &= m(f)^{-1} \mathrm{ad}(f(x)f(y) - f(y)f(x))(z) \\ &= m(f)^{-1} \left[[f(x), f(y)], z \right] \\ &= 2m(f)^{-1} \left(b(f(y)z) f(x) - b(f(x)z) f(y) \right) \\ &= 2f \left(b\left(y, f^{-1}(z)\right) x - b\left(x, f^{-1}(z)\right) y \right) \\ &= f \circ \mathrm{ad} \circ f^{-1}([x, y])(z) = \mathrm{Int}(f) \circ \mathrm{ad}([x, y])(z). \end{aligned}$$

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It is immediate that each $\lambda \in \mathbb{C}^*$ defines a similitude of multiplier λ^2 by scalar product on the elements of V, so that for $x, y \in V$,

$$C(\lambda)(xy) = \lambda^{-2}(\lambda x)(\lambda y) = xy$$

and $C(\lambda)$ acts trivially on $C_0(V, b)$. Then, the quotient groups $\mathrm{PGO}(b) = \mathrm{GO}(b)/\mathbb{C}^*$ and $\mathrm{PGO}^+(b) = \mathrm{GO}^+(b)/\mathbb{C}^*$ act on $C_0(V, q)$ giving rise to homomorphisms

$$\begin{aligned} C &: \mathrm{PGO}(b) \to \mathrm{Aut}(C_0(V, b), \tau_0) \\ C &: \mathrm{PGO}^+(b) \to \mathrm{Aut}(C_0(V, b), \tau_0). \end{aligned}$$

2. Cayley algebras and the triality automorphism

Let $\mathfrak{C}$ be a complex Cayley algebra with conjugation map $\pi : \mathfrak{C} \to \mathfrak{C}$, $\pi(x) = \overline{x}$. The conjugation induces a norm $\mathfrak{n}(x) = x\overline{x}$ and the associated bilinear form $\mathfrak{b}$, $\mathfrak{b}(x, y) = \mathfrak{n}(x + y) - \mathfrak{n}(x) - \mathfrak{n}(y)$ for $x, y \in \mathfrak{C}$, which is nonsingular. The conjugation also defines a new product, $\star$, in $\mathfrak{C}$ given by $x \star y = \overline{xy}$ for $x, y \in \mathfrak{C}$. The bilinear form $\mathfrak{b}$ is associative with respect to $\star$, that is, for each $x, y \in \mathfrak{C}$,

$$\mathfrak{b}(x \star y, z) = \mathfrak{b}(x, y \star z). \tag{4}$$

Each $x \in \mathfrak{C}$ induces endomorphisms $r_x, l_x \in \mathrm{End}(\mathfrak{C})$ defined by $r_x(y) = y \star x$ and $l_x(y) = x \star y$.

A simple computation shows that for all $x, y \in \mathfrak{C}$,

$$x \star (y \star x) = (x \star y) \star x = \mathfrak{n}(x)y. \tag{5}$$

The complex algebra $\mathfrak{C}\oplus\mathfrak{C}$ is also a complex Cayley algebra with the obvious conjugation map $\pi \oplus \pi$ and the bilinear form $\mathfrak{b}\perp\mathfrak{b}$ which makes each summand to be orthogonal to the other.

With these preliminaries one can easily prove the following:

Proposition 2.1. *The homomorphism* $C(\mathfrak{C}, \mathfrak{b}) \to \mathrm{End}(\mathfrak{C} \oplus \mathfrak{C})$ *defined by*

$$x \mapsto \begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix}$$

induces an isomorphism of algebras with involution $\alpha : (C(\mathfrak{C}, \mathfrak{b}), \tau) \to (\mathrm{End}(\mathfrak{C}\oplus \mathfrak{C}), \sigma_{\mathfrak{b}\perp\mathfrak{b}})$ *which reduces to an isomorphism*

$$\alpha_0 : (C_0(\mathfrak{C}, \mathfrak{b}), \tau_0) \to (\mathrm{End}(\mathfrak{C}), \sigma_{\mathfrak{b}}) \times (\mathrm{End}(\mathfrak{C}), \sigma_{\mathfrak{b}}).$$

Proof. From (5), α is well defined. To see that α is compatible with involutions it suffices to see that $\alpha(\tau(xy)) = \mathfrak{b}\perp\mathfrak{b}(\alpha(xy))$ for all $x, y \in \mathfrak{C}$. So let $x, y \in \mathfrak{C}$. From Remark (1), we have that

$$\alpha(\tau(xy)) = \alpha(yx) = \begin{pmatrix} l_y r_x & 0 \\ 0 & r_y l_x \end{pmatrix}$$

and

$$\sigma_{\mathfrak{b}\perp\mathfrak{b}}(\alpha(xy)) = \sigma_{\mathfrak{b}\perp\mathfrak{b}} \begin{pmatrix} l_x r_y & 0 \\ 0 & r_x l_y \end{pmatrix},$$

so, by definition of $\sigma_{\mathfrak{b}\perp\mathfrak{b}}$, $\sigma_{\mathfrak{b}\perp\mathfrak{b}}(\alpha(xy)) = \alpha(\tau(xy))$ if and only if for all $z, t \in \mathfrak{C}$,

$$\mathfrak{b}(l_x r_y(z), t) = \mathfrak{b}(z, l_y r_x(t))$$

or, what is the same,

$$\mathfrak{b}(x \star (z \star y), t) = \mathfrak{b}(z, y \star (t \star x)),$$

which follows from (4).

As $C(\mathfrak{C}, \mathfrak{b})$ is central simple, α in injective so, by dimensions, it is bijective. The second claim about α_0 is immediate from the preceding computations. ☑

Proposition 2.2. *The isomorphism* α_0 *in Proposition 2.1 maps injectively* $[\mathfrak{C}, \mathfrak{C}] \subseteq C(\mathfrak{C}, \mathfrak{b})$ *into* $\mathfrak{o}(\mathfrak{b}) \times \mathfrak{o}(\mathfrak{b})$.

Proof. Let $[x, y] \in [\mathfrak{C}, \mathfrak{C}]$. From the definition of τ and the fact that α_0 commutes with the involutions we have that, for all $z, t \in \mathfrak{C}$,

$$\begin{aligned}\mathfrak{b}\times\mathfrak{b}(z,\alpha_0([x,y])(t)) &= \mathfrak{b}\times\mathfrak{b}(z,\alpha_0(xy)(t)) - \mathfrak{b}\times\mathfrak{b}(z,\alpha_0(yx)(t))\\ &= \mathfrak{b}\times\mathfrak{b}(z,\alpha_0\tau(yx)(t)) - \mathfrak{b}\times\mathfrak{b}(z,\alpha_0\tau(xy)(t))\\ &= \mathfrak{b}\times\mathfrak{b}(z,\sigma\alpha_0(yx)(t)) - \mathfrak{b}\times\mathfrak{b}(z,\sigma\alpha_0(xy)(t))\\ &= \mathfrak{b}\times\mathfrak{b}(\alpha_0(yx)(z),t) - \mathfrak{b}\times\mathfrak{b}(\alpha_0(xy)(z),t)\\ &= -\mathfrak{b}\times\mathfrak{b}(\alpha_0([x,y])(z),t).\end{aligned}$$

Then, by definition of $\mathfrak{o}(\mathfrak{b})$, we have the result. ☑

From the preceding results we have an injective homomorphism

$$\alpha_0 \circ \mathrm{ad}^{-1} : \mathfrak{o}(\mathfrak{b}) \to \mathfrak{o}(\mathfrak{b}) \times \mathfrak{o}(\mathfrak{b}).$$

We denote by $\mathrm{pr}_1, \mathrm{pr}_2 : \mathfrak{o}(\mathfrak{b}) \times \mathfrak{o}(\mathfrak{b}) \to \mathfrak{o}(\mathfrak{b})$ the canonical projections and for each $f \in \mathfrak{o}(\mathfrak{b})$,

$$(f_1, f_2) = \big(\mathrm{pr}_1 \circ \alpha_0 \circ \mathrm{ad}^{-1}(f), \mathrm{pr}_2 \circ \alpha_0 \circ \mathrm{ad}^{-1}(f)\big) \in \mathfrak{o}(\mathfrak{b}) \times \mathfrak{o}(\mathfrak{b}).$$

Denote $d_1 = \mathrm{pr}_1 \circ \alpha_0 \circ \mathrm{ad}^{-1} : \mathfrak{o}(\mathfrak{b}) \to \mathfrak{o}(\mathfrak{b})$ and $d_2 = \mathrm{pr}_2 \circ \alpha_0 \circ \mathrm{ad}^{-1} : \mathfrak{o}(\mathfrak{b}) \to \mathfrak{o}(\mathfrak{b})$. Then, for $f \in \mathfrak{o}(\mathfrak{b})$, $f_1 = d_1(f)$ and $f_2 = d_2(f)$. We also define a Lie algebras homomorphism $d_\pi \in \mathrm{End}(\mathfrak{C})$ associated to the conjugation map π by $d_\pi(f) = \pi \circ f \circ \pi = \mathrm{Int}(\pi)(f)$.

Proposition 2.3. *For any $f \in \mathfrak{o}(\mathfrak{b})$, the elements $f_1, f_2 \in \mathfrak{o}(\mathfrak{b})$ satisfy*

$$\begin{aligned}f(x\star y) &= f_2(x)\star y + x\star f_1(y),\\ f_1(x\star y) &= f(x)\star y + x\star f_2(y),\\ f_2(x\star y) &= f_1(x)\star y + x\star f(y)\end{aligned}$$

for all $x, y \in \mathfrak{o}(\mathfrak{b})$. Moreover, the pair (f_1, f_2) is uniquely determined by any of the three relations.

Proof. The relations follow from straightforward computations. We check that the pair (f_1, f_2) is uniquely determined by the second relation. Uniqueness in the other cases is similar. By linearity, it suffices to see that the only pair of skew-symmetric maps of $(\mathfrak{C}, \mathfrak{b})$ satisfying $f_1(x \star y) = x \star f_2(y)$ for all $x, y \in \mathfrak{C}$ is the pair $(f_1, f_2) = (0, 0)$. Taking $x = 1$ and the definition of $\star$, we have that $f_1(\overline{y}) = \overline{f_2(y)}$, so that $f_2(y) = \overline{f_1(\overline{y})}$. Then, $f_1(xy) = xf_1(y)$, so there exists $a \in \mathbb{C}$ such that $f_1(x) = ax$ for all $x \in \mathfrak{C}$. But the only possibility for a is $a = 0$, because $f_1 \in \mathfrak{o}(\mathfrak{b})$, so $f_1 = 0 = f_2$, as we wanted to prove. ☑

Corollary 2.4. *The endomorphisms d_1 and d_2 of $\mathfrak{o}(\mathfrak{b})$ are Lie algebras automorphisms with $d_1^2 = d_2$ and $d_1^3 = 1$. The endomorphism d_π is also an automorphism of Lie algebras and $d_\pi^2 = 1$.*

Proof. For the first claim, the condition of morphism of Lie algebras and injectivity follow easily from the relations in Proposition 2.3 and applying uniqueness. The second claim follows from the fact that $\pi^2 = 1$ ☑

Proposition 2.5. *The relation* $d_\pi \circ d_1 = d_2 \circ d_\pi$ *holds in* $\mathrm{Aut}(\mathfrak{o}(\mathfrak{b}))$.

Proof. Let $f \in \mathfrak{o}(\mathfrak{b})$. On the one hand, $C(\pi)$ is a similitude with multiplier $m(\pi) = 1$. Then, by definition of $C(\pi)$ and $\alpha_0 \mathrm{ad}^{-1}$, we have that, for $x \in \mathfrak{C}$,

$$\alpha_0 C(\pi) \alpha_0^{-1} \left(\begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & l_{\overline{x}} \\ r_{\overline{x}} & 0 \end{pmatrix},$$

so $\alpha_0 C(\pi) \alpha_0^{-1} = \alpha_0 \mathrm{ad}^{-1}(\mathrm{Int}(\pi))$. Then,

$$\begin{aligned} \alpha_0 C(\pi) \alpha_0^{-1} \left(d_1(f), d_2(f) \right) &= \alpha_0 \mathrm{ad}^{-1} \mathrm{Int}(\pi)(f) \\ &= \alpha_0 \mathrm{ad}^{-1}(\pi f \pi) \\ &= \left(d_1 d_\pi(f), d_2 d_\pi(f) \right). \end{aligned}$$

On the other hand, if $x, y \in \mathfrak{C}$,

$$\alpha(xy) = \begin{pmatrix} l_x r_y & 0 \\ 0 & r_x l_y \end{pmatrix},$$

so

$$\begin{aligned} \alpha_0 C(\pi) \alpha_0^{-1} \left(\begin{pmatrix} l_x r_y & 0 \\ 0 & r_x l_y \end{pmatrix} \right) &= \alpha_0(\overline{xy}) = \begin{pmatrix} l_{\overline{x}} r_{\overline{y}} & 0 \\ 0 & r_{\overline{x}} l_{\overline{y}} \end{pmatrix} \\ &= \begin{pmatrix} \pi r_x l_y \pi & 0 \\ 0 & \pi l_x r_y \pi \end{pmatrix}. \end{aligned}$$

Then,

$$\alpha_0 C(\pi) \alpha_0^{-1}(d_1(f), d_2(f)) = (\pi d_2(f) \pi, \pi d_1(f) \pi).$$

Finally, we have that

$$(\pi d_2(f) \pi, \pi d_1(f) \pi) = \left(d_1 d_\pi(f), d_2 d_\pi(f) \right),$$

which concludes the result. ☑

Corollary 2.6. *The automorphisms* d_π, d_1 *of* $\mathfrak{o}(\mathfrak{b})$ *generate a subgroup of* $\mathrm{Aut}(\mathfrak{o}(\mathfrak{b}))$ *isomorphic to* S_3.

Proof. Immediate from Corollary 2.4 and Proposition 2.5. ☑

In the case in which $n = 8$, $(\mathfrak{C}, \star)$ is the nonassociative algebra of octonions and d_1 is called the *triality* automorphism of the algebra $\mathfrak{o}(\mathfrak{b})$ ([4, §20]). Observe that, in this case, from uniqueness in Proposition 2.3, an element $f \in \mathfrak{o}(\mathfrak{b})$ is fixed by d_1 if and only if, for all $x, \in \mathfrak{C}$,

$$f(x \star y) = f(x) \star y + x \star f(y),$$

that is, $\mathrm{Fix}(d_1) \cong \mathrm{Der}(\mathfrak{C}, \star)$, the Lie algebra of derivations of the algebra of octonions, which is isomorphic to $\mathfrak{g}_2$ (see [7, p. 104]).

Let $F \in \mathrm{GO}^+(\mathfrak{b})$ be a proper similitude of $\mathfrak{C}$. By Proposition 1.5, this similitude induces an automorphism $C(F)$ of $(C_0(\mathfrak{C}), \tau_0)$ such that $\mathrm{ad}C(F)\mathrm{ad}^{-1} = \mathrm{Int}(F)$. Since F is proper, by Proposition 2.1 we have that $\alpha_0 C(F)\alpha_0^{-1} \in \mathrm{Aut}(\mathfrak{o}(\mathfrak{b})) \times \mathrm{Aut}(\mathfrak{o}(\mathfrak{b}))$. In this case, it can be proved the following result, analogue to Porposition 2.3 (see [10]):

Proposition 2.7. *For any proper similitude $F \in \mathrm{GO}^+(\mathfrak{b})$ with multiplier $m(F)$ there exist proper similitudes $F_1, F_2 \in \mathrm{GO}^+(\mathfrak{b})$ such that $\alpha_0 C(F)\alpha_0^{-1} = (\mathrm{Int}(F_1), \mathrm{Int}(F_2))$ and the following relations are satisfied:*

$$\begin{aligned} m(F)^{-1}F(x \star y) &= F_2(x) \star F_1(y), \\ m(F_1)^{-1}F_1(x \star y) &= F(x) \star F_2(y), \\ m(F_2)^{-1}F_2(x \star y) &= F_1(x) \star F(y). \end{aligned}$$

The multipliers verify $m(F_1)m(F_2)m(F) = 1$ and the pair (F_1, F_2) is uniquely determined by F up to a factor $m \in \mathbb{C}^*$.

With the notation of Proposition 2.7, there are two well-defined injective homomorphisms of groups $\rho_1, \rho_2 : \mathrm{GO}^+(\mathfrak{b}) \to \mathrm{PGO}^+(\mathfrak{b})$, $\rho_1(F) = [F_1]$ and $\rho_2(F) = [F_2]$, for each $F \in \mathrm{GO}^+(\mathfrak{b})$. These homomorphisms give rise to automorphisms of $\mathrm{PGO}^+(\mathfrak{b})$

$$\overline{\rho_1} : \mathrm{PGO}^+(\mathfrak{b}) \to \mathrm{PGO}^+(\mathfrak{b}), \quad \overline{\rho_1}([F]) = [F_1]$$

and

$$\overline{\rho_2} : \mathrm{PGO}^+(\mathfrak{b}) \to \mathrm{PGO}^+(\mathfrak{b}), \quad \overline{\rho_2}([F]) = [F_2].$$

We also have the involution

$$\overline{\pi} : \mathrm{PGO}^+(\mathfrak{b}) \to \mathrm{PGO}^+(\mathfrak{b}), \quad \overline{\pi}(F) = [\pi]F[\pi].$$

The relations between $\overline{\rho_1}$, $\overline{\rho_2}$ and $\overline{\pi}$ are given in the following result:

Proposition 2.8. *We have that $\overline{\pi}^2 = 1$, $\overline{\rho_1}^2 = \overline{\rho_2}$, $\overline{\rho_1}^3 = 1$ and $\overline{\pi}\overline{\rho_1} = \overline{\rho_2}\overline{\pi}$.*

Proof. The first claim is obvious from the fact that $\pi^2 = 1$, so we will see the others. Let F be a proper similitude of $(\mathfrak{C}, \mathfrak{b})$. For simplicity we will work on $\mathrm{GO}^+(\mathfrak{b})$ supposing all the multipliers equal to 1. From Proposition 2.7, if $x, y \in \mathfrak{C}$, $F_1(x \star y) = F(x) \star F_2(y)$. Take $x = 1$. Then, $F_1(\overline{y}) = \overline{F_2(y)}$ for all $y \in \mathfrak{C}$. This says that $F_1 = \pi F_2 \pi$ or, what is the same, $\overline{\pi\rho_1} = \overline{\rho_2\pi}$ and we have the last claim.

Applying the second relation in Proposition 2.7 to F_1 we obtain that for all $x, y \in \mathfrak{C}$,

$$(F_1)_1(x \star y) = F_1(x) \star (F_1)_2(y),$$

so taking $y = 1$, we see that, for all $x \in \mathfrak{C}$, $(F_1)_1(x) = \overline{F_1(\overline{x})}$. This says that $\overline{\rho_1}^2 = \overline{\pi\rho_2}^2\overline{\pi} = \overline{\rho_2}$ and we have the second claim.

Finally, from the relations given in Proposition 2.7,

$$\begin{aligned}((F_1)_1)_1(x \star y) &= (F_1)_1(x) \star ((F_1)_1)_2(y)\\ &= (F_1)_1(x) \star (F_2)_2(y)\\ &= F_2(x) \star F_1(y) = F(x \star y),\end{aligned}$$

so $\overline{\rho_1}^3 = 1$. ☑

We will denote $\overline{\rho} = \overline{\rho_1}$. Then, by proposition 2.8, we know that $\overline{\rho_2} = \overline{\rho}^2$, $\overline{\rho\pi} = \overline{\pi\rho}^2$ and $\overline{\rho}^3 = 1 = \overline{\pi}^2$. As a immediate consequence of this, we have the following result.

Corollary 2.9. *The automorphisms* $\{\overline{\pi}, \overline{\rho}\}$ *generate a subgroup of* $\mathrm{Aut}(\mathrm{PGO}^+(\mathfrak{b}))$ *isomorphic to* S_3.

Corollary 2.9 says that S_3 acts on $\mathrm{PGO}^+(\mathfrak{b})$. This action allows us to consider the semidirect product $\mathrm{PGO}^+(\mathfrak{b}) \rtimes S_3$ with the product

$$(F, \alpha) \cdot (F', \alpha') = (F\alpha(F'), \alpha\alpha').$$

From Proposition 1.5, the group $\mathrm{PGO}^+(\mathfrak{b})$ acts on $\mathfrak{o}(\mathfrak{b})$ through inner automorphisms.

Lemma 2.10. *For* $F \in \mathrm{PGO}^+(\mathfrak{b})$, $\alpha \in S_3$ *and* $s \in \mathfrak{o}(\mathfrak{b})$,

$$\mathrm{Int}(F)d_\alpha(s) = d_\alpha(\mathrm{Int}(d_{\alpha^{-1}}(F))(s)).$$

Proof. We know that, for $F \in \mathrm{PGO}^+(\mathfrak{b})$, $\mathrm{ad}^{-1}C(F)\mathrm{ad} = (\mathrm{Int}(F_1), \mathrm{Int}(F_2))$ and $\alpha_0\mathrm{ad}^{-1} = (d_\rho, d_{\rho^2})$. Then,

$$\begin{aligned}\left(d_\rho(\mathrm{Int}(F)(s)), d_{\rho^2}(\mathrm{Int}(F)(s))\right) &= \alpha_0\mathrm{ad}^{-1}(\mathrm{Int}(F)(s))\\ &= (\alpha_0 C(F)\alpha_0^{-1})(\alpha_0\mathrm{ad}^{-1})(s)\\ &= \left(\mathrm{Int}(F_1)d_\rho(s), \mathrm{Int}(F_2)d_{\rho^2}(s)\right).\end{aligned}$$

Then, we have the result for $\alpha \in \{\overline{\rho}, \overline{\rho}^2\}$ and $F \in \mathrm{PGO}^+(\mathfrak{b})$. When $\alpha = \overline{\pi}$ it is obvious. ☑

We define the action of $\mathrm{PGO}^+(\mathfrak{b}) \rtimes S_3$ on $\mathfrak{o}(\mathfrak{b})$ as follows: if $([F], \alpha) \in \mathrm{PGO}^+(\mathfrak{b}) \rtimes S_3$ and $s \in \mathfrak{o}(\mathfrak{b})$,

$$([F], \alpha)(s) = \mathrm{Int}(F) d_\alpha(s).$$

From Lemma 2.10, this action is well defined, so we can see $\mathrm{PGO}^+(\mathfrak{b}) \rtimes S_3$ as a subgroup of $\mathrm{Aut}(\mathfrak{o}(\mathfrak{b}))$. In fact, the following result says that these are the only elements of $\mathrm{Aut}(\mathfrak{o}(\mathfrak{b}))$:

Proposition 2.11. $\mathrm{Aut}(\mathfrak{o}(\mathfrak{b})) = \mathrm{PGO}^+(\mathfrak{b}) \rtimes S_3$.

Proof. See [9, Theorem 5]. ☑

3. The triality automorphism

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. We denote by $\mathrm{Int}(\mathfrak{g})$ the normal subgroup of $\mathrm{Aut}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$. The elements of $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ are called *outer automorphisms*.

We consider this quotient group, which is denoted by $\mathrm{Out}(\mathfrak{g})$. We have the following exact sequence of groups

$$1 \longrightarrow \mathrm{Int}(\mathfrak{g}) \longrightarrow \mathrm{Aut}(\mathfrak{g}) \longrightarrow \mathrm{Out}(\mathfrak{g}) \longrightarrow 1.$$

The following classical result establishes a relation between $\mathrm{Aut}(G)$ and $\mathrm{Aut}(\mathfrak{g})$ in the case in which G is the simply connected complex Lie group with Lie algebra $\mathfrak{g}$. The proposition says that, in this case, we can speak, indistinctly, of automorphisms of G and automorphisms of $\mathfrak{g}$.

Proposition 3.1. *Let $\mathfrak{g}$ be a complex Lie algebra and G the unique connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then there is a natural isomorphism of short exact sequences of groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Int}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Int}(\mathfrak{g}) & \longrightarrow & \mathrm{Aut}(\mathfrak{g}) & \longrightarrow & \mathrm{Out}(\mathfrak{g}) & \longrightarrow & 1. \end{array}$$

The following equivalence relation on $\mathrm{Aut}(\mathfrak{g})$ will also be relevant for us.

Definition 3.2. If $\alpha, \beta \in \mathrm{Aut}(\mathfrak{g})$, we say that $\alpha \sim_i \beta$ if there exists $\theta \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha = \theta \circ \beta \circ \theta^{-1}$.

One has the following result.

Proposition 3.3. *Let $\alpha, \beta \in \mathrm{Aut}(\mathfrak{g})$. If $\alpha \sim_i \beta$, then α and β define the same element in* $\mathrm{Out}(\mathfrak{g})$.

Proof. If $\alpha \sim_i \beta$, then there exists $\sigma \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha = \sigma\beta\sigma^{-1}$. Then $\alpha\beta^{-1} = \sigma\beta\sigma^{-1}\beta^{-1}$. As $\mathrm{Int}(\mathfrak{g})$ is a normal subgroup, $\tau = \beta\sigma^{-1}\beta^{-1} \in \mathrm{Int}(\mathfrak{g})$, so $\alpha\beta^{-1} = \sigma\tau$. From this, $\alpha = (\sigma\tau)\beta$ or, equivalently, $\alpha \sim \beta$. ☑

Proposition 3.3 shows that the obvious map

$$\mathrm{Aut}(\mathfrak{g})/\sim_i \to \mathrm{Out}(\mathfrak{g}) \tag{6}$$

is well defined.

We consider now, for $j \geq 0$,

$$\mathrm{Aut}_j(\mathfrak{g}) = \{\alpha \in \mathrm{Aut}(\mathfrak{g}) \ : \ \alpha \text{ is of order } j\}.$$

An analogous definition for $\mathrm{Out}_j(\mathfrak{g})$,

$$\mathrm{Out}_j(\mathfrak{g}) = \{\alpha \in \mathrm{Out}(\mathfrak{g}) \ : \ \alpha \text{ is of order } j\}$$

and for $(\mathrm{Aut}(\mathfrak{g})/\sim_i)_j$. It is clear that the order of an automorphism of $\mathfrak{g}$ coincides with the order of its class modulo $\sim_i$. Then

$$\mathrm{Aut}_j(\mathfrak{g})/\sim_i = (\mathrm{Aut}(\mathfrak{g})/\sim_i)_j\,.$$

It is clear that, via (6), automorphisms of order $j = 2, 3$ are sent to elements of $\mathrm{Out}(\mathfrak{g})$ of order j or to the identity. This says that $\mathrm{Aut}_j(\mathfrak{g})/\sim_i$ is sent onto $\mathrm{Out}_j(\mathfrak{g}) \cup \{1\}$ via the natural map if $j \in \{2, 3\}$, that is,

$$\mathrm{Aut}_j(\mathfrak{g})/\sim_i \to \mathrm{Out}_j(\mathfrak{g}) \cup \{1\}, \quad j = 2, 3.$$

We will consider this map for $j = 3$, that is,

$$\mathrm{Aut}_3(\mathfrak{g})/\sim_i \to \mathrm{Out}_3(\mathfrak{g}) \cup \{1\}. \tag{7}$$

In our case, $\mathfrak{g} = \mathfrak{o}(\mathfrak{b})$ and $G = \mathrm{Spin}(n, \mathbb{C})$ with $n \equiv 0 \mod 8$. Say $n = 8l$. From Proposition 2.11 we know that $\mathrm{Aut}(\mathfrak{o}(\mathfrak{b})) = \mathrm{PGO}^+(\mathfrak{b}) \rtimes S_3$ and $\mathrm{Out}(\mathfrak{o}(\mathfrak{b})) \cong S_3$. The outer automorphism $\mathcal{T} = \overline{\rho}^{\mathrm{Out}} \in \mathrm{Out}(\mathfrak{o}(\mathfrak{b}))$ is called the *triality automorphism*. We know that $\mathcal{T}^2 = \mathcal{T}^{-1}$ and $\mathcal{T}^3 = 1$. Moreover, $\mathcal{T}$ and $\mathcal{T}^{-1}$ are the only outer automorphisms of order three. Then, there are as many outer automorphisms of order three of $\mathfrak{o}(\mathfrak{b})$ as lifts of $\mathcal{T}$ by the map (7). In order to count them, we will make use of the following immediate result:

Lemma 3.4. *Let $a \in \mathbb{N}$. Then, $8a + 1$ is a perfect square if and only if there exists $b \in \mathbb{N}$ such that $a = \frac{b^2+b}{2}$.*

Proof. If $a = \frac{b^2+b}{2}$, then $8a + 1 = 4b^2 + 4b + 1 = (2b + 1)^2$ is a perfect square. For the converse, suppose that $8a + 1 = c^2$ for some $c \in \mathbb{N}$. Then, c must be odd. Take $b = \frac{c-1}{2}$. Then, it is easy to see that $a = \frac{b^2+b}{2}$. ☑

Proposition 3.5. *The number of elements in the pre-image of $\mathcal{T}$ by the map (7) is*

$$\begin{cases} 2k & \text{if} \quad n = 4k^2 + 4k, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $[F] \in \mathrm{PGO}^+(\mathfrak{b})$ is such that $([F], \overline{\rho})$ is in the pre-image of $\mathcal{T}$ by the map (7) and $F \neq 1$. It is easy to see that, in this case, $F_2F_1F = 1$. We suppose that $m(F) = 1$. If λ is an eigenvalue of F then $|\lambda|^2 = m(F) = 1$. Let $x \in \mathfrak{C}$ be an eigenvector of eigenvalue λ. From the relations given in Proposition 2.7 one can see that, then, x is also an eigenvector of $F - 1$, F_2 or eigenvalue λ and $\overline{x}$ is an eigenvector of F, F_1 and F_2 of eigenvalue $\overline{L}$. From this and the fact that $x = F_2F_1F(x) = \lambda^3 x$, so $\lambda^3 = 1$, we obtain that 1, λ and $\lambda^2 = \overline{\lambda} = \lambda^{-1}$ are the only eigenvalues of F and that the dimension of the subspace of eigenvectors of eigenvalue λ is equal to the corresponding dimension for λ^2. Let r be this dimension.

Let x, y be nonzero elements of $\mathfrak{C}$ with $F(x) = \lambda x$ and $F(y) = \lambda^2 y$. Then, from Proposition 2.7 it is easy to see that $F(x \star y) = x \star y$. From this observation we conclude that the dimension of the subspace of eigenvectors of eigenvalue 1 is r^2. Counting dimensions, we have that $r^2 + 2r = n$. This equation in r has integer solutions if and only if $n + 1 = 8l + 1$ is a perfect square. From Lemma 3.4, this occurs if and only if there exists k such that $l = \frac{k^2+k}{2}$ or, what is the same, if $n = 4k^2 + 4k$. In this case, there are $2k$ lifts. Otherwise there are only one lift, given by $F = 1$. ☑

***Remark* 3.6.** In terms of the dimension n (always with $n \equiv 0 \mod 8$), Proposition 3.5 says that the number of elements in the pre-image of $\mathcal{T}$ by the map (7) is equal to $\sqrt{n+1} - 1$ if $n + 1$ is a perfect square (which occurs if and only if $n = 4k^2 + 4k$ for some $k \in \mathbb{N}$); 1 otherwise.

In Proposition 3.5 we generalize the result given by Wolf and Gray in [20, Theorem 5.5]. They proved that in the case in which $n = 8$ (that is, for $G = \mathrm{Spin}(8, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}(8, \mathbb{C})$), the map (7) has two lifts. This is a particular case of our result for $l = 1$. They also showed that these two lifts have as subalgebras of fixed points $\mathfrak{g}_2$ (we saw that $\mathrm{Fix}(d_\rho) = \mathrm{Der}(\mathfrak{C}, \star)$ which, for $n = 8$, is $\mathfrak{g}_2$) and $\mathfrak{a}_2$.

In the case in which $n + 1$ is not a perfect square, Proposition 3.5 says that the only possibility for a subalgebra of fixed points of an outer automorphism of order three is $\mathrm{Der}(\mathbb{C}, \star)$. In [18] it is proved that, for dimension greater or equal to 8, the dimension of $\mathrm{Der}(\mathbb{C}, \star)$ is always 14 and then $\mathrm{Fix}(\mathcal{T}) \cong \mathfrak{g}_2$. In terms of the group, this says that $\mathrm{Fix}(\mathcal{T}) \cong G_2$, where $\mathcal{T}$ is seen as an automorphism of G. If $n + 1$ is a perfect square, we obtain that one of the $2k$ lifts of $\mathcal{T}$ by the map (7) has $\mathfrak{g}_2$ as subalgebra of fixed points. So we have proved the following result:

Proposition 3.7. *For $\mathfrak{g} = \mathfrak{o}(\mathfrak{b})$ and $G = \mathrm{Spin}(n, \mathbb{C})$, there is an outer automorphism of order three of the group G with subgroup of fixed points isomorphic to G_2. If $n+1$ is not a perfect square, then G_2 is the subgroup of fixed points of every outer automorphism of order three of G.*

4. Principal G_2, $\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{Spin}(n, \mathbb{C})$-bundles

Let G be a complex reductive Lie group. Let X be a compact complex algebraic curve. It is well-known that a principal G-bundle over X is a complex manifold, E, equipped with a holomorphic projection map $\pi : E \to X$ and a holomorphic right action of G on E which preserves the projection π. In these terms, a complex vector bundle of rank n is a principal $\mathrm{GL}(n, \mathbb{C})$-bundle and a complex vector bundle equipped with a skew-symmetric form is a principal $\mathrm{O}(n, \mathbb{C})$-bundle. A general notion of stability for principal G-bundles is studied in [6].

It is also well known that in the case of principal G-bundles, stability implies simplicity, that is, if E is a stable principal G-bundle then the only automorphisms of E are those given by the center of G, $Z(G)$ (see [15, Proposition 3.2]).

From now on we will consider the moduli space of principal G-bundles over X, $M(G)$, which is the complex algebraic variety of isomorphism classes of polystable principal G-bundles over X, and the moduli space of stable principal G-bundles, $M_s(G)$, the open subvariety of nonsingular points in $M(G)$.

In this section we recall the natural way in which a principal G_2-bundle can be understood as an orthogonal bundle and we will study the specific way in which the notion of stability can be written for this kind of bundles.

The group G_2, of rank 2 and dimension 14, has two irreducible representations, called the fundamental representations. These are the adjoint representation, of dimension 14, and its action on the imaginary octonions, of dimension 7. The last representation is an orthogonal representation

$$\rho : G_2 \to \mathrm{SO}(7, \mathbb{C}).$$

Via this representation, G_2 can be seen as the group of automorphisms of $\mathbb{C}^7$ which preserve a non-degenerate 3-form (see [2]). Then, a principal G_2 bundle is a rank 7 complex vector bundle, E, over X together with a holomorphic global non-degenerate 3-form $\omega \in H^0(X, \bigwedge^3 E^*)$.

An appropriate notion of stability for principal G_2-bundles is given in the following definition.

Definition 4.1. A principal G_2-bundle is semistable (resp. stable) if and only if for each isotropic subbundle E' of E we have $\deg E' \leq 0$ (resp. $\deg E' < 0$).

Observe that isotropic subbundles in the definition above must be of rank 1 or 2 because these are the only allowed ranks for isotropic subbundles of a rank 7 vector bundle equipped with a nondegenerate 3-form.

Now, if $i : G_2 \to \mathrm{Spin}(8,\mathbb{C})$ is the inclusion of groups, $\pi : \mathrm{Spin}(8,\mathbb{C}) \to \mathrm{SO}(8,\mathbb{C})$ the $2:1$ covering map and $j : \mathrm{SO}(8,\mathbb{C}) \hookrightarrow \mathrm{SL}(8,\mathbb{C})$, then $j \circ \pi \circ i : G_2 \to \mathrm{SL}(8,\mathbb{C})$ is a faithful 8-dimensional representation of G_2, so it is the direct sum of the fundamental 7-dimensional representation of G_2 and the abelian 1-dimensional representation, $G_2 \to \mathrm{SL}(7,\mathbb{C}) \oplus \mathbb{C}$. This map admits a factorization through $\mathrm{SO}(7,\mathbb{C}) \oplus \mathbb{C}$. If we consider the natural inclusion $k : \mathrm{SL}(8,\mathbb{C}) \hookrightarrow \mathrm{SL}(n = 8l, \mathbb{C})$, we see that the map $k \circ j \circ \pi \circ i$ admits a factorization through $\mathrm{SO}(7,\mathbb{C}) \oplus \mathbb{C}^{n-7}$.

All this shows that the principal $\mathrm{SO}(n,\mathbb{C})$-bundle associated to the image in $M(\mathrm{Spin}(n,\mathbb{C}))$ of E is

$$(E \oplus \mathcal{O}^{n-7}, Q \oplus 1), \tag{8}$$

the orthogonal bundle associated to E via the homomorphism of groups $G_2 \to \mathrm{SO}(7,\mathbb{C})$ stated before.

In [14], appropriate notions os stability, semistability and polystability are given for principal $\mathrm{SO}(n,\mathbb{C})$-bundles, which reduce those given for general G-bundles in [16] and [17].

Definition 4.2. A special orthogonal bundle E is called stable (resp. semistable) if for every isotropic subbundle E' of E we have $\deg E' < 0$ (resp. $\deg E' \leq 0$.

Definition 4.3. A special orthogonal bundle E is called polystable if it can be written as the orthogonal direct sum of stable orthogonal bundles

Using Jordan-Hölder filtration, we can associate to each semistable $\mathrm{SO}(n,\mathbb{C})$-bundle a unique (upto isomorphisms) polystable $\mathrm{SO}(n,\mathbb{C})$-bundle. If an orthogonal bundle E is semistable, then there is a filtration of E by isotropic subbundles of degree 0

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k$$

such that each of the bundles E_{i+1}/E_i is stable as a vector bundle for $i \leq k-1$ and $E_k^{\perp}/E_k$ is stable as an orthogonal bundle. This filtration is not necessarily unique, but the orthogonal bundles $G(E_i) = H(E_{i+1}/E_i)$ and $G_k = E_k^{\perp}/E_k$ are uniquely determined upto isomorphism and order. Thus the isomorphism class of the polystable orthogonal bundle

$$G(E) = H\left(\bigoplus_{i=1}^{k-1} G(E_i)\right) \oplus G_k$$

is well defined (it is usually called the graded object associated to E). Two semistable orthogonal bundles E and F are said to be S-equivalent if the corresponding polystable bundles $G(E)$ and $G(F)$ are isomorphic. One can see the moduli space of principal G-bundles as the moduli of isomorphism of polystable G-bundles or, equivalently, as the moduli of S-equivalence classes of semistable bundles.

Denote by $\pi : \mathrm{Spin}(n,\mathbb{C}) \to \mathrm{SO}(n,\mathbb{C})$ the natural double cover. As both groups have the same Lie algebra and there is a bijection between Borel subgroups and Borel subalgebras of a group (because Borel subgroups are connected), Borel subgroups of $\mathrm{Spin}(n,\mathbb{C})$ correspond exactly to Borel subgroups of $\mathrm{SO}(n,\mathbb{C})$ via π. Moreover, $\ker\pi$ is contained in every Borel subgroup of $\mathrm{Spin}(n,\mathbb{C})$, so the same is true for parabolic subgroups. From this, it is not difficult to verify from the notion of stability given in [16] for general reductive groups (see [1]) that a principal $\mathrm{Spin}(n,\mathbb{C})$-bundle E is stable (resp. semistable, polystable) if and only if the corresponding $\mathrm{SO}(n,\mathbb{C})$-bundle is so.

5. The action of $\mathrm{Out}(G)$ on $M(G)$

Let G be a complex reductive Lie group and let $M(G)$ be the moduli space of polystable principal G-bundles over X.

We consider the action of the group $\mathrm{Aut}(G)$ of automorphisms of the Lie group G on the set of principal G-bundles over X in the following way.

Definition 5.1. Let E be a principal G-bundle. If $A \in \mathrm{Aut}(G)$ and $\{(U_i, \varphi_i)\}_i$ is a trivializing covering of E, then $\{(U_i, \mathrm{id}_{U_i} \times A \circ \varphi_i)\}_i$ is a trivializing cover of a certain principal G-bundle, where

$$\pi^{-1}(U_i) \xrightarrow{\varphi_i} U_i \times G \xrightarrow{\mathrm{id}_{U_i} \times A} U_i \times G.$$

We define $A(E)$ to be this principal G-bundle.

In fact, if $\{\psi_{ij}\}_{ij}$ is a family of transition functions of E associated to the covering given in the definition, the transition functions of the new bundle $A(E)$ associated to this covering are $\{A \circ \psi_{ij}\}_{ij}$. Observe that these functions verify the cocycle conditions. For $x \in U_i \cap U_j$,

$$A \circ \psi_{ij}(x) \cdot A \circ \psi_{ji}(x) = A\left(\psi_{ij}(x) \cdot \psi_{ji}(x)\right) = A(e) = e$$

and similarly for the other conditions.

Proposition 5.2. *Let E be a principal G-bundle and let A be an inner automorphism of G. Then $A(E)$ and E are isomorphic.*

Proof. Let $g_0 \in G$ and let $A = i_{g_0}$ be the inner automorphism of G associated to g_0. Let $\widetilde{\psi}_{ij}$ be the transition functions of $A(E)$ (having fixed a trivializing covering as before). Then, for $x \in X$, we have that

$$\begin{aligned}\widetilde{\psi}_{ij}(x) &= \widetilde{\psi}_j(x)\widetilde{\psi}_i(x)^{-1} = \left(A \circ \psi_j(x)\right)\left(A \circ \psi_i(x)\right)^{-1} \\ &= A\left(\psi_j(x)\right)\left(A\left(\psi_i(x)\right)\right)^{-1} = g_0\psi_j(x)g_0^{-1}\left[g_0\psi_i(x)^{-1}g_0^{-1}\right] \\ &= g_0\psi_{ij}(x)g_0^{-1}.\end{aligned}$$

This says that E and $A(E)$ are defined by conjugated transition functions, so they must be isomorphic. ☑

***Remark* 5.3.** In fact, it is easy to see that, in the conditions of the preceding proposition, the map $f : E \to A(E)$ defined by $f(e) = eg_0^{-1}$ is an isomorphism of principal G-bundles (see [11]).

Let $\mathrm{Int}(G)$ be the group of inner automorphisms of G, which is a normal subgroup of $\mathrm{Aut}(G)$. Let $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$ be the quotient, which is also a group. We have the exact sequence

$$1 \longrightarrow \mathrm{Int}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

Then, we have that $\mathrm{Out}(G)$ acts on the set of isomorphism classes of principal G-bundles over X in the following way: if $\sigma \in \mathrm{Out}(G)$ and $A \in \mathrm{Aut(G)}$ is an automorphism of G representing σ, then $\sigma(E) = A(E)$.

Our goal now is to prove that $\mathrm{Out}(G)$ acts on the moduli space of principal G-bundles, $M(G)$.

Let E be a principal $\mathrm{Spin}(n, \mathbb{C})$-bundle. Thanks to the equivalence between stability for Spin-bundles and SO-bundles established above, $A(E)$ will be stable (resp. semistable, polystable) if and only if it is so seen as an $\mathrm{SO}(n, \mathbb{C})$-bundle. Since the action of an automorphism, A, of $\mathrm{Spin}(n, \mathbb{C})$ gives rise to a bijective correspondence between isotropic subbundles of E and isotropic subbundles of $A(E)$ preserving the degrees. Then, we have the following.

Proposition 5.4. *If E, is a stable (resp. semistable) principal G-bundle and $A \in \mathrm{Aut}(G)$, then $A(E)$ is stable (resp. semistable).*

Proposition 5.5. *If E is a polystable principal G-bundle and $A \in \mathrm{Aut}(G)$, then $A(E)$ is a polystable principal G-bundle.*

6. Principal G-bundles and triality

Let G be a complex reductive Lie group. If H is a complex subgroup of G, then the inclusion map $H \hookrightarrow G$ induces a map at the level of moduli spaces $M(H) \to M(G)$. We denote by $\widetilde{M(H)}$ the image of $M(H)$ by the preceding map.

Take $G = \mathrm{Spin}(n, \mathbb{C})$, where $n \equiv 8 \mod 8$. In this section, we will characterize the subspace of fixed points in $M(G)$ for the action of the triality automorphism. This characterization is in the spirit of the results given in [1] for Higgs bundles in the particular case in which $G = \mathrm{Spin}(8, \mathbb{C})$.

Theorem 6.1. *Let $\mathcal{T}$ be an element of $\mathrm{Out}(\mathrm{Spin}(n, \mathbb{C}))$ of order three with $\mathcal{T} \neq 1$. Let $M^{\mathcal{T}}(\mathrm{Spin}(n, \mathbb{C}))$ be the subset of fixed points in $M(\mathrm{Spin}(n, \mathbb{C}))$ and $M_s^{\mathcal{T}}(\mathrm{Spin}(n, \mathbb{C}))$ be the subset of fixed points in $M_s(\mathrm{Spin}(n, \mathbb{C}))$ for the action induced by $\mathcal{T}$. Then*

$$M_s^{\mathcal{T}}(\mathrm{Spin}(n, \mathbb{C})) \subseteq \widetilde{M(\mathrm{Fix}(\mathcal{T}))} \subseteq M^{\mathcal{T}}(\mathrm{Spin}(n, \mathbb{C})).$$

Proof. Let A be a lifting of $\mathcal{T}$ for the equivalence relation $\sim_i$. Recall that, if f, g are automorphisms of $\mathrm{Spin}(n,\mathbb{C})$, we say that $f \sim_i g$ if they are conjugate by an inner automorphism of $\mathrm{Spin}(n,\mathbb{C})$. Take $E \in M_s^{\mathcal{T}}(\mathrm{Spin}(n,\mathbb{C}))$. We will see that

$$E \in M(\widetilde{\mathrm{Fix}(\mathcal{T})}).$$

There exists an isomorphism $f : E \to A(E)$. Then the corresponding homomorphisms $A(f) : A(E) \to A^2(E)$ and $A^2(f) : A^2(E) \to A^3(E) = E$ are isomorphisms. If we compose them, we obtain an endomorphism

$$A^2(f) \circ A(f) \circ f : E \to E$$

of E and, since E is simple, there exists $z \in Z(\mathrm{Spin}(n,\mathbb{C}))$ such that

$$A^2(f) \circ A(f) \circ f = z. \tag{9}$$

Let express f in local coordinates. Let U be a trivializing open set of X for E and $A(E)$ simultaneously and φ and $\overline{\varphi}$ their respective trivialization maps in U. Denote by π the projection map of E and by $\widetilde{\pi}_1$ the projection map of $A(E)$. Then by (9) we have that

$$A^2\left(\psi f \varphi^{-1}(x,1)\right) \cdot A\left(\psi f \varphi^{-1}(x,1)\right) \cdot \left(\psi f \varphi^{-1}(x,1)\right) = z.$$

That is, if $g = \psi f \varphi^{-1}(x,1)$,

$$A^2(g) \cdot A(g) \cdot g = z \in Z(\mathrm{Spin}(n,\mathbb{C})). \tag{10}$$

We now prove that it must be $z = 1$. We have seen that, if $z \in Z(G)$, then $A(z), A^2(z) \in Z(\mathrm{Spin}(n,\mathbb{C}))$. Applying A to (10) we obtain the following new identities:

$$\begin{aligned} g \cdot A^2(g) \cdot A(g) &= A(z) \in Z(\mathrm{Spin}(n,\mathbb{C})), \\ A(g) \cdot g \cdot A^2(g) &= A^2(z) \in Z(\mathrm{Spin}(n,\mathbb{C})). \end{aligned} \tag{11}$$

Multiplying by g on the left in (10) and using that $z \in Z(\mathrm{Spin}(n,\mathbb{C}))$,

$$gA^2(g)A(g)g = gz = zg.$$

Using that $A^2(g)A(g)g \in Z(\mathrm{Spin}(n,\mathbb{C}))$,

$$g^2A^2(g)A(g) = gz,$$

so

$$gA^2(g)A(g) = z,$$

that is, $A(z) = z$. As $Z(\mathrm{Fix}(A)) \cong \{1\}$ and $z \in \mathrm{Fix}(A)$ we have that $z = 1$.

Then it is easy to see that g is an element such that $R_g \circ A$ is an automorphism of order three of $\mathrm{Spin}(n,\mathbb{C})$ (not as a group, but as a variety). Let a be a fixed point for $R_g \circ A$. Then for this element, $f(ea) = ea$, that is, f admits fixed points. Take

$$E_H = \{e \in E \ : \ f(e) = e\} \subseteq E.$$

The subvariety E_H of E is invariant under the action of $\mathrm{Fix}(A)$. To see this, take $e \in E_H$ and $g \in H$. Then as $A(g) = g$, by definition of the action of $\mathrm{Spin}(n, \mathbb{C})$ on $A(E)$,

$$f(eg) = f(e)A^{-1}(g) = eg,$$

so $eg \in E_H$. Moreover, the action of $\mathrm{Fix}(A)$ is simply transitive on each fibre of E_H. Take $e_1, e_2 \in E_H$ elements of the same fibre. Then by simple transitivity of the action of $\mathrm{Spin}(n, \mathbb{C})$ on each fibre of E, there exists a unique element $g \in \mathrm{Spin}(n, \mathbb{C})$ such that $e_2 = e_1 g$. Taking images by f we have that

$$e_2 = f(e_2) = f(e_1 g) = e_1 A^{-1}(g),$$

because $f(e_1) = e_1$ and $f(e_2) = e_2$. From this, we must have $A^{-1}(g) = g$. Taking the image by A, we have that $A(g) = g$, so $g \in H$, as we wanted to prove.

All this proves that E_H is a reduction of structure group of E to $\mathrm{Fix}(A)$ via the inclusion map $E_H \hookrightarrow E$.

If a principal $\mathrm{Spin}(n, \mathbb{C})$-bundle E is a fixed point for the action of the automorphism A, the preceding reasoning allows us to assign to each isomorphism $f : E \to A(E)$ a reduction of the structure group of E to $\mathrm{Fix}(A)$. Observe that from the proof we have that this reduction of the structure group is, in fact, the variety of fixed points of the isomorphism f. This proves the first claim.

For the second, suppose that $E \in M(\widetilde{\mathrm{Fix}(A)})$. Then E admits a reduction of the structure group to $\mathrm{Fix}(A)$, $E_{\mathrm{Fix}(A)}$, and E is isomorphic to $A(E)$ via an isomorphism $f : E \to A(E)$ such that $E_{\mathrm{Fix}(A)}$ can be seen as the subvariety of E given by the fixed points of f. To see this, observe that the morphism $A : \mathrm{Fix}(A) \to \mathrm{Fix}(A)$ is the identity, so it makes sense to consider the principal $\mathrm{Fix}(A)$-bundle $A(E_{\mathrm{Fix}(A)})$, which coincides with $E_{\mathrm{Fix}(A)}$ (it is the same as a variety and the action of $\mathrm{Fix}(A)$ is the same, because A acts trivially on $\mathrm{Fix}(A)$, but we consider $E_{\mathrm{Fix}(A)}$ embedded in E and $A(E_{\mathrm{Fix}(A)})$ embedded in $A(E)$). If $\sigma : E_{\mathrm{Fix}(A)} \to E$ is a reduction of the structure group of E to $\mathrm{Fix}(A)$, then $A(\sigma) : A(E_{\mathrm{Fix}(A)}) \to A(E)$ is a reduction of the structure group of $A(E)$ to $A(\mathrm{Fix}(A)) = \mathrm{Fix}(A)$ and we have that

$$E \cong E_{\mathrm{Fix}(A)} \times_{\mathrm{Fix}(A)} \mathrm{Spin}(n, \mathbb{C})$$

and

$$A(E) \cong A(E_{\mathrm{Fix}(A)}) \times_{\mathrm{Fix}(A)} \mathrm{Spin}(n, \mathbb{C}) \cong E_{\mathrm{Fix}(A)} \times_{\mathrm{Fix}(A)} \mathrm{Spin}(n, \mathbb{C}),$$

where the action of $\mathrm{Spin}(n, \mathbb{C})$ on $\mathrm{Fix}(A)$ in the second case is given by a combination of the product in $\mathrm{Spin}(n, \mathbb{C})$ and the action of A:

$$[e, g] \diamond h = [e, gA^{-1}(h)],$$

where $e \in E_{\mathrm{Fix}(A)}$ and $g \in \mathrm{Spin}(n, \mathbb{C})$. Then we have that E and $A(E)$ are isomorphic. We can define the following morphism of principal $\mathrm{Spin}(n, \mathbb{C})$-bundles:

$$f : E \to A(E), \quad f([e, g]) = [e, A^{-1}(g)],$$

where we are considering $E \cong E_{\mathrm{Fix}(A)} \times_{\mathrm{Fix}(A)} \mathrm{Spin}(n, \mathbb{C})$ and $A(E) \cong A(E_{\mathrm{Fix}(A)}) \times_{\mathrm{Fix}(A)} \mathrm{Spin}(n, \mathbb{C})$, that is, $e \in E_{\mathrm{Fix}(A)}$ and $g \in \mathrm{Spin}(n, \mathbb{C})$.

It is clear that f is a well defined morphism. If we take $[eh, h^{-1}g]$ other representative of $[e, g]$ in E ($g \in \mathrm{Spin}(n, \mathbb{C})$, $h \in \mathrm{Fix}(A)$), then

$$f([eh, h^{-1}g]) = [eh, A^{-1}(h^{-1}g)] = [e \diamond A(h), A^{-1}(h)^{-1} A^{-1}(g)] = [e, A^{-1}(g)] = f([e, g]).$$

It is also clear that f respects the action of $\mathrm{Spin}(n, \mathbb{C})$. Take $[e, g] \in E$ and $h \in \mathrm{Spin}(n, \mathbb{C})$. Then

$$\begin{aligned} f([e, g]h) = f([e, gh]) &= [e, A^{-1}(gh)] = [e, A^{-1}(g)A^{-1}(h)] \\ &= [e, A^{-1}(g)] \diamond h = f([e, g]) \diamond h. \end{aligned}$$

It is also clear that the subvariety of fixed points of f is

$$\{[e, 1] \ : \ e \in \mathrm{Fix}(A)\},$$

that is, the image of the canonical embedding of $E_{\mathrm{Fix}(A)}$ into $E \cong E_{\mathrm{Fix}(A)} \times_{\mathrm{Fix}(A)} \mathrm{Spin}(n, \mathbb{C})$. ☑

This result says that if a principal $\mathrm{Spin}(n, \mathbb{C})$-bundle in $M(\mathrm{Spin}(n, \mathbb{C}))$ is fixed by the action of $\mathcal{T}$, then the bundle reduces its structure group to $\mathrm{Fix}(\mathcal{T})$. Moreover, every stable and simple principal bundle fixed by $\mathcal{T}$ is of that form. In the case in which the rank n verifies that $n + 1$ is not a perfect square, this says that

$$M_s^{\mathcal{T}}(\mathrm{Spin}(n, \mathbb{C})) \subseteq \widetilde{M(G_2)} \subseteq M^{\mathcal{T}}(\mathrm{Spin}(n, \mathbb{C})).$$

But, from the observation made in (8), we observe that G_2-bundles are never simple when seen as $\mathrm{Spin}(n, \mathbb{C})$-bundles, so $M_s^{\mathcal{T}}(\mathrm{Spin}(n, \mathbb{C}))$ is empty. In this case, the theorem states that

$$\widetilde{M(G_2)} \subseteq M^{\mathcal{T}}(\mathrm{Spin}(n, \mathbb{C})).$$

When we do not restrict the rank to those for which $n + 1$ is not a perfect square, we have more possibilities for the group $\mathrm{Fix}(A)$. In the next section we will study what happens when $n = 8$.

7. Spin$(8,\mathbb{C})$-bundles and triality

Let $\mathcal{T} \in \mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ be an element of order three and let E be a principal Spin$(8,\mathbb{C})$bundle fixed by the action of $\mathcal{T}$, that is, if $A \in \mathrm{Aut}(\mathrm{Spin}(8,\mathbb{C}))$ is an automorphism of Spin$(8,\mathbb{C})$ representing $\mathcal{T}$, then $E \cong A(E)$. We may assume A to be of order three. To see this, observe that there are two automorphisms of order three of Spin$(8,\mathbb{C})$ not related by inner automorphisms, that are the triality automorphism and its inverse and each of them belongs to an element of order three of $\mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$. We consider the subgroup $\mathrm{Fix}(A)$ of fixed points of A. By Proposition 3.5, there are only two possibilities for $\mathrm{Fix}(A)$ depending on the lifting of the triality automorphism that we have taken modulo conjugation by inner automorphisms. These two possibilities are G_2 or $\mathrm{PSL}(3,\mathbb{C})$ (see [20, Theorem 5.5]). The differential of $\mathcal{T}$ is an automorphism of order three of $\mathfrak{so}(8,\mathbb{C})$. We consider the corresponding decomposition of $\mathfrak{so}(8,\mathbb{C})$ into eigenspaces for $d\mathcal{T}$

$$\mathfrak{so}(8,\mathbb{C}) = \mathfrak{h}_1 \oplus \mathfrak{h}_\mu \oplus \mathfrak{h}_{\mu^2},$$

where μ is a primitive cubic root of unity, $\mathfrak{h}_\mu$ is the vector subspace of $\mathfrak{g} = \mathfrak{so}(8,\mathbb{C})$ corresponding to the eigenvalue μ of $d\mathcal{T}$ and $\mathfrak{h}_{\mu^2}$ is the vector subspace corresponding to μ^2. The subspace $\mathfrak{h}_1$ is the subalgebra of fixed points of $d\mathcal{T}$, so $\mathfrak{h}_1 \cong \mathfrak{g}_2$ or $\mathfrak{h}_1 \cong \mathfrak{sl}(3,\mathbb{C})$.

From Theorem 6.1, we have that

$$M_s^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C})) \subseteq \widetilde{M(\mathrm{Fix}(\mathcal{T}))} \subseteq M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C})).$$

In this particular case, we know from Proposition 3.5 and the proof of Theorem 6.1 that stable fixed points in $M(\mathrm{Spin}(8,\mathbb{C}))$ for the action of any automorphism of order three admit a reduction of structure group to G_2 or $\mathrm{PSL}(3,\mathbb{C})$ and every reduction of these types is fixed by the action of $\mathcal{T}$. In other words, we have that

$$M_s^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C})) \subseteq \widetilde{M(G_2)} \cup \widetilde{M(\mathrm{PSL}(3,\mathbb{C}))} \subseteq M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C})).$$

In this section we will give a complete characterization of the subvariety of fixed points for the triality automorphism when the structure group is Spin$(8,\mathbb{C})$.

The following auxiliary result says that a principal Spin$(8,\mathbb{C})$-bundle fixed by the action of an outer automorphism of order three of Spin$(8,\mathbb{C})$ reduces its structure group to the centralizer of an element of Spin$(8,\mathbb{C})$ that is not in the centre of Spin$(8,\mathbb{C})$.

Proposition 7.1. *Let $\tau \in \mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ be a non-trivial element of order three and E be a principal* Spin$(8,\mathbb{C})$*-bundle with $E \cong \tau(E)$ via an isomorphism $f_0 : E \to \tau(E)$ such that $f = \tau^2(f_0) \circ \tau(f_0) \circ f_0 : E \to E$ is an automorphism of E not coming from the centre of* Spin$(8,\mathbb{C})$*. Then, there exists an element $a \in \mathrm{Spin}(8,\mathbb{C})$ with $a \notin Z(\mathrm{Spin}(8,\mathbb{C}))$ such that E admits a reduction of the structure group to the centralizer of a in* Spin$(8,\mathbb{C})$*, $Z(a)$.*

Proof. Take A a lifting of τ by $\sim_i$. Let E be a principal $\mathrm{Spin}(8,\mathbb{C})$-bundle fixed by the action of τ, that is, $E \cong A(E)$. As E is fixed by A, there exists an isomorphism $f_0 : E \to A(E)$. By hypothesis, $f = f_0 \circ A(f_0) \circ A^2(f_0)$ is an automorphism of E not given by an element of the centre of $\mathrm{Spin}(8,\mathbb{C})$.

Fix $x \in X$ and $e_0 \in E_x$ and, for them, consider the inclusion of groups $i : \mathrm{Aut}E \to \mathrm{Spin}(8,\mathbb{C})$. The element $i(f) \in \mathrm{Spin}(8,\mathbb{C})$ is not in the centre of $\mathrm{Spin}(8,\mathbb{C})$ (in other case, f would be given by an element of the centre of $\mathrm{Spin}(8,\mathbb{C})$ and it is not the case by hypothesis). The principal $\mathrm{Spin}(8,\mathbb{C})$-bundle E admits a reduction of the structure group to $Z(i(f))$, the centralizer in $\mathrm{Spin}(8,\mathbb{C})$ of the element $i(f)$. To see that, consider the subspace of E

$$E_0 = \{e \in E \ : \ f(e) = ei(f)\}.$$

Then E_0 is a reduction of the structure group of E to $Z(i(f))$. To see this, take $g \in Z(i(f))$ and $e \in E_0$. Then

$$f(eg) = f(e)g = ei(f)g = egi(f),$$

so $eg \in E_0$. This proves that $Z(i(f))$ acts on E_0. Moreover, if $e, e' \in E_0$ are in the same fibre, there exists a unique $g \in \mathrm{Spin}(8,\mathbb{C})$ such that $e' = eg$. We have to see that $g \in Z(i(f))$. We have that

$$f(e') = f(eg) = f(e)g = ei(f)g$$

and, as $e' \in E_0$,

$$f(e') = e'i(f) = egi(f).$$

Then $egi(f) = ei(f)g$. As the action of $\mathrm{Spin}(8,\mathbb{C})$ is simply transitive, $i(f)g = gi(f)$, that is, $g \in Z(i(f))$, as we wanted to see. ☑

The following is the main result of this section. It describes the variety of fixed points in $M(\mathrm{Spin}(8,\mathbb{C}))$ for the action of an outer automorphism of order three.

Theorem 7.2. *Let $\mathcal{T}$ be an element of order three of* $\mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ *with* $\mathcal{T} \neq 1$. *Let* $M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C}))$ *be the subset of fixed points on* $M(\mathrm{Spin}(8,\mathbb{C}))$ *for the action induced by* $\mathcal{T}$. *Then*

$$M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C})) = \widetilde{M(G_2)} \cup M(\widetilde{\mathrm{PSL}(3,\mathbb{C})})$$

Proof. Take A a lifting of τ by $\sim_i$. Let $\mu \in \mathbb{C}^*$ be a primitive cubic root of unity. The automorphism A induces a decomposition of $\mathfrak{g} = \mathfrak{so}(8,\mathbb{C})$ into eigenspaces

$$\mathfrak{so}(8,\mathbb{C}) = \mathfrak{h}_1 \oplus \mathfrak{h}_\mu \oplus \mathfrak{h}_{\mu^2},$$

where $\mathfrak{h}_\eta$ is the eigenspace of $\mathfrak{so}(8,\mathbb{C})$ of eigenvalue η for $d\tau$ ($\eta \in \{1, \mu, \mu^2\}$). The subspace $\mathfrak{h}_1$ is in fact the Lie algebra of fixed points of dA.

Let $E \in M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C}))$. Suppose, as a first step that E is stable. As E is fixed by $\mathcal{T}$, there exists an automorphism of E, $f_0 : E \to A(E)$. If $f = f_0 \circ A(f_0) \circ A^2(f_0)$ is an automorphism of E given by an element of the centre of $\mathrm{Spin}(8,\mathbb{C})$, then we are in the situation of the preceding proposition. Suppose this does not happen. Then fix $x \in X$ and $e_0 \in E_x$ and, for them, consider the inclusion of groups $i : \mathrm{Aut}E \to \mathrm{Spin}(8,\mathbb{C})$. By Proposition 7.1, the principal $\mathrm{Spin}(8,\mathbb{C})$-bundle E admits a reduction of the structure group to $Z(i(f))$, the centralizer in $\mathrm{Spin}(8,\mathbb{C})$ of the element $i(f)$ and, from the proof of that proposition, we have that this reduction is given by

$$E_0 = \{e \in E \ : \ f(e) = ei(f)\}.$$

From results by Ramanan in [14], as E admits a reduction of structure group to the centralizer of a non-central element, then E is of the form

$$E = (V_1 \otimes V_2^*) \oplus (V_3 \otimes V_4^*)$$

for certain stable vector bundles of rank 2. The bundles $V_1 \otimes V_2^*$ and $V_3 \otimes V_4^*$ are stable principal $\mathrm{SO}(4,\mathbb{C})$-bundles and the direct sum is orthogonal. Moreover, the triality automorphism acts on the subgroup $\mathrm{GL}(2,\mathbb{C})^4$ of $\mathrm{Spin}(8,\mathbb{C})$ by fixing one of the components and interchanging the other three. This means that a stable fixed point for the action of A, E is of the form $(W \otimes V) \oplus (V \otimes V)$, that is, induces a reduction of the structure group of E to $\mathrm{Fix}(A)$.

This completes the case in which E is stable.

The polystable case reduces to the stable case. To see this observe that from the Jordan-Hölder reduction we have that a polystable principal $\mathrm{Spin}(8,\mathbb{C})$-bundle reduces to a stable principal H-bundle where H is the centralizer of a torus of $\mathrm{Spin}(8,\mathbb{C})$ (for details, see [6]). It is easy to see that the centralizer of a maximal torus of $\mathrm{SO}(8,\mathbb{C})$ is of the form $\mathrm{S}(\mathrm{O}(2,\mathbb{C})^4)$. This proves that the centralizer of a torus of $\mathrm{SO}(8,\mathbb{C})$ is always a subgroup of $\mathrm{S}(\mathrm{O}(4,\mathbb{C}) \times \mathrm{O}(4,\mathbb{C}))$ and we are in the preceding situation. ☑

Acknowledgements. I want to thank the Instituto de Ciencias Matematicas (CSIC-UAM-UC3M-UCM) and the School of Education ‘Fray Luis de Leon’ (University of Valladolid) for supporting this research. Thanks are also due to Oscar Garcia-Prada and S. Ramanan for several and very useful discussions and comments. Many of the ideas of this paper come from them.

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(Recibido en mayo de 2015. Aceptado en diciembre de 2015)

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