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Free subgroups of the parametrized modular group

Subgrupos libres del grupo modular parametrizado

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Abstract. We study free subgroups of index four of the parametrized modular group Π, the subgroup of $\mathrm{SL}(2,\mathbb{Z}[\xi])$ generated by $\left(\begin{smallmatrix}1 & \xi\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. There are eight free subgroups, four of which are normal and four are non-normal. Then we study the intersections of the normal subgroups. We give canonical presentations in terms of generators and relations. At the end of the paper we study connections between Π and the Bianchi groups, the two-parabolic group and a group from relativity theory.

Key words and phrases. Parametrized modular group, free subgroups, Bianchi groups, Picard group, discrete relativity theory.

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Resumen. Estudiamos los subgrupos libres de índice cuatro del grupo modular parametrizado Π, que es el subgrupo de $\mathrm{SL}(2,\mathbb{Z}[\xi])$ generado por $\left(\begin{smallmatrix}1 & \xi\\ 0 & 1\end{smallmatrix}\right)$ y $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. Hay ocho subgrupos libres, cuatro de los cuales son normales y los otros cuatro no lo son. Luego estudiamos las intersecciones de estos subgrupos. Damos presentaciones canónicas en término de generadores y relaciones. Al final del artículo estudiamos conexiones entre Π y los grupos de Bianchi, el grupo dos-parabólico y un grupo de la teoría de la relatividad.

Palabras y frases clave. Grupo modular parametrizado, subgrupos libres, grupos de Bianchi, grupo de Picard, teoría de la relatividad discreta.

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1. Introduction

The parametrized modular group Π is defined in [10] as the subgroup of $\mathrm{SL}(2,\mathbb{Z}[\xi])$ generated by

$$A = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

where $\mathbb{Z}[\xi]$ is the polynomial ring over $\mathbb{Z}$ with ξ as indeterminate. In the last section we describe some connections with the Picard group and other Bianchi groups using the results of R. G. Swan [14]. Furthermore we sketch the relation to discrete relativity theory and knot theory.

The previous paper [10] studied analytical properties of the singular set of Π and the enumeration of the elements of Π, see Lemma 2.1 below. The present paper investigates Π more in the spirit of combinatorial group theory [9] [7].

The exponent sums of a word $W \in \Pi$ with respect to the generators (1) are

$$\sigma(W) := (\text{sum of exponents of } A \text{ in } W), \tag{2}$$

which defines a homomorphism of Π into the additive group $\mathbb{Z}$, and

$$\tau(W) := (\text{sum modulo 4 of exponents of } B \text{ in } W), \tag{3}$$

which defines a homomorphism of Π into the additive group $\mathbb{Z}/4\mathbb{Z}$, note that $B^4 = I$.

In particular we shall study the subgroups

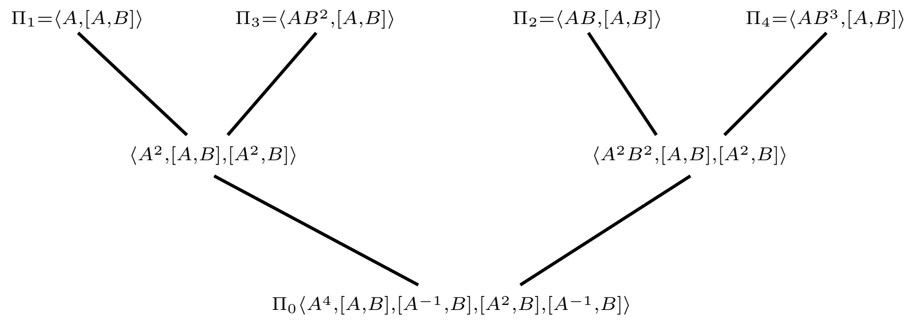
$$\Pi_k := \{W \in \Pi : \tau(W) \equiv (k-1)\sigma(W) \mod 4\} \quad (k = 1, 2, 3, 4) \tag{4}$$

and their common subgroup

$$\Pi_0 := \{W \in \Pi : \sigma(W) \equiv \tau(W) \equiv 0 \mod 4\}. \tag{5}$$

We prove that each Π_k is a rank two free normal subgroup of index four in Π and that Π_0 is a rank five free normal subgroup of index 4 in Π_k $(k = 1, 2, 3, 4)$.

Our main results are summarized in the following subgroup diagram where $[A, B]$ denotes the commutator.



See Theorem 2.2 for the first row, Theorem 3.3 for the second row and Theorem 3.1 for the third row. The other four intersections $\Pi_1 \cap \Pi_2$ and so on are equal to Π_0 by Proposition 3.2.

The presentations in this diagram are canonical in the sense of [9, p.140]: If X is a free group of rank n and Y is a subgroup of rank $m > n$ then there are generators $x_1, \ldots, x_n$ of X and generators $y_1, \ldots, y_m$ of Y such that

$$y_\nu = x_\nu^{d_\nu} z_\nu \ (1 \le \nu \le n), \quad y_\nu = z_\nu \, (n < \nu \le m)$$

where z_ν is a word in Y and

$$\sigma_{x_\nu}(z_\mu) = 0 \ \text{ for } 1 \le \nu \le n, 1 \le \mu \le m,$$

where $\sigma_{x_\nu}(z_\mu)$ is the exponent sums of x_v in the word z_μ.

We study other index four free subgroups of Π that are not normal subgroups.

2. The subgroups Π_i for $1 \le i \le 8$

The derivation of our presentations relies on the following result. See formulas (2.6) and (2.7) in [10], note that any negative sign in W is absorbed in $l \in \mathbb{Z}$ because $B^2 = -I$.

Lemma 2.1. *All words $W \in \Pi$ with $W \neq \pm I, \pm B$ have the form*

$$W = B^e A^{j_n} V \text{ with } V = BA^{j_{n-1}} \cdots A^{j_1} B^l \tag{6}$$

where $e \in \{0, 1\}$, $l \in \{0, 1, 2, 3\}$, $j_\nu \in \mathbb{Z}$ and $j_n \neq 0$.

First we study the groups Π_k defined in (4). See Section 4.3 for the connection of Π_1 with the two-parabolic group.

Theorem 2.2. *Let $k = 1, 2, 3, 4$. The group Π_k is a free normal subgroup of index 4 in Π with the free presentation*

$$\Pi_k = \langle AB^{k-1}, [A, B] \,\rangle. \tag{7}$$

***Remark* 2.3.** Let $\Gamma_k = \langle AB^{k-1}, [A, B] \,\rangle$. The generators in (7) can be replaced by other pairs of generators which we state in the following four lines.

$$\begin{aligned}
&\Gamma_1 = \langle A, BAB^{-1} \rangle \ \text{ because } [A, B] = A \cdot (BAB^{-1})^{-1}, \\
&\Gamma_2 = \langle AB, BA \rangle \text{ because } [A, B] = AB \cdot (BA)^{-1}, \\
&\Gamma_3 = \langle -A, -BAB^{-1} \rangle \text{ because } [A, B] = (-A) \cdot (-BAB^{-1})^{-1}, \\
&\Gamma_4 = \langle -AB, -BA \rangle \text{ because } [A, B] = (-AB) \cdot (-BA)^{-1}.
\end{aligned}$$

Proof. (a) Since σ and τ are homomorphisms into additive groups it is clear from (4) that Π_k is a normal subgroup. Furthermore

$$\Pi_k B^j := \{W \in \Pi : \tau(W) \equiv (k-1)\sigma(W) + j \mod 4\} \ \ (j = 0, 1, 2, 3)$$

are distinct cosets of Π_k and their union is Π. Hence Π_k has index 4.

(b) Since $\sigma(B^j) = 0$ and $\tau(B^j) = j \not\equiv 0$ for $j = 1, 2, 3$, it follows from (4) that $B^j \notin \Pi_k$ for all k. If $W \in \Pi_k, W \neq \pm I, \pm B$, then W has the form (6). It follows from [10, Lemma 2.2] that all these words are different. Hence Π_k is a free group. Let $\Gamma_k = \langle AB^{k-1}, [A, B]\,\rangle$.

(c) Now we show that, for $W \in \Pi$,

$$W \in \Gamma_k \implies \tau(W) \equiv (k-1)\sigma(W) \mod 4. \tag{8}$$

We have $\sigma(AB^{k-1}) = 1$, $\tau(AB^{k-1}) = k - 1$ and $\sigma([A, B]) = \tau([A, B]) = 0$, which proves (8).

(d) Now we prove the converse, namely that, for $W \in \Pi$,

$$\tau(W) \equiv (k-1)\sigma(W) \mod 4 \implies W \in \Gamma_1. \tag{9}$$

We shall however use the alternative forms of the generators listed in Remark 2.3 above. We proceed by induction on the number n of occurrences of the symbol A in the representation (6). If $n = 0$ then $\sigma(W) = 0$ and (9) is trivial. Suppose that (9) holds when the number of occurrences of A is $< n$. Now let there be n occurrences of the symbol A. Below we shall show that there exists $U \in \Gamma_k$ such that $W' = U^{-1}W$ has less than n occurrences of the symbol A. By (4) applied to U we have $\tau(U) \equiv (k-1)\sigma(U) \mod 4$. Using the left-hand side of (9) we conclude that $\tau(W') \equiv (k-1)\sigma(W') \mod 4$. By the induction hypothesis we have $W' \in \Gamma_k$. It follows that $W = UW' \in \Gamma_k$. Now we turn to the construction of U for the different four cases. Let $W = B^e A^{j_n} V$ with $V = BA^{j_{n-1}} \cdots A^{j_1} B^l$ as described in (6). (d1) Let $k = 1$. We define

$$U := B^e A^{j_n} B^{-e} = (B^e A B^{-e})^{j_n} \in \Pi_1,$$

and we have $W' = U^{-1}W = B^e V$ which is shorter than W. (d2) Let $k = 2$. We define

$$U := B^e A^{j_n} B^{j_n - e}$$

and use that $B^2 = -I$. If $j_n = 2q$

$$U = B^e (AA)^q \left(B^2\right)^q B^{-e} = B^e (A(-I)A)^q B^{-e} = B^e (AB \cdot BA)^q B^{-e} \in \Pi_2.$$

If $j_n = 2q + 1$ then

$$U := B^e (AA)^q A \left(B^2\right)^q B B^{-e} = B^e (AB \cdot BA)^q (AB) B^{-e} \in \Pi_2.$$

Therefore, in both cases

$$W' = B^{e-j_n} A^{-j_n} B^{-e} B^e A^{j_n} V = B^{e-j_n} V$$

so W' is shorter than W. (d3) Finally let $k = 3$ or $k = 4$. We see from the remark after Theorem 2.2 that the generators to be obtained are the same as for the cases $k = 1$ and $k = 2$ except for different signs. This difference only changes the exponent l of B in (6) so that we can argue as above. ☑

Proposition 2.4. *Let $j = 1, 2, 3, 4$. The group*

$$\Pi_{j+4} := \langle AB^{j-1}, -[A, B] \rangle \tag{10}$$

is a free subgroup of index 4 in Π and satisfies

$$B\Pi_{j+4}B^{-1} = \Pi_{j'+4} \tag{11}$$

with $j' = j + 2$ mod 4 if $j \neq 2$ and $j' = 4$ if $j = 2$.

Proof. We write $C := [A, B]$. Since Π_j is a free group there is a unique homomorphism $\varphi_j : \Pi_j \to \Pi$ such that

$$\varphi_j(AB^{j-1}) = AB^{j-1},\ \varphi_j(C) = -C, \tag{12}$$

see [9, p.48]. Hence we have $\Pi_{j+4} = \varphi_j(\Pi_j)$ by (10). Since the generators of Π_j and Π_{j+4} only differ by the sign of C we have $\varphi_j(W) = \pm W$ for $W \in \Pi_j$. Now suppose that $\varphi_j(W) = I$. Then $W = \pm I$ where $-I$ is not possible because Π_j is a free group. It follows that $W = I$. Hence φ_j is an isomorphism so that Π_{j+4} is also a free group. As in part (a) of the proof of Theorem 7 we can prove that Π_{j+4} has index 4 in Π. Since $BCB^{-1} = C^{-1}$ it follows from (10) that

$$B\Pi_{j+4}B^{-1} = \langle BAB^{j-2}, -C^{-1} \rangle = \langle BAB^{j-2}, -C \rangle,$$

in the last step we replaced the generator $-C^{-1}$ by its inverse. By another Tietze transformation we can replace BAB^{j-2} by

$$-C \cdot BAB^{j-2} = ABA^{-1}B^{-1} \cdot BAB^{j-2} \cdot B^2 = AB^{j+1} = AB^{j-1+2}$$

and (11) follows from (7). ☑

***Remark* 2.5.** We mention, without proof, that the eight groups Π_i, for $1 \leq i \leq 8$, are the only index four free subgroups of Π.

3. The intersections of these subgroups

Now we turn to the subgroups of Π_k $(k = 1, 2, 3, 4)$. First we study the group defined by (5).

Theorem 3.1. *The group* Π_0 *is a free normal subgroup of index* 16 *in* Π *and has the free presentation*

$$\Pi_0 = \langle A^4, [A, B], [A^{-1}, B], [A^2, B], [A^{-2}, B]\rangle. \tag{13}$$

Proof. (a) It follows from (4) and (5) that Π_0 is a normal subgroup. The 16 sets $\{W : \sigma(W) = j, \tau(W) = k\}$ $(j, k = 0, 1, 2, 3)$ form a complete coset system of Π_0 in Π. Hence Π_0 has index 16 in Π. Since Π_1 has index 4 in Π it follows that Π_0 has index 4 in Π_1. The free group Π_1 has rank 2 by (8). Hence it follows [9, Th.2.10] that Π_0 is free of rank $4(2-1)+1 = 5$. Therefore the 5 generators in (13) are free generators.

(b) Let Γ be the group with the presentation in (13). Each of the words W in (13) satisfies $\sigma(W) \equiv \tau(W) \equiv 0 \mod 4$. Hence it follows from (5) that $\Gamma \subset \Pi_0$. All $W \in \Pi$ with $W \neq \pm I, \pm B$ have the form (6). We shall prove $\Pi_0 \subset \Gamma$ again by induction on the number n of occurrences of the symbol A. In view of (13) we have $A^4 \in \Gamma$ and

$$BA^4B^{-1} = BA^2 \cdot A^2B^{-1} = [A^2, B]^{-1}A^4[A^{-2}, B] \in \Gamma.$$

It follows that

$$A^{4q} \in \Gamma,\ BA^{4q}B^{-1} \in \Gamma\ (q \in \mathbb{Z}). \tag{14}$$

Suppose that $W \in \Pi_0 \Longrightarrow W \in \Gamma$ is true if W has $< n$ occurrences of A. Let $W = B^eA^{j_n}V$, with $V = BA^{j_{n-1}}B \ldots A^{j_1}B^l$ as described in (6). We write

$$j_n = 4q + r,\ q \in \mathbb{Z},\ r = -1, 0, 1, 2.$$

If $e = 0$, then

$$W = A^{4q+r}V = A^{4q} \cdot A^rBA^{-r}B^{-1} \cdot BA^rA^{j_{n-1}}B \cdots A^{j_1}B^l = A^{4q}[A^r, B]V'$$

where $V' = A^{r+j_{n-1}}B \cdots A^{j_1}B^l$ has the form (6) with $n-1$ ocurrences of A. Since the factors of V' belong to Γ by (14), it follows that $W \in \Gamma$. If $e = 1$, then

$$W = BA^{4q+r}V = BA^{4q}B^{-1} \cdot BA^rB^{-1}A^{-r} \cdot A^rBV = BA^{4q}B^{-1} \cdot [A^r, B]^{-1}V'$$

where $V' = A^rBV = A^rB^2A^{j_{n-1}}B\ cdotsA^{j_1}B^l = A^{r+j_{n-1}}B \cdots A^{j_1}B^{l+2}$ has the form (6) with $n-1$ ocurrences of A. By (14) the factors before V' are in Γ. Hence $W \in \Gamma$. ☑

Proposition 3.2. *The group* Π_0 *satisfies*

$$\Pi_0 = \Pi_1 \cap \Pi_2 = \Pi_2 \cap \Pi_3 = \Pi_3 \cap \Pi_4 = \Pi_4 \cap \Pi_1 \tag{15}$$

and is a normal subgroup of index 4 *in each* Π_k $(k = 1, 2, 3, 4)$.

Proof. We abbreviate

$$\{\sigma \equiv m, \tau \equiv n\} := \{W \in \Pi : \sigma(W) \equiv m, \tau(W) \equiv n \mod 4\}. \tag{16}$$

It follows from (8) that

$$\begin{aligned}
\Pi_1 \cap \Pi_2 &= \{\tau \equiv 0\} \cap \{\tau \equiv \sigma\} = \{0 \equiv \tau \equiv \sigma\} = \Pi_0,\\
\Pi_2 \cap \Pi_3 &= \{\tau \equiv \sigma\} \cap \{\tau \equiv 2\sigma\} = \{\sigma \equiv 0 \equiv \tau\} = \Pi_0,\\
\Pi_3 \cap \Pi_4 &= \{\tau \equiv 2\sigma\} \cap \{\tau \equiv 3\sigma\} = \{\sigma \equiv 0, \tau \equiv 0\} = \Pi_0,\\
\Pi_4 \cap \Pi_1 &= \{\tau \equiv 3\sigma\}\{\tau \equiv 0\} = \{\tau \equiv 0, \sigma \equiv 0\} = \Pi_0.
\end{aligned}$$

We see from (15) that Π_0 is a subgroup of all Π_k which is normal because all definitions are in terms of σ and τ. ☑

We see from Proposition 3.2 that four of the six possible intersections of the groups Π_k are equal to Π_0. The remaining two intersections however lead to new groups.

Theorem 3.3. *We have the free presentations*

$$\Pi_1 \cap \Pi_3 = \langle A^2, [A, B], [A^2, B]\rangle, \;\; \Pi_2 \cap \Pi_4 = \langle A^2B^2, [A, B], [A^2, B]\rangle \tag{17}$$

and Π_0 *has index* 2 *in these two groups.*

Proof. (a) First we show that $\supset$ holds in (17). It follows from (4) that, with the notation (16),

$$\Pi_1 \cap \Pi_3 = \{\sigma \equiv 0, 2, \; \tau \equiv 0\}, \;\; \Pi_2 \cap \Pi_4 = \{\sigma \equiv 0, 2, \; \tau \equiv \sigma\}. \tag{18}$$

The generators that occur in (17) satisfy these conditions. (b) Now we prove that $\subset$ holds in (17). To simplify the proof we write

$$\begin{aligned}
\Gamma &:= \langle A^2, [A, B], [A^2, B]\rangle, \; s = +1 \qquad \text{in the case } \Pi_1 \cap \Pi_3\\
\Gamma &:= \langle A^2B^2, [A, B], [A^2, B]\rangle, \; s = -1 \qquad \text{in the case } \Pi_2 \cap \Pi_4.
\end{aligned} \tag{19}$$

Then the assertion (17) becomes

$$\Gamma = \{W \in \Pi : \sigma(W) \equiv 0, 2, \; \tau(W) \equiv \frac{1}{2}(1 - s)\sigma(W) \mod 4\}. \tag{20}$$

First we derive an identity. Since

$$sA^{-2}[A^2, B] = sA^{-2}A^2BA^{-2}B^{-1} = B(sA^{-2})B^{-1}$$

we obtain from (19) and (20) that

$$B(sA^{-2})^q B^{-1} \in \Gamma \, (q \in \mathbb{Z}). \tag{21}$$

We shall use the notation of Lemma 6 and proceed by induction on n. Let $n = 1$. In the case $e = 0$ it follows from (20) with $q \in \mathbb{Z}$ that $W = A^{j_1}B^{l_1} = A^{2q}B^{(1-s)q} \in \Gamma$. In the case $e = 1$ we obtain from (19) and (21) that $W = BA^{j_1}B^{l_1} = B(sA^2)^q B^{-1} \in \Gamma$. Now we assume that our assertion holds for $n - 1$. There are four cases where always $q \in \mathbb{Z}$.

If $W = A^{2q}BA^{j_{n-1}}B \cdots$ then we write

$$W = (sA^2)^q V, \; V = s^q BA^{j_{n-1}} \cdots .$$

If $W = A^{2q+1}BA^{j_{n-1}}B \cdots$ then we write

$$W = (sA^2)^q [A, B]V, \; V = s^q BA^{j_{n-1}+1} \cdots .$$

If $W = BA^{2q}BA^{j_{n-1}}B \cdots$ then we write

$$W = (sA^2)^q B^{-1}V, \; V = s^q BA^{j_{n-1}} \cdots .$$

If $W = BA^{2q+1}BA^{j_{n-1}}B \cdots$ then we write

$$W = (sA^2)^q B^{-1}[A, B]V, V = s^q BA^{j_{n-1}+1} \cdots .$$

We check that V satisfies (20) in all four cases so that $V \in \Gamma$. In the first two cases, the factor before V lies in Γ because of (19). In the last two cases we also use (21) to obtain the same conclusion. Since $V \in \Gamma$ by the induccion hypothesis we see that $W \in \Gamma$ holds in all cases. (c) Since the groups $\Pi_1 \cap \Pi_3$ and $\Pi_2 \cap \Pi_4$ lie properly between the groups Π_k on the one hand and their subgroup Π_0 of index 4 on the other hand, it follows that Π_0 has index 2 in our groups. ☑

Proposition 3.4. *The commutator subgroup* Π' *has infinite index in* Π.

Proof. Let $\Gamma := \{W \in \Pi : \sigma(W) = \tau(W) = 0\}$, here we do not consider congruences. Then $\Pi' \subset \Gamma$ and all cosets ΓA^{4k} are disjoint. We conclude that $|\Pi : \Pi'| \geq |\Gamma : \Pi'| = \infty$. ☑

The situation is often quite different in other contexts. For instance for the Picard group (see below), the first three commutator subgroups have finite index [3, Th.1].

4. Connection with other groups

The groups $\Pi(\zeta)$ are obtained by replacing the indeterminate ξ in Π by the complex number ζ. For instance $\Pi(1)$ and $\Pi(2)$ lead to classical modular groups. Many groups are obtained by combining groups $\Pi(\zeta)$ with different values of ζ as we will see below.

4.1. *The Bianchi groups.* Let $d \in \mathbb{N}$ be square-free and let O_d be the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{-d})$. We now consider the Bianchi groups $\mathrm{SL}(2, O_d)$ but only for the d that allow an euclidean algorithm, namely $d = 1, 2, 3, 7, 11$, see [14] [2, Chapter 4]. The case $d = 1$ is the Picard group, here considered in $\mathrm{SL}(2, \mathbb{C})$, see for instance [2, Chapt. 5] [3] [5]. It will be convenient to enlarge our groups $\Pi(\zeta)$ by adding the generator $A(1)$.

Proposition 4.1. *Let $\Pi^+(\zeta)$ be the group generated by $A(1), A(\zeta)$ and B.*

(1) If $d = 1$ then $\Pi^+(1 + i) = \mathrm{SL}(2, O_1)$.

(2) If $d = 3$ then $\Pi^+(\omega) = \mathrm{SL}(2, O_3)$ where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$.

(3) If $d = 2, 7, 11$ then $\mathrm{SL}(2, O_d) = \Pi^+(\omega)$ where $\omega = i\sqrt{d}$ for $d = 2$ and where $\omega = \frac{1}{2}(1 + i\sqrt{d})$ for $d = 7, 11$.

Proof. The cases $d = 2, 7, 11$ are due to Swan but the cases $d = 1, 3$ are new, Swan had an additional generator, see [14, p.64-71] or [2, Chapt.4]. In a later paper it will be proved the cases $d = 1, 3$. ☑

4.2. *Discrete space-time.* The Schild group Σ is the subgroup of $\mathrm{SL}(2, \mathbb{C})$ that leaves the discrete space-time grid $\{(t, x, y, z) : t, x, y, z \in \mathbb{Z}\}$ invariant under the Lorentz transformation

$$X = \begin{pmatrix} t + z & x + iy \\ x - iy & t - z \end{pmatrix} \mapsto SXS^*, \quad S \in \mathrm{SL}(2, \mathbb{C}). \tag{22}$$

See [13][6][4]. The Schild group Σ is generated by

$$A(1 + i), \begin{pmatrix} (1 - i)/2 & (1 - i)/2 \\ -(1 + i)/2 & (1 + i)/2 \end{pmatrix}, \begin{pmatrix} 0 & -(1 - i)/\sqrt{2} \\ (1 + i)/\sqrt{2} & 0 \end{pmatrix}.$$

The last two matrices generate a subgroup of order 24.

Proposition 4.2. *The group*

$$\Sigma_1 := \langle A(1 + i), A(1 - i), B, Q\rangle, \quad Q := \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$$

is a subgroup of index 6 in the Schild group Σ and is also a subgroup of index 3 in the Picard group $\mathrm{SL}(2, O_1)$. Furthermore $\langle A(1+i), A(1-i), B\rangle$ is a subgroup of index 2 in Σ_1.

Proof. The first two assertions were proved in [4, Sect.4]. The group Σ_1 is similar to the Picard group discussed in Section 4.1, only the generators $A(1), A(i)$ are replaced by $A(1+i), A(1-i)$. This will be proved in a later paper. ☑

For the Lorentz transformation (22), the generators B and Q of Σ_1 have a simple physical interpretation, namely

$$B : x \mapsto -x, y \mapsto y, z \mapsto -z, \quad Q : x \mapsto -x, y \mapsto -y, z \mapsto z$$

whereas the time t remains unchanged. But $t \mapsto 2t + x \mp y - z$ holds for $A(1 \pm i)$. We remark that the groups $A(1)$ and $A(i)$ are not subgroups of Σ_1.

4.3. *The two-parabolic group.* We consider the polynomial ring $\mathbb{Z}[x]$. The two-parabolic group is the subgroup of $\mathrm{SL}(2, \mathbb{Z}[x])$ generated by $X := \left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)$ and $Y := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$; see e.g. [12][11]. Up to conjugation in $\mathrm{SL}(2, \mathbb{Z}[x])$, this is the only subgroup generated by two parabolic matrices with distinct fixed points, see [11].

The subgroup Π_1 of Π was studied in Section 2. It is generated by A and $D := BA^{-1}B^{-1}$. If $V := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Pi_1$ then b and c are odd polynomials [10, (2.15)]. Hence

$$\varphi(V) := \begin{pmatrix} a & b/\xi \\ c\xi & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}[x]), \quad x = \xi^2 \tag{23}$$

is well-defined and it can be checked that φ is a homomorphism. It follows from (23) that $\varphi(D) = X, \varphi(A) = Y$. Hence $\varphi(\Pi_1)$ is the two-parabolic group.

The two-parabolic group is of interest in knot theory, see e.g. [1] [8, Chapt.4]. A knot K is a Jordan curve in $\mathbb{R}^3$. In many cases a discrete group $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ is associated with the knot complement $\mathbb{R}^3 \backslash K$. For a 2-bridge knot a "Wirtinger word" $W \in \varphi(\Pi_1)$ partially describes the knot K. To get Γ one chooses x such that the relation $XW = WY$ is satisfied.

For example, for the knot $4_1 = (5, 3)$ we have $W = YX^{-1}Y^{-1}X$ and $x = \frac{1}{2}(1 + i\sqrt{3})$ [11, Example 8]. Going back to Π_1 we have the relation $DAD^{-1}A^{-1}D = AD^{-1}A^{-1}DA$ and obtain $\Gamma = \varphi(\Pi_1(-\frac{1}{2} + \frac{1}{2}i\sqrt{3}))$.

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