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The total component of the partial Schur multiplier of the elementary abelian 3-group

La componente total del multiplicador parcial de Schur del 3-grupo abeliano elemental

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Abstract. In this work we determine the total component of the partial Schur multiplier of elementary abelian 3-groups.

Key words and phrases. partial factor set, total component, partial coboundary.

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Resumen. En este trabajo determinamos la componente total del multiplicador parcial de Schur para los 3-grupos abelianos elementales.

Palabras y frases clave. conjunto factor parcial, componente total, cobordo parcial.

1. Introduction

Structural results on partial representations, their associated domains, and the corresponding partial group algebras of a group G, were obtained in [3, 5, 9, 11, 13] and recently in [10]. With the intention of developing a cohomological theory based on partial actions, the concept of partial projective representations was introduced and developed in [6] and [7], this naturally led to the definition of the partial Schur multiplier $pM(G)$ of G, over a field $\mathbb{K}$, and as in the classical case, a key problem in the theory of partial projective representations of G is the study of the structure of $pM(G)$. The latter contains the Schur multiplier $M(G)$ and, different from it, the set $pM(G)$ is not a group, but it is a semilattice of abelian groups $pM_X(G)$, called components, indexed by certain subsets $X \subseteq G \times G$, which are exactly the invariant sets under the action of a semigroup (see

Section 2). A deeper understanding of the structure of $pM(G)$ was presented in [8, 12], where was shown that each component $pM_X(G)$ of $pM(G)$ is formed by equivalence classes of partially defined functions $\sigma\colon G \times G \to K$, having X as domain, i.e. the so-called partial factor sets of G, it is remarkable to notice that the elements of $pM_X(G)$ are determined by its values in a subset of X, which is the union of the so called *effective* orbits, see [12, Theorem 3.9]. In [4, Theorem 2.13], the authors proved that each component, and then the partial Schur multiplier $pM(G)$, is a union of 2-cohomology groups with values in some, in general non-trivial, partial G-modules, that is with values on a monoid M with a unital partial action defined on it.

The total component $pM_{G\times G}(G)$ of $pM(G)$ is the one formed by equivalence classes of totally defined partial factor sets, this includes the usual Schur multiplier as a subgroup and, some necessary and sufficient conditions for a classical two-cocycle to be a totally defined partial factor set have been given in [7, Section 10] and [15, Section 7]. Moreover, according to [8, Corollary 5.8 (iv)] any component of $pM(G)$ is an epimorphic image of $pM_{G\times G}(G)$, so the total component provide information about the structure of $pM(G)$.

In some recent works this component have been calculated for different families of groups. For instance, in the case of finite cyclic group [8, Corollary 6.4] see also [2, Proposition 6.3], an elementary abelian 2-group [12, Theorem 3.11], the symmetric group [14, Lemma 3.10] and other relevant families of groups [1]. Moreover, some results on the structure of the torsion part of $pM_{G\times G}(G)$ are presented in [15].

The purpose of this work is to characterize $pM_{G\times G}(G)$ when G is the elementary abelian 3-group.

The article is structured as follows. After the introduction, in Section 2 we provide all the necessary preliminaries, fix some notations and recall characterizations of the partial Schur multiplier, partial factor sets, and coboundaries. In Section 3 we focus in elementary 3-abelian groups and after determining the cardinality of the set consisting of effective orbits, we use Proposition 3.1 and Proposition 3.3 to obtain a determine the (total) coboundaries, finally combining our results a description of $pM_{G\times G}(G)$ is obtained in Theorem 3.4.

2. The notions

Partial projective representations of a group G over a field $\mathbb{K}$ were introduced and studied in [6, 7]. As in the classical case, the set $pm(G)$ of the partial factor sets of G appeared naturally, as well as the corresponding partial Schur multiplier $pM(G)$. We recall these concepts for the reader's convenience.

Definition 2.1. Let G be a group, $\mathbb{K}$ an algebraically closed field, and $\mathbb{K}^*$ its multiplicative group.

- A *(unital) partial homomorphism* of a group G with values in $M_n(\mathbb{K})$ is a map $\phi\colon G \to M_n(\mathbb{K})$ preserving the unity and such that $\phi(g)\phi(h)\phi(h^{-1}) = \phi(gh)\phi(h^{-1})$ and $\phi(g^{-1})\phi(g)\phi(h) = \phi(g^{-1})\phi(gh)$, for all $g, h \in G$.
- Denote by $PM_n(\mathbb{K})$ the monoid of projective linear matrices over $\mathbb{K}$ and let $\xi\colon M_n(\mathbb{K}) \to PM_n(\mathbb{K})$ be the natural projection. A *partial projective representation* of G on $M_n(\mathbb{K})$ is a function $\Gamma\colon G \to M_n(\mathbb{K})$ such that the composition $\xi\Gamma\colon G \to PM_n(\mathbb{K})$, is a partial homomorphism.

Given a partial projective representation $\Gamma\colon G \to M_n(\mathbb{K})$ then by [6, Theorem 3] there is a unique partially defined function $\sigma\colon G \times G \to \mathbb{K}^*$, such that

$$\operatorname{dom}\sigma = \{(x, y) \in G \times G \mid \Gamma(x)\Gamma(y) \neq 0\},$$
$$\Gamma(x^{-1})\Gamma(x)\Gamma(y) = \Gamma(x^{-1})\Gamma(xy)\sigma(x, y)$$

and

$$\Gamma(x)\Gamma(y)\Gamma(y^{-1}) = \Gamma(xy)\Gamma(y^{-1})\sigma(x, y),$$

for every $(x, y) \in \operatorname{dom}\sigma$.

Definition 2.2. The function σ associated with a partial projective representation Γ as above is called a *factor set* of Γ or a *partial factor set of* G.

By [6, Corollary 5], factor sets of partial projective representations of G form a commutative inverse monoid $pm(G)$, with respect to point-wise multiplication. Thus $pm(G)$ is isomorphic to a semilattice of abelian groups.

2.1. The semigroup $\mathcal{T}$ and the partial Schur multiplier

We use semigroup actions to obtain a better description of partial factor sets. Consider the following transformations on $G \times G$:

$$u : (x, y) \mapsto (xy, y^{-1}), \quad v : (x, y) \mapsto (y^{-1}, x^{-1}), \quad t : (x, y) \mapsto (x, 1).$$

It is readily seen that these transformations satisfy the equalities

$$u^2 = v^2 = (uv)^3 = 1,\ t^2 = t,\ ut = t,\ tuvt = tvuv,\ tvt = 0. \tag{1}$$

where 0 stands for the map $(x, y) \mapsto (1, 1)$.

In [7, Section 6] the authors introduced the abstract monoid $\mathcal{T}$ generated by symbols u, v and t with relations (1).

There is a disjoint union $\mathcal{T} = \mathcal{S} \cup t\mathcal{S} \cup vt\mathcal{S} \cup uvt\mathcal{S} \cup 0$, where

$$\mathcal{S} = \langle u, v \mid u^2 = v^2 = (uv)^3 = 1\rangle$$

is a group isomorphic to the symmetric group S_3. Then we have a left action of $\mathcal{T}$ on $G \times G$ defined by the following maps:

$$t(x, y) = (x, 1), \quad u(x, y) = (xy, y^{-1}) \quad \text{and} \quad v(x, y) = (y^{-1}, x^{-1}),$$

for any $x, y \in G$. The $\mathcal{T}$-invariant subsets X of $G \times G$, that is, the elements of $C(G) = \{X \subseteq G \times G \mid \mathcal{T}X \subseteq X\}$ form a meet semilattice which helps us to characterize partial factor sets. Indeed, by [6, Theorem 5] we have that $pm(G) = \bigcup_{X \in C(G)} pm_X(G)$, where $pm_X(G)$ consists of the factor sets with fixed domain $X \in C(G)$.

On the semigroup $pm(G)$ a congruence $\sim$ is defined by:

$$\sigma \sim \tau \Leftrightarrow \sigma(x,y) = \eta(x)\eta(xy)^{-1}\eta(y)\tau(x,y),\ x, y \in G$$

for some function $\eta\colon G \to K^*$. The semigroup $pM(G) = pm(G)/\sim$ is called *the partial Schur multiplier of G.*

Equivalent partial factor sets have the same domain and by [7, Theorem 5] the semigroup $pM(G)$ is a semilattice of the abelian groups $pM_X(G) = pm_X(G)/\sim$. These groups are called *components* of $pM(G)$, the component $pM_{G\times G}(G)$ is called *total*, and elements in $pM(G)$ are denoted by $\text{cls}(\sigma)$, for $\sigma \in pm(G)$.

Theorem 2.3 ([8, Theorem 5.6])**.** *Let τ be a partial factor set of G with domain X. Then there is a partial factor set σ, equivalent to τ, such that for all $(a, b) \in X$*

$$\sigma(a,b)\sigma(b^{-1},a^{-1}) = 1_{\mathbb{K}}, \tag{2}$$

$$\sigma(a,b) = \sigma(b^{-1}a^{-1},a) = \sigma(b,b^{-1}a^{-1}), \tag{3}$$

$$\sigma(a,1) = 1_{\mathbb{K}}, \tag{4}$$

for any $(a, b) \in X$. Conversely, let $\sigma : G \times G \to \mathbb{K}$ be a partially defined map with domain $X \in C(G)$ such that (2)-(4) *are satisfied for all $(a, b) \in X$. Then σ is a partial factor set of G.*

Since S_3 is isomorphic to a subgroup of $\mathcal{T}$, there is an action of S_3 in $G \times G$ induced by the action of $\mathcal{T}$. Thus, the S_3-orbit of $(x, y) \in G \times G$ is:

$$S_3(x,y) = \{(x,y),(xy,y^{-1}),(y,y^{-1}x^{-1}),(y^{-1},x^{-1}),(y^{-1}x^{-1},x),(x^{-1},xy)\}.$$

Consequently, the Orbit-Stabilizer Theorem implies that each S_3-orbit contains $1, 2, 3$ or 6 elements. In [12], the orbits with 2 or 6 elements were called *effective.* Then, the non-effective orbits are of the form $\{(1,y),(y,y^{-1}),(y^{-1},1)\},\ y \in G$, and the value of any partial factor set on non-effective orbits is $1_{\mathbb{K}}$ according to (2) and (4).

***Remark* 2.4.** We conclude that any σ in $pm_X(G)$ is completely determined by its values in $X \cap U$, where $U = \{(x, y) \in G \times G \mid x, y, xy \neq 1\}$.

Now to determine $pM_X(G)$ we recall the next.

Corollary 2.5 ([8, Corollary 5.8])**.** *Let $X \in C(G)$. Then:*

(1) *The kernel* $N_X = \{\sigma \in pm_X(G) \mid \sigma \sim 1\}$ *of the natural epimorphism of* $pm_X(G) \to pM_X(G)$ *consists of those* $\sigma\colon G \times G \to K$ *for which there is* $\rho : G \times G \to K^*$ *satisfying the following conditions:*

$$\rho(1) = 1_{\mathbb{K}}, \quad \rho(a)\rho(a^{-1}) = 1_{\mathbb{K}},\ \forall\, a \in G \quad \text{with} \quad (a,1) \in X \tag{5}$$

and

$$\sigma(a,b) = \begin{cases} \rho(a)\rho(b)\rho(ab)^{-1}, & \text{if}(a,b) \in X, \\ 0, & \text{if}(a,b) \notin X. \end{cases} \tag{6}$$

(2) *Let* $s = s(G, X)$ *be the cardinality of the set of effective* S_3*-orbits of* X *and* $\{(a_i, b_i)\}_{1 \leq i \leq s}$ *a full set of representatives of these orbits. Then the map*

$$\phi : (K^*)^s \ni x \mapsto \sigma_x \in pm_X(G),$$

in which $x = (x_i)_{1 \leq i \leq s}$ *and* $\sigma_x(a_i, b_i) = x_i$*, is an isomorphism of multiplicative groups.*

(3) *For every domain* $Y \in C(G)$ *such that* $Y \supseteq X$*, there is an epimorphism* $\psi_X^Y\colon pM_Y(G) \to pM_X(G)$*. In particular,* $pM_X(G)$ *is an epimorphic image of the total component* $pM_{G\times G}(G)$*.*

The partial factor sets σ in N_X are called *coboundaries* in $pm_X(G)$. In this case we write $\sigma = \partial\rho$, where ρ is a function satisfying (5) and (6). Thus to calculate the total component, it is useful to consider the epimorphism

$$\psi : (K^*)^s \ni x \mapsto \text{cls}(\sigma_x) \in pM_{G\times G}(G),$$

where $s = s(G, G \times G)$.

3. Elementary abelian 3-group

From now on in this work it is assumed that G is the elementary abelian 3-group

$$G = \langle x_1, x_2, \ldots, x_n \mid x_i^3 = [x_i, x_j] = 1,\ 1 \leq i, j \leq n\rangle.$$

We are interested in determining the component $pM_{G\times G}(G)$ of $pM(G)$ over an algebraically closed field $\mathbb{K}$.

Using Remark 2.4 we see that the elements of $pM_{G\times G}(G)$ are determined by its values in the set

$$A = (G \setminus 1) \times (G \setminus 1) \setminus \Delta,$$

where $\Delta = \{(x, x^2) \mid x \in G \setminus \{1\}\}$. The action of S_3 on A is given by:

$$g\colon (x, y) \to (xy, y^2) \quad h\colon (x, y) \to (y^2, x^2),$$

and the orbit of the element $(x,y)\in A$ is

$$S_3(x,y)=\{(x,y),\,(x^2y^2,x),\,(y,x^2y^2),\,(y^2,x^2),\,(x^2,xy),\,(xy,y^2)\}.$$

Observe that $|A|=(3^n-1)(3^n-2)$. On the other hand, for $a\in G\setminus\{1\}$, $|S_3(a,a)|=2$, and if $b\in G\setminus\{1,a,a^2\}$, we have $|S_3(a,b)|=6$. Thus, there are:

$$M_n=\frac{3^n-1}{2}+\frac{(3^n-1)(3^n-2)-(3^n-1)}{6}=\frac{3^n(3^n-1)}{6},$$

different effective S_3-orbits $S_3(x,y)$. There exists an epimorphism:

$$\psi\colon(\mathbb{K}^*)^{M_n}\ni\mu\longrightarrow\operatorname{cls}(\sigma_\mu)\in pM_{G\times G}(G), \tag{7}$$

where $\mu=(\mu_1,\cdots,\mu_{M_n})$, $\sigma=\sigma_\mu\in p\tilde{m}_{G\times G}(G)$ and its value in the i^{th} orbit is determined by μ_i. Hence we need to find $\ker\psi=\{\mu\in(\mathbb{K}^*)^{M_n}\,|\,\operatorname{cls}(\sigma_\mu)=1\}$.

Proposition 3.1. *Let $\sigma=\sigma_\mu$ be a coboundary, then σ is completely determined by the values ω_x, where $\omega_1=1_{\mathbb{K}}$ and for $x\in G\setminus\{1\}$, ω_x is a fixed cubic root of $\sigma(x,x)$ such that $\omega_x\omega_{x^2}=1$.*

Proof. Let $\rho\colon G\to\mathbb{K}^*$, such that $\sigma(a,b)=\rho(a)\rho(b)\rho(ab)^{-1}$ for all $a,b\in G$. As $\sigma=\sigma_\mu$, we obtain $\rho(1)=\rho(a)\rho(a^2)=1_K$ for all $a\in G$. In particular, if $a\in G\setminus\{1\}$, $\sigma(a,a)=\rho(a)^3$. Hence, $\rho(a)=\dfrac{1_{\mathbb{K}}}{\rho(a^2)}=\omega_a$, where $\omega_a\in\mathbb{K}^*$ verifies $\omega_a^3=\sigma(a,a)$. Now, suppose $(x,y)\in A$ and $x\neq y$, then:

$$\sigma(x,y)=\rho(x)\rho(y)\rho(xy)^{-1}=\rho(x)\rho(y)\rho(x^2y^2)=\omega_x\omega_y\omega_{x^2y^2}.$$

☑

***Remark* 3.2.** Calculating the number L_n of the independent $\sigma(x,x)$ values, we have

$$L_n=\frac{3^n-1}{2}.$$

From the proof of Proposition 3.1 we conclude that any coboundary σ satisfying (2)-(4) verifies:

$$\sigma(x,y)=\omega_x\omega_y\omega_{x^2y^2}, \tag{8}$$

where the $\omega_x\in\mathbb{K}^*$ are such that:

$$\omega_1=1_{\mathbb{K}} \tag{9}$$

and

$$\omega_x\omega_{x^2}=1_{\mathbb{K}}. \tag{10}$$

Taking into account those conditions, we can get the converse assertion to Proposition 3.1. Set $I_{L_n}=\{1,\ldots,L_n\}$ and $G_{L_n}=\{u_i\in G\setminus 1\,|\,i\in L_n, u_i\notin\{u_j,u_j^2\}\text{ if }i\neq j\}$. Then $|G_{L_n}|=L_n$ and $x\in G_{L_n}$ if, and only if, $x^2\notin G_{L_n}$.

Proposition 3.3. *Let $\phi : I \to \mathbb{K}^*$ be a map. For $i \in I_{L_n}$ and $u_i \in G_{L_n}$ define $\sigma(u_i, u_i) = \phi(i)$, pick $\omega_{u_i} \in \mathbb{K}^*$ such that $\omega_{u_i}^3 = \phi(i)$ and extend σ on $G \times G$ by (8), (9) and (10). Then σ is a coboundary.*

Proof. For a function $\rho\colon G \to \mathbb{K}^*$ and $u_i \in G_{L_n}$, write $\rho(1) = \omega_1 = 1_{\mathbb{K}}$ and $\rho(u_i) = \dfrac{1_K}{\rho(u_i^2)} = \omega_{u_i}$, in particular $\rho(u_i^2) = \omega_{u_i}^2$. Then:

$$\sigma(u_i, u_i) = \phi(i) = \omega_{u_i}^3 = \omega_{u_i}\omega_{u_i}\omega_{u_i} = \rho(u_i)\rho(u_i)\rho(u_i) = \frac{\rho(u_i)\rho(u_i)}{\rho(u_i^2)},$$

$$\sigma(u_i^2, u_i^2) = \frac{1_{\mathbb{K}}}{\sigma(u_i, u_i)} = \frac{1_{\mathbb{K}}}{\phi(i)} = \frac{\rho(u_i)^2}{\rho(u_i)\rho(u_i)} = \frac{\rho(u_i^2)\rho(u_i^2)}{\rho(u_i)}, \text{ and for } u \neq v$$

$$\sigma(u, v) = \frac{\omega_u\omega_v}{\omega_{uv}} = \frac{\rho(u)\rho(v)}{\rho(uv)}.$$

Now we can describe the component $pM_{G\times G}(G)$. Denote by $\mathcal{O}r$ the set of all S_3-orbits of A. Let $\mathcal{O}r_1$ be the subset of all S_3-orbits which contain pairs of the form (u, u) and $\mathcal{O}r_2 = \mathcal{O}r \setminus \mathcal{O}r_1$. We know that

$$pm_{G\times G}(G) \cong \prod_{A\in\mathcal{O}r} \mathbb{K}_A^* = (\mathbb{K}^*)^{M_n},$$

(here $\mathbb{K}_A^* = \mathbb{K}^*$). So $pm_{G\times G}(G) \cong P_1 \oplus P_2$, where

$$P_1 = \prod_{A\in\mathcal{O}r_1} \mathbb{K}_A^* = (K^*)^{L_n},\ P_2 = \prod_{B\in\mathcal{O}r_2} \mathbb{K}_B^* = (K^*)^{M_n-L_n}.$$

The kernel of the epimorphism ψ given by (7) is the subgroup $Q \subset (\mathbb{K}^*)^{M_n}$, consisting of tuples

$$\mu = \left(\left(\mu_{(u,u)}\right)_{S_3(u,u)\,\in\,\mathcal{O}r_1}, (\mu_B)_{B\in\mathcal{O}r_2} \right),$$

where $\mu_{(u,u)}$ $(S_3(u, u) \in \mathcal{O}r_1)$ belongs to $\mathbb{K}^*$, while $\sigma_\mu(u, v)$ for $(u, v) \in B \in \mathcal{O}r_2$ are calculated by Proposition 3.1. From this we conclude $pM_{G\times G}(G) \cong (\mathbb{K}^*)^{M_n-L_n}$, hence

$$pM_{G\times G}(G) \cong (\mathbb{K}^*)^{(3^{n-1}-1)\left(\frac{3^n-1}{2}\right)}.$$

Thus we have obtained the following. ☑

Theorem 3.4. *Let G be the elementary abelian 3-group with n generators and $\mathbb{K}$ an algebraically closed field. Then the total component $pM_{G\times G}(G)$ of the partial Schur multiplier of G is isomorphic to $(\mathbb{K}^*)^{(3^{n-1}-1)\left(\frac{3^n-1}{2}\right)}$.*

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