

Intersection numbers of geodesic arcs

Números de Intersección de Arcos Geodésicos

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ABSTRACT. For a compact surface S with constant curvature $-\kappa$ (for some $\kappa > 0$) and genus $g \geq 2$, we show that the tails of the distribution of the normalized intersection numbers $i(\alpha, \beta)/l(\alpha)l(\beta)$ (where $i(\alpha, \beta)$ is the intersection number of the closed geodesics α and β and $l(\cdot)$ denotes the geometric length) are estimated by a decreasing exponential function. As a consequence, we find the asymptotic average of the normalized intersection numbers of pairs of closed geodesics on S . In addition, we prove that the size of the sets of geodesic arcs whose T -self-intersection number is not close to $\kappa T^2/(2\pi^2(g-1))$ is also estimated by a decreasing exponential function. And, as a corollary of the latter, we obtain a result of Lalley which states that most of the closed geodesics α on S with $l(\alpha) \leq T$ have roughly $\kappa l(\alpha)^2/(2\pi^2(g-1))$ self-intersections, when T is large.

Key words and phrases. geodesics, geodesic flow, geodesic currents, intersection number, mixing, ergodicity.

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RESUMEN. Para una superficie S con curvatura constante $-\kappa$ (con $\kappa > 0$) y género $g \geq 2$, mostramos que las colas de la distribución de $i(\alpha, \beta)/l(\alpha)l(\beta)$ (donde $i(\alpha, \beta)$ es el número de intersección de las geodésicas cerradas α y β) se puede estimar con una función exponencial decreciente. Como consecuencia, encontramos el promedio asintótico de los números de intersecciones normalizados de los pares de geodésicas cerradas en S . Además, demostramos que el tamaño de los conjuntos de geodésicas cuyo número de T -auto-intersecciones no es cercano $\kappa T^2/(2\pi^2(g-1))$ también decrece exponencialmente rápido. Y, como corolario de este último, obtenemos un resultado de Lalley que afirma que la mayoría de las geodésicas cerradas α en S con $l(\alpha) \leq T$ tienen aproximadamente $\kappa l(\alpha)^2/(2\pi^2(g-1))$ auto-intersecciones, cuando T es grande.

Palabras y frases clave. geodésica, flujo geodésico, corrientes geodésicas, número de intersección, mezcla, ergodicidad.

1. Introduction

Let S be a compact surface of constant curvature $-\kappa$, for some $\kappa > 0$, and genus $g \geq 2$. A *geodesic (parametrized by the arc length)* on S is a smooth locally distance-minimizing curve $\gamma : \mathbb{R} \rightarrow S$. For every $x \in S$ and every unit vector v tangent to S at x , there is a unique geodesic $\gamma_{(x,v)}$ on S such that $\gamma_{(x,v)}(0) = x$ and $\dot{\gamma}_{(x,v)}(0) = v$, where $\dot{\gamma}(t)$ denotes the unit vector tangent to γ at $\gamma(t)$. The restriction $\gamma' = \gamma|_{[a,b]}$ for $-\infty \leq a < b \leq \infty$ is called a *geodesic arc* or *segment* and its *length* is $l(\gamma') = b - a$. The geodesic γ is *closed* if there exists $l > 0$ such that $\gamma([0, l]) = \gamma(\mathbb{R})$, and in this case, we say that $l(\gamma) = \min\{l \mid \gamma([0, l]) = \gamma(\mathbb{R})\}$.

Two geodesics γ and η on S are *identical* if they both have the same trace, that is, there is $r \neq 0$ such that either (i) $\gamma(t) = \eta(t + r)$ and $\dot{\gamma}(t) = \dot{\eta}(t + r)$ or (ii) $\gamma(t) = \eta(r - t)$ and $\dot{\gamma}(t) = -\dot{\eta}(r - t)$, for every $t \in \mathbb{R}$. Let $[\gamma]$ be the equivalence class formed by all geodesics on S that are identical to γ . We choose a representative geodesic from each class and form a set that we denote by $\mathbb{G}$. Let $C\mathbb{G}$ be the subset of $\mathbb{G}$ consisting of the geodesics that are closed. Let $C\mathbb{G}_T = \{\gamma \in C\mathbb{G} : l(\gamma) \leq T\}$ and $N(T)$ be the cardinality of $C\mathbb{G}_T$. H. Huber proved in [8, Theorem 10] that the number $N(T)$ satisfies the asymptotic formula $N(T) \sim e^{\sqrt{\kappa}T}/\sqrt{\kappa}T$, that is, $\lim_{T \rightarrow \infty} T\sqrt{\kappa}N(T)e^{-T\sqrt{\kappa}} = 1$.

Definition 1.1. Let $T > 0$, and γ and η be geodesics on S . The *T -intersection number of γ and η* is denoted by $i_T(\gamma, \eta)$ and defined by

$$i_T(\gamma, \eta) = \#\{r \in [0, T] \mid \gamma(r) = \eta(t); \dot{\gamma}(r), \dot{\eta}(t) \text{ non-parallel, for some } t \in [0, T]\}.$$

In particular, $i_T(\gamma, \gamma)$ is the *T -self-intersection number of γ* .

Remark 1.2. If $l(\gamma), l(\eta) \leq T$ and both γ and η are either closed geodesics or geodesic arcs then

$$i_T(\gamma, \eta) = i(\gamma, \eta),$$

where $i(\gamma, \eta)$ is the (*geometric*) *intersection number* of γ and η . In addition, $i_T(\gamma, \gamma) = i(\gamma, \gamma)$ is the *self-intersection number of γ* .

The intersection numbers have been of interest to many researchers and here are some of the most relevant results so far achieved. Lalley showed in [13, Theorem 1] that for T large enough, the self-intersection number of most of the closed geodesics α with $l(\alpha) \leq T$ is about $\kappa l(\alpha)^2/(2\pi^2(g-1))$. Later, Pollicott and Sharp generalized this result to self-intersections of closed geodesics with an angle in a given interval (see [16, Theorem 1]). Recently, Chas and Lalley in [6, Main Theorem] proved that if a free homotopy class of curves on a surface with boundary is chosen at random from among all classes of word length m , then the distribution of the self-intersection numbers appropriately scaled approaches the Gaussian distribution, for m “large enough.” Furthermore, Lalley also showed in [12, Theorem 1] that the random variable

$[N_T - \kappa T^2 / (2\pi^2(g-1))] / T$ has a limit distribution as $T \rightarrow \infty$, where N_T is the number of self-intersections of a closed geodesic on S of length $\leq T$ randomly chosen.

In this paper, we prove that the tails of the distribution of the *normalized intersection numbers* of the pairs of elements of $C\mathbb{G}$, that is $i(\alpha, \beta) / l(\alpha)l(\beta)$ for $\alpha, \beta \in C\mathbb{G}$, are estimated by a decreasing exponential function.

Theorem 1.3. *Let $\epsilon > 0$. There exists $\delta > 0$ such that*

$$\frac{1}{N(R)N(T)} \# \left\{ (\alpha, \beta) \in C\mathbb{G}_R \times C\mathbb{G}_T : \left| \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} - \frac{\kappa}{2\pi^2(g-1)} \right| \geq \epsilon \right\} = O(e^{-\delta R}),$$

as $R \rightarrow \infty$, with $T \geq R$.

Theorem 1.3 allows us to show that the average of the normalized intersection numbers of pairs of closed geodesics of length at most R and T is asymptotically equal to $\kappa / (2\pi^2(g-1))$.

Corollary 1.4. $\frac{1}{N(R)N(T)} \sum_{\substack{\alpha \in C\mathbb{G}_R \\ \beta \in C\mathbb{G}_T}} \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} \sim \frac{\kappa}{2\pi^2(g-1)},$ as $R, T \rightarrow \infty$.

In order to introduce our other results we need the following definitions.

Let $T^1(S) = \{\mathbf{v} = (x, v) \mid x \in S, v \in T_x(S), \|v\| = 1\}$ be the *unit tangent bundle* of S . Let ϑ denote the *Riemannian measure* on $T^1(S)$, i.e., the volume measure of $T^1(S)$. In addition, let $\bar{\vartheta}$ denote the normalized Riemannian measure, that is, $\bar{\vartheta} = \frac{1}{\vartheta(T^1(S))} \vartheta = \frac{\kappa}{2\pi^2(g-1)} \vartheta$.

By identifying the unit tangent bundle of S with the set of geodesics on S we prove that the size (or $\bar{\vartheta}$ -measure) of the subset of $T^1(S)$ consisting of vectors whose corresponding geodesics have the normalized T -self-intersection number not close to $\kappa / (2\pi^2(g-1))$ is bounded by a decreasing exponential function.

Theorem 1.5. *Let $\epsilon > 0$. There exists $\delta > 0$ such that*

$$\bar{\vartheta} \left\{ \mathbf{v} \in T^1(S) : \left| \frac{i_T(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}})}{T^2} - \frac{\kappa}{2\pi^2(g-1)} \right| \geq \epsilon \right\} = O(e^{-\delta T}), \text{ as } T \rightarrow \infty.$$

As a consequence of Theorem 1.5, we obtain the result given by Lalley in [13, Theorem 1].

Corollary 1.6 (Lalley). *For every $\epsilon > 0$,*

$$\lim_{T \rightarrow \infty} \frac{1}{N(T)} \# \left\{ \gamma \in C\mathbb{G}_T : \left| i(\gamma, \gamma) - \frac{\kappa l(\gamma)^2}{2\pi^2(g-1)} \right| < \epsilon l(\gamma)^2 \right\} = 1.$$

The outline of this paper is the following. Section 2 is the collection of definitions and results needed in the demonstrations of Theorems 1.3 and 1.5, and Section 3 contains the proofs of these theorems as well as the proofs of Corollary 1.4 and Corollary 1.6. For detailed explanation of all the concepts (or a different approach on them) used in this work, please see [3], [9], [11], [15] and [16].

2. Preliminaries

2.1. Measure of Maximum Entropy

The map $\varphi : T^1(S) \times \mathbb{R} \rightarrow T^1(S)$ defined by $\varphi(\mathbf{v}, t) = \varphi^t \mathbf{v} = (\gamma_{\mathbf{v}}(t), \dot{\gamma}_{\mathbf{v}}(t))$ is the *geodesic flow over S* . Let $h_{top}(\varphi)$ be the *topological entropy* of φ . A measure μ on $T^1(S)$ is φ -invariant if $\mu(\varphi^t(E)) = \mu(E)$, for every $t \in \mathbb{R}$ and every Borel set E of $T^1(S)$. For instance, the measure ϑ is φ -invariant. Denote by $\mathcal{P}_{\varphi}$ the set of φ -invariant probability measures on $T^1(S)$ equipped with the weak*-topology, and for $\mu \in \mathcal{P}_{\varphi}$, let $h_{\mu}(\varphi)$ denote its *measure theoretic entropy with respect to φ* (please see [9, §4.3] for definitions.) The Variational Principle (proven by T.N.T. Goodman) in [7, Main Theorem] states that $h_{top}(\varphi) = \sup_{\mu \in \mathcal{P}_{\varphi}} h_{\mu}(\varphi)$. In fact, Bowen proved in [5] that in our case this supremum is actually a maximum and is uniquely achieved by the normalized Riemannian measure $\bar{\vartheta}$, with $h_{\bar{\vartheta}}(\varphi) = h_{top}(\varphi) = \sqrt{\kappa}$. Therefore, $\bar{\vartheta}$ coincides with the *measure of maximum entropy on $T^1(S)$* . Moreover, $\bar{\vartheta}$ also coincides with the *Margulis-Bowen measure* from [14], and in this work, we use the characterization of this measure given by Bowen in [4].

The (φ) -orbit of $\mathbf{v} \in T^1(S)$ is the set $\{\varphi^t \mathbf{v} \mid t \in \mathbb{R}\}$. These orbits form a partition of $T^1(S)$. Note that there is a one-to-one correspondence between the set of orbits and the set $\mathbb{G}$. The vector $\mathbf{v} \in T^1(S)$ and its orbit are *periodic* if there exists $l > 0$ such that $\varphi^l \mathbf{v} = \mathbf{v}$, the number l is a *period* and the minimal period is precisely $l(\gamma_{\mathbf{v}})$.

For a periodic orbit γ , Bowen defined the *occupation measure* ζ_{γ} on $T^1(S)$ by

$$\zeta_{\gamma}(E) = \int_0^{l(\gamma)} \chi_E(\varphi^t \mathbf{v}) dt, \quad (1)$$

for $\mathbf{v} \in \gamma$ and E a Borel set of $T^1(S)$. In addition, Bowen proved in [4, (5.5)] the following.

Theorem 2.1 (Bowen). *The periodic orbits of the geodesic flow φ are equidistributed with respect to the measure of maximum entropy $\bar{\vartheta}$ as the period tends to $+\infty$. More precisely, for any Borel set E with $\bar{\vartheta}(\partial E) = 0$,*

$$\bar{\vartheta}(E) = \lim_{T \rightarrow \infty} \frac{1}{N(T)} \sum_{\gamma \in C\mathbb{G}_T} \frac{\zeta_{\gamma}}{l(\gamma)}(E).$$

2.2. Geodesic Currents

Let $\mathbb{L} = T^1(S)/\sim$, with $(x, v) \sim (x, -v)$, be the *line bundle* of S and $\mathcal{F}$ be the *foliation* of $\mathbb{L}$ by φ -orbits. A (*geodesic*) *current* μ on S is a positive transverse invariant measure for the geodesic foliation $\mathcal{F}$. The set of currents on S equipped with the weak* topology is denoted by $\mathcal{C}$ and called the *space of currents on S* .

Given any φ -invariant measure μ , we can consider the associated transverse measure $\tilde{\mu}$ for the foliation $\mathcal{F}$. Each $\tilde{\mu} \in \mathcal{C}$ is normalized by the requirement that (locally) $\mu = \tilde{\mu} \times dt$, where dt is the one-dimensional Lebesgue measure along leaves in $\mathcal{F}$. The current associated to ϑ is called the *Liouville current* on S . In this paper, we identify the measure μ with the current $\tilde{\mu}$.

The basic example of a current is the one associated to a closed geodesic γ on S . To this geodesic γ corresponds a compact leaf $\tilde{\gamma}$ of $\mathcal{F}$. We associate to it the current μ_γ which induces on each transverse manifold V the Dirac measure at the point $V \cap \tilde{\gamma}$. Such current corresponds to the φ -invariant measure ζ_γ , as defined in (1).

Observe that it is always possible to add two geodesic currents, and to multiply a geodesic current by a non-negative real number. Then, the space $\mathcal{C}$ appears as the completion of the space of real multiples of homotopy classes of closed curves by the following fact, proven by Bonahon in [3, Proposition 2], which we state although we will not make use of it in this paper.

Proposition 2.2. *The uniform space $\mathcal{C}$ is complete, and the real multiples of homotopy classes of closed curves are dense in it.*

2.3. Intersection Form

Starting from the bundle $\mathbb{L} \rightarrow S$, we consider the Whitney sum $\mathbb{L} \oplus \mathbb{L} \rightarrow S$. In other words, $\mathbb{L} \oplus \mathbb{L}$ is the 4-dimensional manifold of triples $(x, \lambda_1, \lambda_2)$, where $x \in S$ and λ_1 and λ_2 are two lines in the tangent space $T_x(S)$. Forgetting the first or the second line defines two projections p_1 and p_2 from $\mathbb{L} \oplus \mathbb{L}$ to $\mathbb{L}$. We consider the two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of codimension 2 in $\mathbb{L} \oplus \mathbb{L}$, whose leaves are the preimages of the leaves of $\mathcal{F}$ by, respectively, p_1 and p_2 . These foliations are transverse outside the diagonal $\Delta = \{(x, v, v) \mid (x, v) \in \mathbb{L}\}$ of $\mathbb{L} \oplus \mathbb{L}$.

Recall that a *transverse invariant measure* on $\mathbb{L}$ is an assignment of a positive measure to each transversal τ to $\mathbb{L}$, supported on $\tau \cap \mathbb{L}$ and invariant under homotopy. Let μ and ν be two currents. Through p_1 , μ induces a transverse invariant measure $\hat{\mu}_1$ on $\mathcal{F}_1$, which, by transversality of $\mathcal{F}_1$ and $\mathcal{F}_2$, gives outside Δ a measure on each leaf of $\mathcal{F}_2$. Similarly, ν induces outside Δ a measure $\hat{\nu}_2$ on each leaf of $\mathcal{F}_1$. Consider then the product measure $\hat{\mu}_1 \times \hat{\nu}_2$ on $\mathbb{L} \oplus \mathbb{L} \setminus \Delta$. The total mass of this measure is finite. The *intersection form* of μ and ν is

$$\iota(\mu, \nu) = \hat{\mu}_1 \times \hat{\nu}_2(\mathbb{L} \oplus \mathbb{L} \setminus \Delta).$$

Remark 2.3. The *normalized Liouville current* denoted by $\bar{\vartheta}$ (which corresponds to the normalized Riemannian measure) that satisfies the equality

$$\iota(\vartheta, \bar{\vartheta}) = 1.$$

By identifying the closed geodesic α on S with the current ζ_α , Bonahon proved the following facts in [2, Theorem 4.1] and [3, Proposition 15].

Theorem 2.4 (Bonahon). *The intersection form function $\iota : \mathcal{C} \times \mathcal{C} \rightarrow [0, \infty)$ is a continuous extension of the intersection number function. In particular, for α and β closed geodesics on S ,*

$$\iota(\zeta_\alpha, \zeta_\beta) = i(\alpha, \beta).$$

In addition,

$$\iota(\vartheta, \vartheta) = \frac{2\pi^2(g-1)}{\kappa} = \frac{\vartheta(T^1(S))}{2}$$

and

$$\iota(\bar{\vartheta}, \bar{\vartheta}) = \frac{\kappa}{2\pi^2(g-1)} = \frac{2}{\vartheta(T^1(S))}.$$

3. Results

The proofs of Theorem 1.3 and 1.5 are based on both the continuity of the intersection form function (given by Theorem 2.4) and a deviation result given by Y. Kifer [11].

For $T > 0$ and $\mathbf{v} \in T^1(S)$, Y. Kifer defined the occupation measure $\zeta_{\mathbf{v}}^T$ by

$$\zeta_{\mathbf{v}}^T(E) = \int_0^T \chi_E(\varphi^t \mathbf{v}) dt,$$

for every Borel set E of $T^1(S)$.

Note that if γ is a periodic orbit, we have $\zeta_{\mathbf{v}}^{l(\gamma)} = \zeta_\gamma$, for $\mathbf{v} \in \gamma$.

Fact 1. The intersection form can be extended to the whole set of (positive) finite measures (not necessarily φ -invariant). By abuse of notation, we denote this extension also by ι . Such extension satisfies the following condition

$$\iota(\zeta_{\mathbf{v}}^T, \zeta_{\mathbf{w}}^T) = i_T(\gamma_{\mathbf{v}}, \gamma_{\mathbf{w}}), \text{ for } \mathbf{v}, \mathbf{w} \in T^1(S) \text{ and } T > 0.$$

Since $T^1(S)$ is a compact metric space and φ is a hyperbolic dynamical system, the deviation results of Y. Kifer in [10, Theorem 3.4] and [11, Theorem 2.1], respectively, can be translated into our setting in the following way.

Theorem 3.1 (Kifer). *For any closed subset K of $\mathcal{P}$, the space of probability measures on $T^1(S)$,*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \bar{\nu} \left\{ \mathbf{v} \in T^1(S) : \frac{\zeta_{\mathbf{v}}^T}{T} \in K \right\} \leq - \inf_{\mu \in K} I(\mu),$$

where

$$I(\mu) = \begin{cases} h_{\text{top}}(\varphi) - h_{\mu}(\varphi), & \mu \in \mathcal{P}_{\varphi} \\ \infty, & \text{otherwise} \end{cases}.$$

Remark 3.2. Note that, for any subset $Z \subset \mathcal{P}$, $\inf_{\mu \in Z} I(\mu) = \inf_{\mu \in Z \cap \mathcal{P}_{\varphi}} I(\mu)$, since $I(\mu) = \infty$, for $\mu \notin \mathcal{P}_{\varphi}$.

Theorem 3.3 (Kifer). *Let $\mathcal{U}$ be an open neighborhood of the measure of maximal entropy $\bar{\nu}$ in the set of φ -invariant probability measures on $T^1(S)$. Then*

$$\lim_{T \rightarrow \infty} \frac{1}{N(T)} \#\{\gamma \in C\mathbb{G}_T : \zeta_{\gamma}/l(\gamma) \notin \mathcal{U}\} = O(e^{-\delta T}),$$

as $T \rightarrow \infty$, where $\delta = \inf_{\mu \in \mathcal{U}^c} \{h_{\text{top}}(\varphi) - h_{\mu}(\varphi)\}$.

Proof of Theorem 1.5. Let $\epsilon > 0$. Consider the set

$$K := \{\mu \in \mathcal{P} : |\imath(\mu, \mu) - \imath(\bar{\nu}, \bar{\nu})| \geq \epsilon\}. \quad (2)$$

By Theorem 2.4 and Fact 1,

$$\left\{ \mathbf{v} \in T^1(S) : \left| \frac{i_T(\gamma_{\mathbf{v}}, \gamma_{\mathbf{v}})}{T^2} - \frac{\kappa}{2\pi^2(g-1)} \right| \geq \epsilon \right\} \quad (3)$$

$$= \left\{ \mathbf{v} \in T^1(S) : \frac{\zeta_{\mathbf{v}}^T}{T} \in K \right\}. \quad (4)$$

By (3), it is enough to prove that there exists $\delta > 0$ such that

$$\bar{\nu} \left\{ \mathbf{v} \in T^1(S) : \frac{\zeta_{\mathbf{v}}^T}{T} \in K \right\} = O(e^{-\delta T}), \text{ as } T \rightarrow \infty. \quad (5)$$

In order to prove 5 we first see that the set K defined in (2) is a closed subset of $\mathcal{P}$ since the intersection form function $\imath$ is continuous by Theorem 2.4. Therefore, by Theorem 3.1,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \bar{\nu} \left\{ \mathbf{v} \in T^1(S) : \frac{\zeta_{\mathbf{v}}^T}{T} \in K \right\} \leq - \inf_{\mu \in K} I(\mu).$$

Now we consider two cases taking Remark 3.2 into account.

(i) $K \cap \mathcal{P}_\varphi = \emptyset$.

In this case, $\inf_{\mu \in K} I(\mu) = \infty$. Thus,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \bar{\vartheta} \left\{ \mathbf{v} \in T^1(S) : \frac{\zeta_{\mathbf{v}}^T}{T} \in K \right\} \leq -\infty.$$

Hence, for every $r > 0$, $M(T) := \bar{\vartheta} \left\{ \mathbf{v} \in T^1(S) : \zeta_{\mathbf{v}}^T \in K \right\} \leq e^{-r}$, consequently, $M(T) = 0 = O(e^{-\delta T})$, as $T \rightarrow \infty$, for any $\delta > 0$.

(ii) $K \cap \mathcal{P}_\varphi \neq \emptyset$.

Since $\bar{\vartheta}$ is the unique probability measure of $T^1(S)$ where the maximum entropy $h_{\text{top}}(\varphi) = \sqrt{\kappa}$ is achieved and $\bar{\vartheta} \notin K$, then taking δ as

$$\delta = \inf_{\mu \in K} I(\mu) = \inf_{\mu \in K \cap \mathcal{P}_\varphi} I(\mu) = \inf_{\mu \in K \cap \mathcal{P}_\varphi} (\sqrt{\kappa} - h_\mu(\varphi)),$$

we have that $\delta > 0$ is such that

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \bar{\vartheta} \left\{ \mathbf{v} \in T^1(S) : \frac{\zeta_{\mathbf{v}}^T}{T} \in K \right\} \leq -\delta.$$

Therefore, from cases (i) and (ii) we conclude (5), and consequently our result.

□

Proof of Corollary 1.6. Let $T, \epsilon > 0$ and

$$\mathcal{O}(T, \epsilon) := \left\{ \gamma \in C\mathbb{G}_T : \left| \frac{i(\gamma, \gamma)}{l(\gamma)^2} - \frac{\kappa}{2\pi^2(g-1)} \right| < \epsilon \right\}.$$

Consider

$$\mathcal{U} = K^c = \{ \mu \in \mathcal{P} : |\imath(\mu, \mu) - \imath(\bar{\vartheta}, \bar{\vartheta})| < \epsilon \}.$$

Then, $\mathcal{O}(T, \epsilon) = \{ \gamma \in C\mathbb{G}_T : \zeta_\gamma \in \mathcal{U} \}$.

By Theorem 3.3, we have

$$\frac{1}{N(T)} \# \{ \gamma \in C\mathbb{G}_T : \zeta_\gamma / l(\gamma) \notin \mathcal{U} \} = O(e^{-\delta T}).$$

Consequently,

$$\lim_{T \rightarrow \infty} \frac{1}{N(T)} \# \left\{ \gamma \in C\mathbb{G}_T : \left| \frac{i(\gamma, \gamma)}{l(\gamma)^2} - \frac{\kappa l(\gamma)^2}{2\pi^2(g-1)} \right| < \epsilon l(\gamma)^2 \right\} = 1.$$

□

Proof of Theorem 1.3. Consider the function $\iota : \mathcal{P} \times \mathcal{P} \rightarrow \iota(\mathcal{P} \times \mathcal{P})$. This function is continuous since it is the restriction of the intersection form function ι , which is continuous by Theorem 2.4, to $\mathcal{P} \times \mathcal{P}$ a closed subset of $\mathcal{C} \times \mathcal{C}$.

Therefore, for $\epsilon > 0$, the set $\mathcal{Z} = \iota^{-1}\left(\frac{\kappa}{2\pi^2(g-1)} - \epsilon, \frac{\kappa}{2\pi^2(g-1)} + \epsilon\right)$ is an open subset of $\mathcal{P} \times \mathcal{P}$ since it is the preimage under ι of the ball of radius ϵ centered at $\iota(\bar{\vartheta}, \bar{\vartheta}) = \kappa/(2\pi^2(g-1))$.

Let $R, T > 0$ with $R \leq T$ and

$$\mathcal{W}_{R,T} = \left\{ (\alpha, \beta) \in C\mathbb{G}_R \times C\mathbb{G}_T : \left(\frac{\zeta_\alpha}{l(\alpha)}, \frac{\zeta_\beta}{l(\beta)} \right) \in \mathcal{Z} \right\}.$$

By Theorem 2.4,

$$\left\{ (\alpha, \beta) \in C\mathbb{G}_R \times C\mathbb{G}_T : \left| \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} - \frac{\kappa}{2\pi^2(g-1)} \right| \geq \epsilon \right\} \quad (6)$$

$$= C\mathbb{G}_R \times C\mathbb{G}_T \setminus \mathcal{W}_{R,T}. \quad (7)$$

Hence, by (6), it is enough to prove that there exists $\delta > 0$ such that

$$\frac{\#C\mathbb{G}_R \times C\mathbb{G}_T \setminus \mathcal{W}_{R,T}}{N(R)N(T)} = O(e^{-\delta R}).$$

Since $(\bar{\vartheta}, \bar{\vartheta}) \in \mathcal{Z}$ and $\mathcal{Z}$ is an open set of the product topology of $\mathcal{P} \times \mathcal{P}$, there exist $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}$ open neighborhoods of $\bar{\vartheta}$ in $\mathcal{P}$ such that $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{Z}$.

$$\text{Let } \mathcal{U}_R = \left\{ \alpha \in C\mathbb{G}_R : \frac{\zeta_\alpha}{l(\alpha)} \in \mathcal{U} \right\} \text{ and } \mathcal{V}_T = \left\{ \beta \in C\mathbb{G}_T : \frac{\zeta_\beta}{l(\beta)} \in \mathcal{V} \right\}.$$

Given that both $\mathcal{U}$ and $\mathcal{V}$ are open neighborhoods of $\bar{\vartheta}$ on $\mathcal{P}$, Theorem 3.1, guarantees the existence of $\delta_1, \delta_2 > 0$ depending on $\mathcal{U}$ and $\mathcal{V}$, respectively, such that

$$\frac{\#C\mathbb{G}_R \setminus \mathcal{U}_R}{N(R)} = \frac{1}{N(R)} \#\{\gamma \in C\mathbb{G}_R : \zeta_\gamma/l(\gamma) \notin \mathcal{U}\} = O(e^{-\delta_1 R})$$

and

$$\frac{\#C\mathbb{G}_T \setminus \mathcal{V}_T}{N(T)} = \frac{1}{N(T)} \#\{\gamma \in C\mathbb{G}_T : \zeta_\gamma/l(\gamma) \notin \mathcal{V}\} = O(e^{-\delta_1 T}).$$

Thus, since $\mathcal{U}_R \times \mathcal{V}_T \subseteq \mathcal{W}_{R,T}$, we get as $R, T \rightarrow \infty$,

$$\begin{aligned} \frac{\#C\mathbb{G}_R \times C\mathbb{G}_T \setminus \mathcal{W}_{R,T}}{N(R)N(T)} &\leq \frac{\#C\mathbb{G}_R \times C\mathbb{G}_T \setminus \mathcal{U}_R \times \mathcal{V}_T}{N(R)N(T)} \\ &\leq \frac{\#C\mathbb{G}_R \setminus \mathcal{U}_R \cdot \#C\mathbb{G}_T \setminus \mathcal{V}_T}{N(R)N(T)} \\ &\quad + \frac{\#C\mathbb{G}_R \setminus \mathcal{U}_R \cdot \#\mathcal{V}_T}{N(R)N(T)} + \frac{\#C\mathbb{G}_T \setminus \mathcal{V}_T \cdot \#\mathcal{U}_R}{N(R)N(T)} \\ &= O(e^{-\delta_1 R})O(e^{-\delta_2 T}) + O(e^{-\delta_1 R}) + O(e^{-\delta_2 T}) \\ &= O(e^{-\delta_1 R}). \end{aligned}$$

□

For the proof of Corollary 1.4, we need a bound for the intersection number of pairs of closed geodesics on S . Here, we provide a universal bound for the normalized intersection numbers of pairs of closed geodesics. It is worth noting that such bound can also be deduced by the techniques used by A. Basmajian in [1].

The *injectivity radius* at a point $x \in S$ is the largest radius for which the exponential map at x is a diffeomorphism. The *injectivity radius of S* , which we denote by ϱ , is the infimum of the injectivity radii of all points of S . By the definition of ϱ , the least length of an essential loop on S is 2ϱ .

Proposition 3.4. *Let α and β be two closed geodesics on S . Then*

$$\frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} \leq \frac{1}{\varrho^2}.$$

Proof. Let $\bar{\alpha}$ be a sub-arc of α with $l(\bar{\alpha}) < \varrho$ and such that $i(\bar{\alpha}, \beta) \geq i(\alpha^*, \beta)$ for any sub-arc α^* of α with $l(\alpha^*) < \varrho$. Hence,

$$i(\alpha, \beta) \leq \left\lceil \frac{l(\alpha)}{\varrho} \right\rceil i(\bar{\alpha}, \beta). \quad (8)$$

Let $\{x_1, \dots, x_n\}$ be the ordered set of points of intersection of $\bar{\alpha}$ and β , with $\beta^{-1}(x_i) \leq \beta^{-1}(x_{i+1})$, for $1 \leq i \leq n-1$, and $n = i(\bar{\alpha}, \beta)$. Let β_k be the sub-arc of β from x_k to x_{k+1} , for $1 \leq k \leq n-1$, and, β_n be the sub-arc of β from x_n to x_1 . Similarly, let $\bar{\alpha}_k$ be the sub-arc of $\bar{\alpha}$ from x_k to x_{k+1} , for $1 \leq k \leq n-1$, and $\bar{\alpha}_n$ be the sub-arc of $\bar{\alpha}$ from x_n to x_1 .

Consider γ_k the concatenation of $\bar{\alpha}_k$ and β_k , for $1 \leq k \leq n$. Thus, γ_k is an essential loop of S , for $1 \leq k \leq n$.

Hence, $2\varrho \leq l(\gamma_k) = l(\bar{\alpha}_k) + l(\beta_k) \leq l(\bar{\alpha}) + l(\beta_k) < \varrho + l(\beta_k)$, which implies $\varrho < l(\beta_k)$, for $1 \leq k \leq n$.

Consequently, $n\varrho = n \sum_{i=1}^n \varrho < \sum_{i=1}^n l(\beta_k) \leq l(\beta)$. Therefore, $n = i(\bar{\alpha}, \beta) < \frac{l(\beta)}{\varrho}$. Thus, by (8), we conclude $(\alpha, \beta) \leq \left\lceil \frac{l(\alpha)}{\varrho} \right\rceil i(\bar{\alpha}, \beta) \leq \frac{l(\alpha)}{\varrho} \frac{l(\beta)}{\varrho} = \frac{l(\alpha)l(\beta)}{\varrho^2}$. $\square$

Proof of Corollary 1.4. Let $\epsilon > 0$. For $R, T > 0$ with $R \leq T$, consider the set $\mathcal{W}_{R,T}$ defined in (6) from the proof of Theorem 1.3. In addition, let $\delta, J, C > 0$ be constants satisfying the conclusion of such theorem, that is, for $J \leq R \leq T$, we have

$$\frac{\#C\mathbb{G}_R \times C\mathbb{G}_T \setminus \mathcal{W}_{R,T}}{N(R)N(T)} \leq \frac{C}{e^{\delta R}}.$$

Moreover, let J be such that $Ce^{-\delta R} < \epsilon$, whenever $R > J$.

By Proposition 3.4, we have $\sup_{C\mathbb{G}_R \times C\mathbb{G}_T} \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} \leq \frac{1}{\varrho^2}$. Therefore, for $J < R \leq T$, we have

$$\begin{aligned} & \left| \frac{2\pi^2(g-1)}{\kappa N(R)N(T)} \left(\sum_{\substack{\alpha \in C\mathbb{G}_R \\ \beta \in C\mathbb{G}_T}} \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} \right) - 1 \right| \\ &= \left| \frac{2\pi^2(g-1)}{\kappa N(R)N(T)} \sum_{\substack{\alpha \in C\mathbb{G}_R \\ \beta \in C\mathbb{G}_T}} \left(\frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} - \frac{\kappa}{2\pi^2(g-1)} \right) \right| \\ &\leq \frac{2\pi^2(g-1)}{\kappa N(R)N(T)} \left(\sum_{(\alpha, \beta) \in \mathcal{W}_{R,T}} \left| \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} - \frac{\kappa}{2\pi^2(g-1)} \right| \right. \\ &\quad \left. + \sum_{(\alpha, \beta) \in C\mathbb{G}_R \times C\mathbb{G}_T \setminus \mathcal{W}_{R,T}} \left| \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} - \frac{\kappa}{2\pi^2(g-1)} \right| \right) \\ &\leq \frac{2\pi^2(g-1)}{\kappa N(R)N(T)} \left(\# \mathcal{W}_{R,T} \cdot \epsilon \right. \\ &\quad \left. + \#C\mathbb{G}_R \times C\mathbb{G}_T \setminus \mathcal{W}_{R,T} \cdot \sup_{\substack{\alpha \in C\mathbb{G}_R \\ \beta \in C\mathbb{G}_T}} \left| \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} - \frac{\kappa}{2\pi^2(g-1)} \right| \right) \\ &< \frac{2\pi^2(g-1)}{\kappa} \left(\epsilon + \frac{C}{e^{\delta R}} \left[\frac{1}{\varrho^2} + \frac{\kappa}{2\pi^2(g-1)} \right] \right) \\ &< \frac{2\pi^2(g-1)}{\kappa} \left(1 + \frac{1}{\varrho^2} + \frac{\kappa}{2\pi^2(g-1)} \right) \epsilon. \end{aligned}$$

Given that ϵ was chosen arbitrarily, we conclude that

$$\lim_{R, T \rightarrow \infty} \frac{2\pi^2(g-1)}{\kappa N(R)N(T)} \sum_{(\alpha, \beta) \in C\mathbb{G}_R \times C\mathbb{G}_T} \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} = 1,$$

or equivalently,

$$\frac{1}{N(R)N(T)} \sum_{(\alpha, \beta) \in C\mathbb{G}_R \times C\mathbb{G}_T} \frac{i(\alpha, \beta)}{l(\alpha)l(\beta)} \sim \frac{\kappa}{2\pi^2(g-1)},$$

as $R, T \rightarrow \infty$.

✓

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