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Central quasipolar rings

Anillos casi-polares centrales

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Abstract. In this paper, we introduce a kind of quasipolarity notion for rings, namely, an element a of a ring R is called *central quasipolar* if there exists $p^2 = p \in R$ such that $a+p$ is central in R, and the ring R is called *central quasipolar* if every element of R is central quasipolar. We give many characterizations and investigate general properties of central quasipolar rings. We determine the conditions that some subrings of upper triangular matrix rings are central quasipolar. A diagonal matrix over a local ring is characterized in terms of being central quasipolar. We prove that the class of central quasipolar rings lies between the classes of commutative rings and Dedekind finite rings, and a ring R is central quasipolar if and only if it is central clean. Further we show that several results of quasipolar rings can be extended to central quasipolar rings in this general setting.

Key words and phrases. Quasipolar ring, central quasipolar ring, clean ring, central clean ring.

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Resumen. En este trabajo, se presenta una noción de un tipo de casi-polaridad en anillos, esto es, un elemento a de un anillo R se dice *casi-polar central* si existe $p^2 = p \in R$ tal que $a + p$ es central en R, y el anillo R es llamado *casi-polar central* si todo elemento de R es casi-polar central. Se dan algunas caracterizaciones y se investigan propiedades generales de los anillos centrales casi-polares. Se determinan las condiciones bajo las cuales algunos subanillos de anillos de matrices triangulares superiores son casi-polares centrales. Una matriz diagonal sobre un anillo local se caracteriza en términos de ser casi-polar central. Se demuestra que la clase de anillos casi-polares centrales se encuentra dentro de la clase de los anillos conmutativos y los anillos finitos de Dedekind, y un anillo R es casi-polar central si es limpio central. Además se muestra que varios resultados de anillos casi-polares se pueden extender a anillos casi-polares centrales en un contexto general.

Palabras y frases clave. Anillo casi-polar, anillo casi-polar central, anillo limpio, anillo limpio central.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let R be a ring and $a \in R$. The *commutant* and *double commutant* of a in R are defined by $comm(a) = \{b \in R \mid ab = ba\}$ and $comm^2(a) = \{b \in R \mid bc = cb$ for all $c \in comm(a)\}$ respectively, and $R^{qnil} = \{a \in R \mid 1 + ax$ is invertible for each $x \in comm(a)\}$. In [11], a is called *quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a + p$ is invertible and $ap \in R^{qnil}$. The ring R is called *quasipolar* if every element of R is quasipolar. General properties of quasipolar rings can be found in [3, 6, 14]. Quasipolar rings are generalized to J-quasipolar in [4]. It is said that an element $a \in R$ is J*-quasipolar* if there exists an idempotent $p \in R$ such that $p \in comm^2(a)$ and $a + p \in J(R)$ where $J(R)$ is the Jacobson radical of R. The ring R is J*-quasipolar* if every element of R is J-quasipolar. Recently, nil-quasipolar rings are introduced in [7]. An element $a \in R$ is said to be *nil-quasipolar* if there exists $p^2 = p \in comm^2(a)$ such that $a+p$ is nilpotent, R is called *nil-quasipolar* in case each of its elements is nil-quasipolar. Motivated by these concepts, in this paper, we define central quasipolarity notion for rings and study basic properties of this class of rings. We present some examples to show that there is no implication between quasipolarity and central quasipolarity of rings. It is also proved that the class of central quasipolar rings lies between those of commutative rings and Dedekind finite rings. Note that being Dedekind finite is still an open problem for strongly clean rings. It is seen that being a commutative ring and being a central quasipolar abelian ring coincide. On the other hand, being a central quasipolar ring and being a central clean ring are the same. We show that central quasipolarity is inherited by homomorphic images, ring direct summands and finite ring direct sums. We characterize an endomorphism of a module in terms of the central quasipolarity. Corner rings, polynomial extensions and trivial extensions of rings are also studied in terms of central quasipolarity. Lastly, we give some characterizations of a central quasipolar diagonal matrix over a local ring.

In what follows, $\mathbb{Z}$ denotes the ring of integers. For a positive integer n, let $Mat_n(R)$ denote the ring of all $n \times n$ matrices and $T_n(R)$ the ring of all $n \times n$ upper triangular matrices over a ring R, and $T(R, R)$ the trivial extension of R by R. We write $R[x]$ and $J(R)$ for the polynomial ring and the Jacobson radical of R, respectively.

2. Central Quasipolar Rings

In this section, we investigate general properties of central quasipolar rings. The structure and several illustrative examples of this class of rings are given.

We now begin with our main definition.

Definition 2.1. Let R be a ring. An element a of R is called *central quasipolar* if there exists $p^2 = p \in R$ such that $a+p$ is central. The ring R is called *central quasipolar* if every element of R is central quasipolar.

Note that the idempotent p relating to a in Definition 2.1 is called *central spectral idempotent* which is borrowed from spectral theory in Banach algebras [9] and [11]. Although the idempotent which is attributed to $a \in R$ where R is a quasipolar (J-quasipolar, nil-quasipolar, respectively) ring is unique, this is not the case for central quasipolar rings as the following shows.

***Remark* 2.2.** Let R be a ring and $a \in R$ central quasipolar. Then the central spectral idempotent of a need not be unique.

Proof. $0 \in R$ is central quasipolar with central spectral idempotents 0 and 1. ☑

***Remark* 2.3.** Note that all versions of quasipolarity require spectral idempotents p belong to $comm^2(a)$. In fact, for central quasipolarity, this condition is unnecessary since the central spectral idempotent p with $a+p$ central always belongs to $comm^2(a)$.

The following examples illustrate that neither condition being a quasipolar ring or being a central quasipolar ring imply the other in general. These examples follow from a routine computation.

Examples 2.4. The following statements hold for a ring R.

(1) Every commutative ring is central quasipolar.

(2) Let R be a commutative local ring. Then the upper triangular matrix ring $T_2(R)$ is quasipolar but not central quasipolar.

(3) $\mathbb{Z}$ is a central quasipolar ring which is not quasipolar.

While every commutative ring is central quasipolar, despite all our efforts, we have not succeeded in giving a counterexample for the converse.

Question. Is there a central quasipolar ring which is not commutative?

Our next endeavor is to find conditions under which a central quasipolar ring is commutative, also quasipolarity implies central quasipolarity. We show that local rings, abelian rings and unit-central rings play important roles in these directions. Recall that a ring R is called *unit-central* [10], if all unit elements are central in R, and R is said to be *abelian* if every idempotent of R is central.

Theorem 2.5. *A ring R is commutative if and only if it is central quasipolar and abelian.*

Proof. The necessity is clear. For the sufficiency, let R be a central quasipolar abelian ring and $a \in R$. Then there exists $p^2 = p \in R$ with $a + p$ central, say $a+p = y$. For any $r \in R$, since R is abelian, $ar = (y-p)r = yr-pr = ry-rp = r(y-p) = ra$. Thus a is central in R. Therefore R is commutative. ☑

Proposition 2.6. *Every quasipolar unit-central ring is central quasipolar.*

Proof. Clear by definitions. ☑

The next result shows that central quasipolarity of elements a and $-1-a$ in a ring R coincide.

Lemma 2.7. *Let R be a ring and $a \in R$. Then a is central quasipolar if and only if $-1-a$ is central quasipolar.*

Proof. For the necessity, let a be central quasipolar. Then there exists $p^2 = p \in R$ such that $a + p$ is central. Hence $(1-p)^2 = 1-p$. So $(-1-a) + (1-p)$ is central. Thus $-1-a$ is central quasipolar. For the sufficiency, assume that $-1-a$ is central quasipolar. There exists $q^2 = q \in R$ such that $-1-a+q$ is central. Then $(1-q)^2 = 1-q$ and $a + (1-q)$ is central. Therefore a is central quasipolar. ☑

Now we prove that every corner ring of a central quasipolar ring inherits this property.

Proposition 2.8. *If R is a central quasipolar ring, then so is eRe for all $e^2 = e \in R$.*

Proof. Let $r \in eRe$. Since R is central quasipolar, there exists $p^2 = p \in R$ such that $r+p$ is central in R. Then $ep = pe$. Hence $(epe)^2 = epe$. Since $r+p$ is central in R, $r+epe$ is central in eRe. Therefore eRe is central quasipolar. ☑

In [12], Nicholson defined clean elements and clean rings, also in [13] Nicholson introduced strongly clean rings. Other generalizations of clean notion of rings investigated by many authors ([1], [2], [5], [8]). Namely, an element $a \in R$ is called *clean* provided that there exists $e^2 = e \in R$ and $u \in U$ such that $a = e + u$. In this direction, we call an element $a \in R$ is *central clean* if there exists an idempotent $e^2 = e$ and a central element u of R such that $a = e + u$, and a ring R is called *central clean* if each element of R is central clean. We now give a relation between central quasipolar and central clean elements.

Theorem 2.9. *Let R be a ring and $a \in R$. Then a is central quasipolar if and only if a is central clean.*

Proof. Assume that $a \in R$ is central quasipolar. Then there exists $p^2 = p \in R$ such that $a + p = b$ is central. Since $a = 1 - p + b - 1$ and b is central, $b - 1$ is central. Thus a is central clean. Conversely, assume that a is central clean. Then there exists $e^2 = e \in R$ and central u such that $a = e+u$. Hence $a+1-e = 1+u$ is also central in R. Therefore a is central quasipolar. ☑

The following result is an immediate consequence of Theorem 2.9, and it reveals that the concepts of central cleanness and central quasipolarity for rings are the same.

Corollary 2.10. *A ring R is central quasipolar if and only if R is central clean.*

Lemma 2.11. *Let R be a ring, $a \in R$ and u invertible in R. Then a is central quasipolar if and only if uau^{-1} is central quasipolar.*

Proof. Assume that a is central quasipolar. There exists $e^2 = e \in R$ such that $a + e = b$ is central. Given any $x \in R$, $[uau^{-1} + ueu^{-1}]x = [u(a + e)u^{-1}]x = (a + e)x = x(a + e) = x[uau^{-1} + ueu^{-1}]$. So $uau^{-1} + ueu^{-1}$ is central. The converse is clear. ☑

Central quasipolar property is inherited by homomorphic images as shown below.

Proposition 2.12. *Every homomorphic image of a central quasipolar ring is central quasipolar.*

Proof. Let $f : R \to S$ be a ring epimorphism, R a central quasipolar ring and $s \in S$. There exists $a \in R$ such that $f(a) = s$ and $a+p$ is central in R for some $p^2 = p \in R$. The image of any central element of R is contained in the center of S. Then $f(a + p) = f(a) + f(p)$ is central and $f(p)^2 = f(p)$. Then $f(p)$ is a central spectral idempotent for $s = f(a)$. This completes the proof. ☑

By using Proposition 2.12, we obtain the next result related to a ring decomposition of central quasipolar rings.

Corollary 2.13. *Every ring direct summand of a central quasipolar ring is central quasipolar.*

Proof. Let $R = I \oplus K$ be a ring decomposition of a central quasipolar ring R. Consider the natural projection $\pi : R \to I$. By Proposition 2.12, I is central quasipolar. ☑

Proposition 2.14. *Let $\{R_i\}_{i\in I}$ be a class of rings for a finite index set $I = \{1, 2, \ldots , n\}$ and R denote $\prod_{i=1}^{n} R_i$. Then R is central quasipolar if and only if each R_i is central quasipolar for $i \in I$.*

Proof. One way is clear from Corollary 2.13. We may assume that $n = 2$ and R_1 and R_2 are central quasipolar. Let $a = (x_1, x_2) \in R$. There exists idempotents $p_i \in R_i$ such that $x_i + p_i$ is central in R_i for $(i = 1, 2, \dots, n)$. Then $p = (p_1, p_2, \dots, p_n)$ is an idempotent in R and $a + p$ is central in R. Hence R is central quasipolar. ☑

Recall that a ring R is called *Dedekind finite* (or *directly finite*) if for a, $b \in R$, $ab = 1$ implies $ba = 1$. It is still an open problem whether strongly clean rings are Dedekind finite. When we deal with central quasipolar rings, in the next result, we give an affirmative answer for an analogues of this question. This result also shows that the class of central quasipolar rings lies between the classes of commutative rings and Dedekind finite rings.

Theorem 2.15. *Every central quasipolar ring is Dedekind finite.*

Proof. Let a, $b \in R$ with $ab = 1$. There exists $p^2 = p, q^2 = q \in R$ such that $a + p$ and $b + q$ are central in R. Then $a(b + q) = (b + q)a = ab + aq = ba + qa = 1 + aq = ba + qa$. Multiplying the latter from the left by a we have $a^2q = aqa$ and so $qa^2q = qaqa$. Then $1 + aq = ba + qa$ implies

$$1 - ba = qa - aq \tag{1}$$

is an idempotent. Hence

$$\begin{aligned} qa - aq &= (qa - aq)^2 \\ &= qaqa - qa^2q - aqa + aqaq \\ &= -aqa + aqaq \\ &= -aqa(1 - q) \\ &= -a^2q(1 - q) \\ &= 0 \end{aligned}$$

Thus $qa = aq$. We have $1 = ba$ by (1.6). This completes the proof. ☑

In [13], Nicholson gives several characterizations of strongly clean rings through the endomorphism ring of a module. Analogously, in [4], Cui and Chen do the same for J-quasipolar rings. In this vein, we have the following result for central quasipolar rings.

Theorem 2.16. *Let M be an R-module and $\alpha \in E = End_R(M)$. Then the following are equivalent.*

(1) α is central quasipolar.

(2) There exists $\pi^2 = \pi \in E$ such that $\pi \in comm^2(\alpha)$, $\alpha\pi$ is central in $\pi E\pi$ and $(1 + \alpha)(1 - \pi)$ is central in $(1 - \pi)E(1 - \pi)$.

(3) M has a decomposition $M = P \oplus Q$ where P and Q are β-invariant for every $\beta \in comm_E(\alpha)$, $\alpha\mid_P$ is central in $End_R(P)$ and $(1+\alpha)\mid_Q$ is central in $End_R(Q)$.

(4) M has a decomposition $M = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ for some $n \geq 1$, where P_i is β-invariant for every $\beta \in comm_E(\alpha)$, $\alpha\mid_{P_i}$ is central quasipolar in $End_R(P_i)$ for each i.

Proof. (1) $\Rightarrow$ (2) Since α is central quasipolar in E, there exists $\pi^2 = \pi \in comm^2_E(\alpha)$ such that $\alpha + (1-\pi) = \eta$ is central in E. Note that α, π and η commute. Multiplying $\alpha + (1-\pi) = \eta$ by π yields $\alpha\pi = \eta\pi = \pi\eta$ is central in $\pi E\pi$. Further, $1 + \alpha = \pi + \eta$, and so $(1+\alpha)(1-\pi) = \eta(1-\pi) = (1-\pi)\eta$ is central in $(1-\pi)E(1-\pi)$.

(2) $\Rightarrow$ (3) Let $P = M\pi$ and $Q = M(1-\pi)$. Then $M = P \oplus Q$. Let $\beta \in comm_E(\alpha)$. Since $\pi \in comm^2_E(\alpha)$, $\beta\pi = \pi\beta$. Then P and Q are β-invariant. As $\pi\alpha$ is central in $\pi E\pi$ and $(1-\pi)\alpha(1-\pi)$ is central in $(1-\pi)E(1-\pi)$, $\alpha\mid_P$ is central in $\mathrm{End}_R(P)$ and $(1+\alpha)\mid_Q$ is central in $\mathrm{End}_R(Q)$.

(3) $\Rightarrow$ (4) Clear.

(4) $\Rightarrow$ (1) We may assume that $M = P \oplus Q$ where P and Q are invariant for every $\beta \in comm_E(\alpha)$. By hypothesis, $\alpha|_P$ and $\alpha|_Q$ are central quasipolar and $\alpha|_P(P) \subseteq P$ and $\alpha|_Q(Q) \subseteq Q$. By hypothesis, there exists $e^2 = e \in \mathrm{End}_R(P)$, $e \in comm^2(\alpha|_P)$ and $f^2 = f \in \mathrm{End}_R(Q)$, $f \in comm^2(\alpha|_Q)$ such that $\alpha|_P + e$ is central in $\mathrm{End}_R(P)$, and $\alpha|_Q + f$ is central in $\mathrm{End}_R(Q)$. Then $\alpha = \alpha|_P + \alpha|_Q$ and $g = e+f$ is an idempotent in $\mathrm{End}_R(P)$ and $g \in comm^2(e+f)$ since $comm^2(e+f) = comm^2(e) + comm^2(f)$. On the other hand, $\alpha + g = \alpha|_P + e + \alpha|_Q + f$ is central in $\mathrm{End}_R(M)$ since center of M is the ring direct sum of center of $\mathrm{End}_R(P)$ and center of $\mathrm{End}_R(Q)$. This implies that α is central quasipolar, as asserted. ☑

We close this section by observing the relations between a ring and its polynomial ring extension in terms of central quasipolarity.

Theorem 2.17. *Let R be a ring. Then the following hold.*

(1) If the polynomial ring $R[x]$ is central quasipolar, then R is central quasipolar.

(2) If R is a central quasipolar local ring, then $R[x]$ is central quasipolar.

Proof. (1) Let $a \in R$. Consider $f(x) = a \in R[x]$. There exists $p(x)^2 = p(x) \in R[x]$ such that $a + p(x)$ is central in $R[x]$. Set $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $a_0^2 = a_0$ and $a_1 = a_2 = \cdots = a_n = 0$. Hence $a + a_0$ is central. Therefore R is central quasipolar.

(2) Let R be a central quasipolar local ring. Then R is commutative by Theorem 2.5, and so $R[x]$ is commutative. ☑

3. Central Quasipolarity of Matrices

Let R be any ring and n any positive integer. One may suspect that the matrix ring $Mat_n(R)$ of all $n \times n$ matrices and the ring $T_n(R)$ of all $n \times n$ upper triangular matrices with entries in R are central quasipolar. But the following example illustrates that this is not the case.

Example 3.1. Let A denote the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in Mat_2(\mathbb{Z})$. For any idempotent matrix P, $A+P$ can not be central in $Mat_2(\mathbb{Z})$. In fact, let P be an idempotent with $A+P$ central. Then P has the form $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ for some $a, b, c \in \mathbb{Z}$. By commuting $A+P$ with $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, we have $a = 1/2$. This is a contradiction. Same proof also works for $T_2(\mathbb{Z})$. Therefore $Mat_2(\mathbb{Z})$ and $T_2(\mathbb{Z})$ are not central quasipolar.

Although every corner ring of a central quasipolar ring is also central quasipolar (see Proposition 2.8), the property being a central quasipolar ring does not pass on the matrix rings as is seen from Example 3.1. Therefore this property is not Morita invariant.

For any ring R,

$$T(R,R) = \left\{ \begin{bmatrix} r & t \\ 0 & r \end{bmatrix} \mid r, t \in R \right\}$$

denotes the *trivial extension of* R *by* R as a subring of 2×2 upper triangular matrix ring $T_2(R)$. The next example shows that $T(R,R)$ need not be central quasipolar.

Example 3.2. Let $\mathbb{H}$ denote the ring of the Hamilton quaternions over the field of real numbers. Then $T(\mathbb{H},\mathbb{H})$ is not central quasipolar.

Proof. Let $A = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$. By the same discussion as in Example 3.1, there is no idempotent P with $A+P$ central in $T(\mathbb{H},\mathbb{H})$. ☑

Now we introduce a notation for some subrings of $T_n(R)$ where R is a ring. Let k be a natural number smaller than n. Say

$$T_n^k(R) = \left\{ \sum_{i=j}^{n} \sum_{j=1}^{k} x_j e_{(i-j+1)i} + \sum_{i=j}^{n-k} \sum_{j=1}^{n-k} a_{ij} e_{j(k+i)} : x_j, a_{ij} \in R \right\}$$

where e_{ij}'s are matrix units. Elements of $T_n^k(R)$ are in the form

$$\begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_k & a_{1(k+1)} & a_{1(k+2)} & \dots & a_{1n} \\ 0 & x_1 & x_2 & \dots & x_{k-1} & x_k & a_{2(k+2)} & \dots & a_{2n} \\ 0 & 0 & x_1 & \dots & x_{k-2} & x_{k-1} & \dots & \dots & a_{3n} \\ . & . & . & \dots & . & . & \dots & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ . & . & . & \dots & . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & x_1 \end{bmatrix}$$

where $x_i \in R$, $a_{js} \in R$, $1 \leq i \leq k$, $1 \leq j \leq n \leq k$ and $k+1 \leq s \leq n$.

Example 3.3. Let R be a ring.

(i) If R is commutative, then $T_n^n(R)$ is commutative and so it is central quasipolar.

(ii) Assume that R is not commutative and $A = (a_{ij}) \in T_n^n(R)$. If R is central quasipolar and all a_{ij} are central in R, then A is central quasipolar in $T_n^n(R)$.

(iii) Assume that R is not commutative. Then for any k with $k+1 \leq n$, $T_n^k(R)$ is not central quasipolar.

(iv) If R is central quasipolar, then $A \in T_n^n(R)$ is central quasipolar in $T_n^n(R)$ if and only if $A = (a_{ij})$ is diagonal, i.e., $a_{ij} = 0$ if $i \neq j$.

Proof. (ii) Let $A = (a_{ij}) \in T_n^n(R)$. Set $p^2 = p \in R$ such that $a_{11} + p$ is central in R. Set $P = (x_{ij})$ where $x_{ij} = p$ if $i = j$ and $x_{ij} = 0$ otherwise. Then $A + P$ is central in $T_n^n(R)$.
(iii) Without loss generality, we may assume that $n = 4$ and $k = 2$. Let e_{ij} denote the matrix unit with 1 in the (i,j)-entry and zero elsewhere. Let $A = e_{12} + e_{23} + e_{34}$. It is obvious that any idempotent $P \in T_4^2(R)$ is the zero matrix or the identity matrix. Therefore $A + P$ is not central in $T_4^2(R)$. A similar discussion may be done for $A = e_{13} + e_{24}$ and $A = e_{14}$ to show that for any idempotent $P \in T_4^2(R)$, $A + P$ is not central in $T_4^2(R)$.
(iv) If $A \in T_n^k(R)$ is not diagonal, a similar proof as in (iii), we may conclude that A is not central quasipolar. So assume that $A = (a_{ii}) \in T_n^k(R)$ is diagonal. Let $p_{ii} \in R$ denote the spectral idempotent for a_{ii}. Let P denote the diagonal matrix (p_{ii}). Then P is idempotent and $AP = PA$ since $a_{ii} + p_{ii}$ is central in R. Then $A + P$ is central in $T_n^n(R)$. ☑

By Example 3.1, $Mat_2(R)$ is not central quasipolar in general. We now investigate the central quasipolarity of single 2×2 matrices over local rings.

Theorem 3.4. *Let R be a local ring. If $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is central quasipolar in $Mat_2(R)$, then one of the following conditions holds.*

(1) A is central in $Mat_2(R)$.

(2) $A + I_2$ is central in $Mat_2(R)$.

(3) a and $b + 1$ are central in R.

(4) $a + 1$ and b are central in R.

The converse statement holds if $a = b$.

Proof. Assume that A is central quasipolar. Let $P = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ be an idempotent in $Mat_2(R)$ such that $A + P$ is central in $Mat_2(R)$ and $AP = PA$. Since $AP = PA$ and $A + P$ is central, it is easily checked that $x^2 = x$, $t^2 = t$ and $y = z = 0$. There are some cases for x and t as the following.

Case (i) If $x = t = 0$, then $P = 0$ and A is central in $Mat_2(R)$.

Case (ii) If $x \neq 0$ and $t \neq 0$, then we take $P = I_2$ identity matrix in $Mat_2(R)$ and $A + P$ has the form (2).

Case (iii) If $x = 0$ and $t \neq 0$, then $t = 1$ and we take $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $A + P$ is central, we have the case (3).

Case (iv) If $x \neq 0$ and $t = 0$, then $x = 1$ and we take $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since $A + P$ is central, we have the case (4).

Conversely, assume that $a = b$. Consider $P = I_2$ or $P = 0$ or $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ or $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AP = PA$. If A is central in $Mat_2(R)$, we take $P = 0$ and then $A + P$ is central. If $A + I_2$ is central in $Mat_2(R)$, then A is central and we take $P = I_2$ and then $A + I_2$ is central. If a is central (equivalently, $a + 1$ is central), then we take $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and then $A + P$ becomes central. If $a + 1$ is central (equivalently, a is central) in R, then we take $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for $A + P$ being central in $Mat_2(R)$. This completes the proof. ☑

For a positive integer n and a ring R, consider the set

$$D_n(R) = \{A = (a_{ij}) \in Mat_n(R) | a_{ij} = 0 \ when \ i \neq j\}.$$

Then $D_n(R)$ is a subring of $Mat_n(R)$ consisting of all diagonal matrices of $Mat_n(R)$. We end this paper by giving a characterization of central quasipolar rings R in terms of the subring $D_n(R)$.

Theorem 3.5. *A ring R is central quasipolar if and only if $D_2(R)$ is central quasipolar.*

Proof. It is clear by Proposition 2.14 since R is central quasipolar if and only if so is $R^{(n)}$. ☑

Corollary 3.6. *A ring R is central quasipolar if and only if $D_n(R)$ is central quasipolar where n is a positive integer.*

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