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On non-abelian representations of Baumslag-Solitar groups

Representationes no Abelianas de los Grupos de Baumslag-Solitar

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Abstract. The goal of this paper is to study the set of non-abelian representations $\widetilde{H}$ (nab-rep) of the Baumslag-Solitar groups,

$$BS(n,m) = \left\langle x, y : xy^n x^{-1} = y^m \right\rangle,$$

with n, m non zero integers, into $SL(2,\mathbb{C})$.

We use such information in order to show, which it is well known, that for $|m| > 1$, $BS(1,m)$ is a linear group. Moreover, we prove that its representation image into the Möbius transformations is an elementary and non discrete subgroup.

Key words and phrases. Representations of Baumslag-Solitar groups, Baumslag-solitar groups, Parabolic representations, Elliptic representations, Affine varieties, Affine algebraic sets.

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Resumen. El proposito de este artículo es estudiar el conjunto de las representaciones no abelianas $\widetilde{H}$ (nab-rep) de los grupos de Baumslag-Solitar,

$$BS(n,m) = \left\langle x, y : xy^n x^{-1} = y^m \right\rangle,$$

donde n, m son enteros distintos de cero, en $SL(2,\mathbb{C})$.

Usamos tal información para verificar, que ya es bien sabido, que $BS(1,m)$ es un grupo lineal, para $|m| > 1$. Mas aún, probamos que su representación en las transformaciones de Möbius es un subgrupo elemental y no es discreto.

Palabras y frases clave. Representationes de los grupos de Baumslag-Solitar, Grupos de Baumslag-Solitar, Representaciones parabólicas, Representaciones elipticas, Variedades afines, Conjuntos algebraicos afines.

1. Introduction

Baumslag-Solitar groups constitute an important family of examples or counterexamples in the theory of combinatorial groups, see [2], [7], [5] and [4]. These groups were first introduced by G. Baumslag and D. Solitar in [2] in order to get non-Hopfian one-relator group presentations and they are given by the following short presentation

$$BS(m,n) = \langle x, y : xy^n x^{-1} = y^m \rangle$$

where m and n are non zero integer numbers.

Although the principal focus of the study of non-abelian representations has been on the family of classical knot groups, in particular on the collection of 2-bridge classical knot groups, we want to start a classification of the $PSL(2,\mathbb{C})$ non abelian representations on the family of Baumslag-Solitar groups. Recently, in a joint work with O. Salazar and J. Mira, see [8], we proved that $BS(n, n+1)$ are the only Baumslag-Solitar groups that correspond to groups of 2-bridge non-classical virtual knots. From this we consider that the study of some properties of the Baumslag-Solitar groups becomes an important aim, in particular the classification of their non-abelian representations into the group $PSL(2,\mathbb{C})$. It is well known that there exists an isomorphism between the orientation preserving isometries of $\mathbb{H}^3$ and $PSL(2,\mathbb{C})$.

In this paper we present a collection of algebraic varieties that encode information about the set of equivalence classes of non-abelian representations of $BS(n,m)$. The definition of these algebraic varieties uses a classification of the conjugacy classes of elements of the Möbius group via diagonal matrices and Jordan forms which is not the standard classification of elements of the Möbius group given in [3]. It is worth noting that for $|m| > 1$, $BS(1,m)$ is a linear group. It would seem that this result forms part of the folklore in the scope of Baumslag-Solitar group theory. As a consequence, there is no article in the literature which contains a formal proof of such fact. However, we can get one from *mathoverflow*. From the results displayed in this paper and some restrictions given in [7] and [1] about the residually finiteness of the Baumslag-Solitar groups, we prove that $BS(n,m)$ only admits a faithful representation into $SL(2,\mathbb{C})$ when $n = 1$ and $|m| > 1$. This result overlaps with what was mentioned in the previous paragraph, but we propose a different method that might be of interest for those working in Kleinian and Fuchsian groups. Moreover, if Γ_m denotes the image of $BS(1,m)$, for $|m| > 1$, into $PSL(2,\mathbb{C})$, then we show that Γ_m is an elementary and non-discrete subgroup of Möbius transformations.

This paper is organized as follows. In section 2 we give a short list of preliminaries concerning Möbius transformations. We also recall the definition of $SL(2,\mathbb{C})$-representations, residually finite groups and linear groups. Then, in

Section 3 we show the existence of non-abelian representations of the Baumslag-Solitar groups and provide a classification of them into two families, pseudo-parabolic and pseudo-elliptic representations. We give functions of certain algebraic affine varieties into each of them. Finally, in Section 4 we show that $BS(1, m)$ is a linear group, for $|m| > 1$. Moreover, we prove that its representation image in the Möbius transformations group is an elementary and non discrete subgroup.

2. Notation and preliminary results

In this section we introduce a short list of definitions, notations and some results necessary to understand this article.

2.1. Möbius transformations

The set of Möbius transformations $f(z) = \frac{az+b}{cz+d}$, with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$, acting on $\mathbb{C}_\infty = \mathbb{C} \cap \{\infty\}$, is denoted by $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$. This set has some additional structure; it is a group under composition of transformations. Besides, the homomorphism $\pi : SL(2, \mathbb{C}) \to \mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\pi} \varphi_A : \varphi_A(z) = \frac{az + b}{cz + d},$$

has kernel $\{\pm I\}$, and it provides a natural identification between $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$ and $PSL(2, \mathbb{C})$.

We use $tr(A)$ and A^t to denote the trace and the transpose of a matrix A.

The following definition come from the correspondence between $SL(2, \mathbb{C})$ and the group of Möbius transformations $PSL(2, \mathbb{C})$.

Definition 2.1. [3] Let A $(A \neq I)$ be a matrix in $SL(2, \mathbb{C})$, then A is a *parabolic matrix* if and only if $tr^2(A) = 4$, A is an *elliptic matrix* if and only if $tr^2(A) \in [0, 4)$, A is a *hyperbolic matrix* if and only if $tr^2(A) \in (4, +\infty)$, and A is a *strictly loxodromic matrix* if and only of $tr^2(A) \notin [0, +\infty)$.

The following notation will be useful in order to simplify the proof and certain definitions given in the rest of this paper.

Notation 1. For $\delta \in \mathbb{C}^*$ and $\rho \in \mathbb{C}$,

$$D(\delta) = \begin{pmatrix} \delta & 0 \\ 0 & \delta^{-1} \end{pmatrix} \quad \text{and} \quad D(\rho, \delta) = \begin{pmatrix} \delta & 0 \\ \rho & \delta^{-1} \end{pmatrix}$$

We say that $A \in SL(2, \mathbb{C})$ is a scalar matrix if $A = D(\delta)$, with $\delta^2 = 1$, and $I = D(1)$.

The purpose of the next theorem is to classify the conjugacy classes of the matrices in $SL(2,\mathbb{C})$. This classification involves diagonal matrices and matrices in Jordan form. Another classification using the theory of fixed points of Möbius transformations is given in [3].

Theorem 2.2. *Every non scalar matrix A in $SL(2,\mathbb{C})$ is a conjugate of one of the following matrices: $D(1,1)$, $D(1,-1)$ or $D(\delta)$.*

Moreover, if A is taken in $PSL(2,\mathbb{C})$, then it is conjugate to $D(1,1)$ or $D(\delta)$.

Proof. Let $A \in SL(2,\mathbb{C})$ be a non scalar matrix with characteristic polynomial $ch_A(\lambda) = \lambda^2 - tr(A)\lambda + 1$ and consider the following cases.

Case 1: If $ch_A(\lambda) = (\lambda-\delta)^2$, then, because A is not a scalar matrix, its minimal polynomial must be equal to $ch_A(\lambda)$, hence, from the canonical Jordan form, the matrix A is a conjugate of some matrix of the form $D(1,\delta)$, where $\delta^2 = 1$.

Case 2 If $ch_A(\lambda) = (\lambda-\delta)(\lambda-\delta^{-1})$, then A is a diagonalizable matrix, therefore the matrix A is a conjugate of some matrix of the form $D(\delta)$, where $\delta \neq \delta^{-1}$.

For the last part, note that, in $PSL(2,\mathbb{C})$, $D(1,-1) = D(-1,1)$. Because $D(1,i)D(1,1)D(1,i)^{-1} = D(-1,1)$, then $D(1,-1)$ and $D(1,1)$ are in the same conjugacy class. ☑

Let $A \in SL(2,\mathbb{C})$, we define the norm of A, denoted $\|A\|$, as

$$\|A\| = tr^{1/2}(A\overline{A^t}).$$

The topology on $SL(2,\mathbb{C})$ induced by this norm is denoted by $\Im_{\mathbb{C}}$.

The homomorphism π induces the quotient topology $\Im$ on $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$ with respect to which π is continuous. Besides, $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$ has the topology $\Im^*$ of uniform convergence with respect to the chordal metric on $\mathbb{C}_\infty$. These topologies are the same. (see [3]).

Definition 2.3. A subgroup G of $SL(2,\mathbb{C})$ is *discrete* if and only if the relative topology on G is the discrete topology.

The proof of the following lemma can be found in [3].

Lemma 2.4. *A subgroup G of $SL(2,\mathbb{C})$ is discrete if and only if for each positive k, the set $\{A \in G : \|A\| \leq k\}$ is finite.*

Definition 2.5. A subgroup G of $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$ is said to be *elementary* if and only if any two elements of infinite order have a common fixed point.

A subgroup G of $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$ acts *discontinuously* on a G-invariant disc Δ (or on the half-plane), if for every compact $K \subset \Delta$, $g(K) \cap K = \varnothing$ except for a finite number of $g \in G$.

Definition 2.6. A non-elementary subgroup G of $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$ is a *Kleinian* group if it is discrete. If, moreover, G has a G-invariant disc Δ on which G acts discontinuously, we say that G is a *Fuchsian* group.

The proof of the following theorem can be found in [3].

Theorem 2.7. *Let $f \in \mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$ with $f \neq I$ and not of order two. Let $\theta : \mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty) \to \mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$, where $\theta(g) = gfg^{-1}$. If for some n, $\theta^n(g) = f$, then $\langle f, g\rangle$ is elementary.*

Therefore, any elementary Fuchsian group is either cyclic or it is conjugate to some group $\langle f, g\rangle$, where $g(z) = kz$ $(k > 1)$ and $f(z) = -1/z$.

2.2. $SL(2,\mathbb{C})$-representations of Baumslag-Solitar groups

In this section we will give some background material concerning the representation and character of the Baumslag-Solitar groups.

A representation of $BS(n,m)$ into $SL(2,\mathbb{C})$ is understood as a homomorphism $\rho : BS(n,m) \to SL(2,\mathbb{C})$. The set of all representations is denoted by $R(BS(n,m))$. It is not hard to verify that $R(BS(n,m))$ can be endowed with the structure of an affine algebraic set.

Recall that two *representations* ρ and ρ' are equivalent (conjugate), denoted $\rho \approx \rho'$, if there exists $C \in SL(2,\mathbb{C})$ such that $\rho'(g) = C\rho(g)C^{-1}$ for every $g \in BS(n,m)$.

An immediate consequence of the definition above is, two representations ρ and ρ' of $BS(n,m)$ are equivalent if and only if there exists $C \in SL(2,\mathbb{C})$ such that $\rho'(x) = C\rho(x)C^{-1}$ and $\rho'(y) = C\rho(y)C^{-1}$.

The *character* of a representation ρ is the function $\chi_\rho : BS(n,m) \to \mathbb{C}$ defined by $\chi_\rho(g) = tr(\rho(g))$. Since, equivalent representations have the same character, the map

$$\chi : R(BS(n,m)) \to X(BS(n,m)),$$

where $X(BS(n,m)) = \{\chi_\rho \mid \varphi \in R(BS(n,m))\}$ and $\chi(\rho) = \chi_\rho$ induce a well-defined function $R(BS(n,m))/\approx \to X(BS(n,m))$.

Definition 2.8. Let $\rho \in R(BS(n,m))$. Then ρ is called irreducible if the only subspaces of $\mathbb{C}^2$ which are invariant under $\rho(BS(n,m))$ are $\{0\}$ and $\mathbb{C}^2$. In other case, we say that ρ is reducible.

From the previous definition, a representation $\rho \in R(BS(n,m))$ is reducible if and only if all $\rho(g)$, with $g \in BS(n,m)$, have a common one-dimensional eigenspace. Therefore, a representation $\rho \in R(BS(n,m))$ is reducible if and only if $\rho(x)$ and $\rho(y)$ have a common eigenvector.

The proof of the following theorem can be found in [9].

Theorem 2.9. *Let $\rho \in R(BS(n,m))$, then ρ is reducible if and only if $\chi_\rho(c) = 2$, for each element c of the commutator subgroup of $BS(n,m)$.*

In this case, if Γ denotes the image of ρ, then every element of the commutator subgroup $[\Gamma,\Gamma]$ of Γ is parabolic.

A representation ρ is *abelian* if its image is an abelian subgroup of $SL(2,\mathbb{C})$, and *nonabelian* otherwise. We denote the set of nonabelian representations of $BS(n,m)$ by nab-rep$(BS(n,m))$.

Definition 2.10. Let $\rho \in R(BS(n,m))$. Then ρ is called *parabolic* if $\rho(y)$ is conjugate to $D(1,1)$ or $D(1,-1)$. Besides, if $\rho(y)$ is conjugate to $D(\delta)$, then ρ is *elliptic*, *hyperbolic* or *strictly loxodromic* depending of the subset of $\mathbb{C}$ containing $\delta + \delta^{-1}$.

A representation $\rho : BS(n,m) \to SL(2,\mathbb{C})$ is said to be *faithful* if the kernel of ρ is trivial.

We recall the definition of residually finite groups.

Definition 2.11. A group G is called *residually finite* if for every g and h in G there exist a finite group F and a homomorphism $\varphi : G \to F$ such that $\varphi(g) \neq \varphi(h)$.

The following theorem gives us a complete classification of the Baumslag-Solitar groups in terms of the previous definition. Its proof can be found in [7].

Theorem 2.12. *The group $BS(m,n)$ is residually finite if and only if $|m| = |n|$ or $|m| = 1$ or $|n| = 1$.*

Definition 2.13. A group G is said to be a *linear group* if there exists a faithful representation $\phi : G \to SL(2,\mathbb{C})$

The proof of the next theorem can be found in [6].

Theorem 2.14. *Let R be a field, and let M be a finite set of n by n matrices with elements in R and with non-vanishing determinant. Then the set of matrices in M generate a residually finite group.*

Corollary 2.15. *Linear groups are residually finite.*

Therefore, $BS(n,m)$ does not have faithful representations if $|m| \neq |n|$ and $|m| \neq 1$ and $|n| \neq 1$.

3. Non-abelian representations of $BS(n,m)$

In this section we will construct a collection of algebraic varieties that encode information about the set of equivalence classes of non-abelian representations of $BS(n,m)$.

For an ideal J of $\mathbb{C}[x_1,...,x_n]$, we denote by $\vartheta(J)$ the algebraic variety $\vartheta(J)=\left\{\vec{x}\in\mathbb{C}^n/f(\vec{x})=0 \text{ for every } f\in J\right\}$.

3.1. The case $BS(n,n)$

Let us start this section with the following theorem.

Theorem 3.1. *Let $A\in SL(2,\mathbb{C})$ be a non scalar matrix and $\lambda=tr(A)\neq 0$. Consider the infinite sequence of polynomials in $\mathbb{Z}[x]$, $\{\varphi_k(x)\}_{k=0}^{\infty}$, where $\varphi_0(x)=1$, $\varphi_1(x)=x$ and $\varphi_n(x)=x\varphi_{n-1}(x)-\varphi_{n-2}(x)$, for $n>1$. Then*

$$A^n=\varphi_{n-1}(\lambda)A-\varphi_{n-2}(\lambda)I,$$

for every $n\in\mathbb{N}$.

Proof. From the Cayley-Hamilton theorem $A^2=\lambda A-I$. Therefore

$$A^2=\varphi_1(\lambda)A-\varphi_0(\lambda)I. \tag{1}$$

Now suppose that $A^n=\varphi_{n-1}(\lambda)A-\varphi_{n-2}(\lambda)I$, then $A^{n+1}=\varphi_{n-1}(\lambda)A^2-\varphi_{n-2}(\lambda)A$. From the equation (1),

$$A^{n+1}=\varphi_{n-1}(\lambda)[\lambda A-I]-\varphi_{n-2}(\lambda)A,$$

and so

$$A^{n+1}=(\lambda\varphi_{n-1}(\lambda)-\varphi_{n-2}(\lambda))A-\varphi_{n-1}(\lambda)I.$$

☑

The proof of the following lemma is a direct consequence of the Cayley-Hamilton theorem, so we will omit it.

Lemma 3.2. *Let A in $SL(2,\mathbb{C})$ be a non scalar matrix such that $tr(A)=0$, then $A^{2m}=(-1)^mI$ and $A^{2m+1}=(-1)^mA$.*

Corollary 3.3. *Let A,B in $SL(2,\mathbb{C})$ be non scalar matrices such that $BA^n=A^nB$ and $\lambda=tr(A)$.*

(a) If $\lambda\neq 0$ and $\varphi_{n-1}(\lambda)\neq 0$, then $AB=BA$.

(b) If $\lambda=0$ and $n=2m+1$, then $AB=BA$.

Let A be a non scalar matrix with $\lambda = tr(A) \neq 0$ and $\varphi_{n-1}(\lambda) = 0$. Then $A^n = \pm I$. Therefore A is a diagonalizable matrix, so there exists $W \in SL(2,\mathbb{C})$ and $\delta \in \mathbb{C}^*$ such that $A = WD(\delta)W^{-1}$, with, $\delta^2 \neq 1$, $\delta + \delta^{-1} = \lambda$ and $\delta^n = \pm 1$.

Now, let $B \in SL(2,\mathbb{C})$ be a non-scalar matrix. It is well known that $AB = BA$ implies that $B = WD(\gamma)W^{-1}$, with $\gamma^2 - 1 \neq 0$. We have proven the following theorem.

Theorem 3.4. *Let φ be a representation of $BS(n,n)$, with $\varphi(x) = B$ and $\varphi(y) = A$ non-scalar matrices, and let $\lambda = tr(A) \neq 0$, then:*

(a) If $\varphi_{n-1}(\lambda) = 0$, then A is a diagonalizable matrix. Moreover, $AB = BA$ if and only if A and B are simultaneously diagonalizable.

(b) If $\varphi_{n-1}(\lambda) \neq 0$, then $AB = BA$.

Therefore, if φ is a non abelian representation, then A must be elliptic and $A^n = \pm I$.

The following lemma will give us a easy way to compute the polynomial $\varphi_{n-1}(\lambda)$ for the case of diagonalizable matrices.

Lemma 3.5. *Let $A = WD(\alpha)W^{-1} \in SL(2,\mathbb{C})$ be a non-scalar matrix, with $W \in SL(2,\mathbb{C})$ and $\lambda = \alpha + \alpha^{-1} \neq 0$. Then,*

(a) If n is an even number, then

$$\alpha^{n-1}\varphi_{n-1}(\lambda) = (1 + \alpha^2 + \alpha^4 + ... + \alpha^{n-2})(\alpha^n + 1)$$

and

(b) if n is a odd number, then

$$\alpha^{n-1}\varphi_{n-1}(\lambda) = (1 + \alpha + \alpha^2 + ... + \alpha^{n-1})(\sum_{k=1}^{n} \alpha^{n-k}(-1)^{k-1}).$$

Proof. From Theorem 3.1,

$$A^n = WD(\alpha^n)W^{-1} = W(\varphi_{n-1}(\lambda)D(\alpha) - \varphi_{n-2}I)W^{-1}.$$

Then $D(\alpha^n) = \varphi_{n-1}(\lambda)D(\alpha) - \varphi_{n-2}(\lambda)I$. Therefore $\varphi_{n-1}(\lambda)D(\alpha) - D(\alpha^n) = \varphi_{n-2}(\lambda)I$, and so

$$\varphi_{n-1}(\lambda)\alpha - \alpha^n = \varphi_{n-1}(\lambda)\alpha^{-1} - \alpha^{-n} = \varphi_{n-2}(\lambda).$$

If we multiply both sides by α^n, we obtain $\varphi_{n-1}(\lambda)\alpha^{n+1} - \alpha^{2n} = \varphi_{n-1}(\lambda)\alpha^{n-1} - 1$. This implies that

$$\alpha^{n-1}\varphi_{n-1}(\lambda)(1 - \alpha^2) = (1 - \alpha^n)(1 + \alpha^n).$$

The rest of the proof follows by factorizing polynomials of the form $1 \pm \alpha^n$. ☑

Theorem 3.6. *With the notation above.*

(1) *If n is a even number, then there exists an injective function from the algebraic set $\vartheta(J)$ into nab-rep$(BS(n,n))/\approx$, where J is the ideal of $\mathbb{C}[\alpha,\beta,\gamma,\sigma]$ spanned by*

$$\phi_n(\alpha) = (\alpha^2+1)(1+\alpha^2+\alpha^4+\cdots+\alpha^{n-2})(\alpha^n+1)$$

and $f(\alpha,\beta,\gamma,\sigma) = \beta\gamma\sigma - 1$.

(2) *If n is an odd number, then there exists an injective function from the algebraic set $\vartheta(J)$ into nab-rep$(BS(n,n))/\approx$, where J is the ideal of $\mathbb{C}[\alpha,\beta,\gamma,\sigma]$ spanned by*

$$\phi_n(\alpha) = (1+\alpha+\alpha^2+\cdots+\alpha^{n-1})(\sum_{k=1}^{n}\alpha^{n-k}(-1)^{k-1})$$

and $g(\alpha,\beta,\gamma,\sigma) = \beta\gamma\sigma(\alpha^2+1) - 1$.

Proof. (1) Let $(\alpha,\beta,\gamma,\sigma)\in\vartheta(J)$, and let $H:\langle x,y/\rangle \longrightarrow SL(2,\mathbb{C})$ the canonical homomorphism such that $H(x)=D(\gamma,\beta)$ and $H(y)=D(\alpha)$. Suppose that $n=2t$, $t>0$.

Now, consider the following two cases for the trace, λ, of $D(\alpha)$.

Case 1: If $\lambda=0$, then $D(\alpha)^n=\pm I$, and

Case 2: if $\lambda\neq 0$, then $\varphi_{n-1}(\lambda)=0$. Therefore $D(\alpha)^n=\pm I$.

From the previous cases, and the fact that $H(x)$ and $H(y)$ are not simultaneously diagonalizable, H extends to a nonabelian representation $\widetilde{H}$. So, we can define the function

$$\varsigma:\vartheta(J)\longrightarrow \textit{nab-rep}(BS(n,n))/\approx,$$

where $\varsigma(\alpha,\beta,\gamma,\sigma)=[\widetilde{H}]$ is the equivalence class of the representation $\widetilde{H}$.

Let $\overrightarrow{x}=(x_1,x_2,x_3,x_4)$ and $\overrightarrow{y}=(y_1,y_2,y_3,y_4)$ in $\vartheta(J)$ be such that $\varsigma(\overrightarrow{x})=\varsigma(\overrightarrow{y})$, then there exists $C=(C_{ij})\in SL(2,\mathbb{C})$ with $CD(x_1)C^{-1}=D(y_1)$ and $CD(x_3,x_2)C^{-1}=D(y_3,y_2)$.

By expanding the matrix equality $CD(x_1)C^{-1}=D(y_1)$ we obtain two possibilities, either $x_1=y_1$ or $x_1=y_1^{-1}$. If we suppose that $x_1\neq y_1$ then $C_{11}=C_{22}=0$ and $C_{21}=-C_{12}^{-1}$. If we replace them in the equation $CD(x_3,x_2)C^{-1}=D(y_3,y_2)$, we get that $C_{12}=0$, but this is a contradiction, and so $x_1=y_1$, and therefore $C=I$. Hence, ς is an injective function.

(2) Let $(\alpha,\beta,\gamma,\sigma)\in\vartheta(J)$ and let, again, $H:\langle x,y/\rangle\to SL(2,\mathbb{C})$ be the canonical homomorphism given by $H(y)=D(\alpha)$ and $H(x)=D(\gamma,\beta)$.

Because $g(\alpha,\beta,\gamma,\sigma) = 0$, then $\alpha^2 + 1 \neq 0$, therefore $\lambda = \alpha + \alpha^{-1} \neq 0$ and, from Lemma 3.5, $\varphi_{n-1}(\lambda) = 0$. Therefore H extends to a non-abelian representation $\widetilde{H}$. So, we can define the function

$$\varsigma : \vartheta(J) \longrightarrow \text{nab-rep}(BS(n,n))/\approx,$$

where $\varsigma(\alpha,\beta,\gamma,\sigma) = [\widetilde{H}]$ is the equivalence class of the representation $\widetilde{H}$.

The proof that ς is an injective function is quite similar to what we did in the proof of (1). ☑

We conclude this section with the fact that for every $(\alpha,\beta,\gamma,\sigma) \in \vartheta(J)$ all the representations in the equivalence class $\varsigma(\alpha,\beta,\gamma,\sigma)$ are reducible.

Theorem 3.7. *Every representation in $\varsigma(\alpha,\beta,\gamma,\sigma)$ is reducible.*

Proof. Let $\widetilde{H}$ be the representative element of $\varsigma(\alpha,\beta,\gamma,\sigma)$, where $\widetilde{H}(x) = D(\gamma,\beta)$ and $\widetilde{H}(y) = D(\alpha)$. The eigenspace $E_{\alpha^{-1}} = \{(0,z) \mid z \in \mathbb{C}\}$ corresponds to the eigenvalue α^{-1}, of the matrix $\widetilde{H}(y)$, and has the property that it is an eigenspace of the matrix $\widetilde{H}(x)$. Therefore $E_{\alpha^{-1}}$ is an $\widetilde{H}(BS(n,n))$-invariant subspace of $\mathbb{C}^2$. ☑

3.2. The case $BS(n,m)$, with $n \neq m$

We start this section with the following theorem.

Theorem 3.8. *There are no non-abelian representations $\widetilde{H}$ of $BS(n,m)$ such that $\widetilde{H}(x)$ and $\widetilde{H}(y)$ are both parabolic matrices.*

Proof. Suppose that there exists a non-abelian representations $\widetilde{H}$ of $BS(n,m)$ in which $\widetilde{H}(x)$ and $\widetilde{H}(y)$ are both parabolic matrices. Then $\widetilde{H}$ is a conjugate of the non-abelian representations $\widetilde{K} : BS(n,m) \to SL(2,\mathbb{C})$ such that $K(y) = D(1,1)$ and $K(x) = (\omega_{ij})$, with $(\omega_{11} + \omega_{22})^2 = 4$.

Since $D(1,1)^n = D(n,1)$ and $D(1,1)^m = D(m,1)$, then $(\omega_{ij})D(1,1)^n(\omega_{ij})^{-1} = D(1,1)^m$ if and only if

$$(\omega_{ij})D(n,1) = D(m,1)(\omega_{ij}). \tag{2}$$

Then we prove that (2) is true if and only if $\omega_{12} = 0$ and $n\omega_{22} = m\omega_{11}$. Since $\omega_{11}\omega_{22} - \omega_{21}\omega_{12} = 1$, then $\omega_{11}\omega_{22} = 1$, and so $\omega_{22}^2 = \frac{m}{n}$. Hence, $K(x) = D(\omega_{21}, \omega_{22}^{-1})$, where $\omega_{22}^2 = \frac{m}{n}$. But $\omega_{22} + \omega_{22}^{-1} = \pm 2$ and $\omega_{22}^2 = 1$, that is a contradiction. ☑

Corollary 3.9. *Let $f(z,w) = nz^2 - m$ and denote by L the ideal of $\mathbb{C}[z,w]$ spanned by $f(z,w)$. Then there exists a function from the algebraic variety $\vartheta(L)$ into nab-rep$(BS(n,m))/\approx$.*

Proof. Let $(\alpha, \beta) \in \vartheta(L)$ and consider the map

$$\Psi : \vartheta(L) \to \text{nab-rep}(BS(n,m))/\approx, \quad \Psi((\alpha,\beta)) = [\widetilde{K}],$$

where $\widetilde{K} : BS(n,m) \to SL(2,\mathbb{C})$ is the representation given by $\widetilde{K}(x) = D(\beta, \alpha^{-1})$ and $\widetilde{K}(y) = D(1,1)$. It is not hard to prove that Ψ is a well-defined function. ☑

The function Ψ is not injective because $\Psi(\alpha,\beta) = \Psi(\alpha, \alpha^{-1} + \beta - \alpha)$ and $\beta \neq \alpha^{-1} + \beta - \alpha$.

Theorem 3.10. *With the above notation. The representation $\widetilde{K}$ is reducible.*

Proof. Consider the eigenspace $E = \{(0,z) \mid z \in \mathbb{C}\}$. Then, E is a $\widetilde{K}(BS(n,m))$-invariant one dimensional subspace of $\mathbb{C}^2$. ☑

Theorem 3.11. *If $\widetilde{H} \in$ nab-rep$(BS(n,m))$ and $\widetilde{H}(y)$ is not a parabolic matrix, then $\widetilde{H}(y)$ has finite order.*

Proof. Suppose that $\widetilde{H} \in$ nab-rep$(BS(n,m))$ and $\widetilde{H}(y)$ is not a parabolic matrix. Then $\widetilde{H}$ must be equivalent to a representation $\widetilde{K}$, such that $\widetilde{K}(x) = (\sigma_{ij})$ and $\widetilde{K}(y) = D(\delta)$, where $\delta^2 \neq 1$.

The matrix equality $(\sigma_{ij})D(\delta)^n = D(\delta)^m(\sigma_{ij})$ is true if and only if

$$\sigma_{11}(\delta^{n-m} - 1) = \sigma_{12}(\delta^{n+m} - 1) = \sigma_{21}(\delta^{n+m} - 1) = \sigma_{22}(\delta^{n-m} - 1) = 0.$$

Now, $(\sigma_{ij}) \in SL(2,\mathbb{C})$, then $(\delta^n - \delta^m)(\delta^{n+m} - 1) = 0$, therefore $D(\delta)$ is an elliptic matrix of finite order. Moreover, since $\widetilde{H}$ is a non abelian representation, then we have to add one of the following inequalities $\sigma_{12} \neq 0$ or $\sigma_{21} \neq 0$ ☑

Theorem 3.12. *We suppose that $m > n$. Let $g_1(\alpha,\beta,\gamma,\sigma) = \Phi_{n+m}(\alpha)\,\Phi_{m-n}(\alpha)$ and $g_2(\alpha,\beta,\gamma,\sigma) = \beta\gamma\sigma(\alpha+1) - 1$, and let I the ideal of $\mathbb{C}[\alpha,\beta,\gamma,\sigma]$ spanned by $\{g_1, g_2\}$. Then, there exist a injective function from the algebraic variety $\vartheta(I)$ into nab-rep$(BS(n,m))$.*

Proof. Consider the map

$$\Xi : \vartheta(I) \to \text{nab-rep}(BS(n,m)), \quad \Xi((\alpha,\beta,\gamma,\sigma) = \widetilde{K},$$

where $\widetilde{K} : BS(n,m) \to SL(2,\mathbb{C})$ is the representation given by $\widetilde{K}(x) = D(\gamma,\beta)$ and $\widetilde{K}(y) = D(\alpha)$.

Due to the fact that $\alpha^2 \neq 1$, and from Theorem 3.11, we get that $\widetilde{K}$ is a non-abelian representation, therefore we have shown that Ξ is well defined.

The proof that Ξ is an injective function, is straightforward. ☑

Theorem 3.13. *The representation $\widetilde{K} : BS(n,m) \to SL(2,\mathbb{C})$, given by $\widetilde{K}(x) = D(\gamma,\beta)$ and $\widetilde{K}(y) = D(\alpha)$, is reducible.*

4. On non-abelian faithful representations of $BS(n,m)$

From the Theorem 2.12, the group $BS(n,m)$ could have a non-abelian faithful representation in $SL(2,\mathbb{C})$ only for the cases in which $n=m$ or $n=1$ or $m=1$. But, we know that $BS(n,n)$ does not have non-abelian faithful representations into $SL(2,\mathbb{C})$. (see Theorem 3.6). Therefore there only remains the case $n=1$, because for $m=1$ we have that $BS(1,m)\cong BS(m,1)$.

Before the proof that $BS(1,m)$ is a linear group, consider the following theorem.

Theorem 4.1. *Each element $w\in BS(1,m)$ is uniquely represented by a word of the form $x^{-p}y^kx^q$, where p,k,q are integers and $p,q\geq 0$.*

Proof. From $xyx^{-1}=y^m$ we obtain the infinite family of relators $\{x^tyx^{-t}=y^{m^t}/t\in\mathbb{N}\}$. So, we have the following four kinds of equations:

(a) $x^ty=y^{m^t}x^t$,

(b) $yx^{-t}=x^{-t}y^{m^t}$,

(c) $y^{-1}x^{-t}=x^{-t}y^{-m^t}$ and

(d) $x^ty^{-1}=y^{-m^t}x^t$.

Without loss of generality, we may suppose that the words in $BS(1,m)$ have the form $w=x^{t_1}y^{k_1}...x^{t_n}y^{k_n}$ where $t_i,k_j\in\mathbb{Z}$. We will complete the proof by induction on the length n.

When $n=1$, then $w=x^{t_1}y^{k_1}$. So, if $t_1>0$, then from (a) and (d), $w=x^{-0}y^{k_1m^{t_1}}x^{t_1}$.

Now assume that $\widetilde{w}=x^{t_1}y^{k_1}...x^{t_{n-1}}y^{k_{n-1}}$. then $w=\widetilde{w}x^{t_n}y^{k_n}$. By the induction hypothesis $\widetilde{w}=x^{-a}y^bx^c$, where $a,c\geq 0$, so $w=x^{-a}y^bx^{c+t_n}y^{k_n}$. If $c+t_n\geq 0$, from (a) and (d), $w=x^{-a}y^{b+k_nm^{c+t_n}}x^{c+t_n}$, but if $c+t_n\leq 0$, from (b) and (c), $w=x^{-a}y^bx^{c+t_n}y^{k_n}=x^{-a+c+t_n}y^{bm^{c+t_n}+k_n}$. ☑

Corollary 4.2. *Let $m\in\mathbb{Z}$ such that $|m|>1$, then $BS(1,m)$ is a linear group, and it is residually finite.*

Proof. We give the proof for the case $m>1$. From Theorem 3.8, there exists a representation $\widetilde{K}:BS(1,m)\to SL(2,\mathbb{C})$, such that $\widetilde{K}(x)=D(m^{-1/2})$ and $\widetilde{K}(y)=D(1,1)$.

Let w be a word in $BS(1,m)$ such that $\widetilde{K}(w)=I$. Because w is uniquely represented by a word of the form $x^{-p}y^kx^q$, where $p,k,q\in\mathbb{Z}$ and $p,q\geq 0$, then

$$D(m^{p/2})D(k,1)D(m^{-q/2})=D\left(\frac{k}{m^{p/2+q/2}},m^{p/2-q/2}\right),$$

hence, $\widetilde{K}(w) = I$ if and only if $p = q$ and $k = 0$. Therefore $w = 1$ and so $\widetilde{K}$ is injective. ☑

If $m = -1$, then we have that $D^2(\delta, i) = D(-1)$, therefore $D(\delta, i)$ is a matrix of order 4, as a consequence, neither of the representations given in the proof of Theorem 3.8 are faithful.

Let $\varphi_A = \pi(A)$ and $\varphi_B = \pi(B)$, where $A = D(1, 1)$ and $B = D(m^{-1/2})$. Let Γ_m, with $m > 1$, be the subgroup of $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$ generated by φ_A and φ_B.

Proposition 4.3. The subgroup Γ_m is an elementary subgroup of $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$. Moreover, Γ_m is not a Fuchsian group.

Proof. Since $\theta(\varphi_B) = \varphi_B \varphi_A \varphi_B^{-1} = \varphi_A^m$, then

$$\theta^2(\varphi_B) = (\varphi_B \varphi_A \varphi_B^{-1})\varphi_A(\varphi_B \varphi_A \varphi_B^{-1})^{-1} = \varphi_A,$$

therefore, from Theorem 2.7 we have that Γ_m is an elementary subgroup of $\mathcal{M}_{\mathbb{C}}(\mathbb{C}_\infty)$. Besides, Γ_m is not abelian, and hence it is not cyclic. Since A is not a diagonalizable matrix, Γ_m is not conjugate to some group of the form $\langle f, g\rangle$, where $g(z) = kz$ $(k > 1)$ and $f(z) = -1/z$, then Γ_m is not a Fuchsian group. ☑

Theorem 4.4. *The subgroup Γ_m is not discrete.*

Proof. Let $\varphi \in \langle \varphi_A, \varphi_B\rangle$, then there exists p, q, k in $\mathbb{Z}$, with $p, q \geq 0$ such that $\varphi = \varphi_B^{-p}\varphi_A^k\varphi_B^q$. Due to the fact that

$$D(m^{p/2})D(k, 1)D(m^{-q/2}) = D\left(\frac{k}{m^{p/2+q/2}}, m^{p/2-q/2}\right),$$

then

$$\|\varphi\| = \sqrt{m^{p-q} + m^{-p+q} + k^2 m^{-(p+q)}}.$$

Let $t \geq 0$ such that $m^{p-q} + m^{-p+q} + k^2 m^{-(p+q)} \leq t$. Then

$$m^{2p} + m^{2q} + k^2 \leq t(m^{p+q}).$$

If we take $t = 3$ and $p = q$, then the previous inequality becomes $2m^{2p} + k^2 \leq 3m^{2p}$, and so $k^2 \leq m^{2p}$. It is not hard to prove that there are infinitely many pairs $(k, p) \in \mathbb{Z}^2$ such that $k^2 \leq m^{2p}$. ☑

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