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New Ostrowski's inequalties

Nuevas desigualdades de Ostrwoski

Badreddine Meftah

[1]University of 8 May 1945 Guelma, Guelma, Algeria

Abstract. Some new Ostrowski's inequalities for n-times differentiable mappings which are φ-convex are established.

Key words and phrases. Ostrowski inequality, Hölder inequality, power mean inequality, φ-convex functions.

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Resumen. Se establecen algunas nuevas desigualdades de Ostrowski para asignaciones n-diferenciables que son φ-convexas.

Palabras y frases clave. Desigualdad de Ostrowski, desigualdad de Hölder, desigualdad media de poder, funciones φ-convexas.

1. Introduction

In 1938, A.M. Ostrowski proved an interesting integral inequality, estimating the absolute value of the derivative of a differentiable function by its integral mean as follows

Theorem 1.1. [2] *Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping in the interior I° of I, and $a, b \in I^\circ$, with $a < b$.*

If $|f'| \leq M$ for all $x \in [a, b]$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq M(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right], \quad \forall x \in [a, b] \tag{1}$$

This is well-known as Ostrowski's inequality. In recent years, a number of authors have written about generalizations, extensions and variants of inequality (1).

In [1], Cerone et al. proved the following identity

Lemma 1.2. [1] *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then for all $x \in [a, b]$ we havethe identity*

$$\int_a^b f(t)dt = \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) + (-1)^n \int_a^b K_n(x,t) f^{(n)}(t)dt,$$

where the kernel $K_n : [a, b]^2 \to \mathbb{R}$ is given by

$$K_n(x,t) = \begin{cases} \frac{(t-a)^n}{n!} & \text{if } t \in [a, x] \\ \frac{(t-b)^n}{n!} & \text{if } t \in (x, b] \end{cases}, x \in [a, b]$$

and n is natural number, $n \geq 1$.

We also recall some definitions

Definition 1.3. [3] a function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.4. [2] A function $f : I \to \mathbb{R}$ is said to be is said to be φ-convex, if the following inequality

$$f(tx + (1-t)y) \leq f(y) + t\varphi(f(y), f(x)) \tag{2}$$

holds for all $x, y \in I$ and $t \in [0, 1]$, where I is an interval of $\mathbb{R}$ and $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a bifunction.

***Remark* 1.5.** Obviously if we choose $\varphi(f(y), f(x)) = f(x) - f(y)$, Definition 1.4 recaptures Definition 1.3.

In this paper we establish some new Ostrwoski's inequalities for n-times differentiable mappings which are φ-convex.

2. Main results

In what follows, we assume that $n \in \mathbb{N}$, and $I \subset \mathbb{R}$ be an interval, where $[a, b] \subset I$, and $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a bifunction

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be n-times differentiable on $[a, b]$ such that $f^{(n)} \in L([a, b])$. If $\left|f^{(n)}\right|$ is φ-convex, then the following inequality*

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \\
&\le \frac{(x-a)^{n+1}}{n!}\left(\frac{1}{n+1}\left|f^{(n)}(a)\right| + \frac{1}{n+2}\varphi\left(\left|f^{(n)}(a)\right|, \left|f^{(n)}(x)\right|\right)\right) \\
&+ \frac{(b-x)^{n+1}}{n!}\left(\frac{1}{n+1}\left|f^{(n)}(x)\right| + \frac{1}{(n+1)(n+2)}\varphi\left(\left|f^{(n)}(x)\right|, \left|f^{(n)}(b)\right|\right)\right)
\end{aligned} \tag{3}$$

holds for all $x \in [a, b]$.

Proof. From Lemma 1.2, and properties of modulus, we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right| \\
\le &\int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right| du + \int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right| du \\
= &\frac{(x-a)^{n+1}}{n!}\int_0^1 t^n \left|f^{(n)}((1-t)a + tx)\right| dt \\
&+ \frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x + tb)\right| dt.
\end{aligned} \tag{4}$$

Since $\left|f^{(n)}\right|$ is φ-convex, (4) gives

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq\frac{(x-a)^{n+1}}{n!}\int_0^1 t^n\left(\left|f^{(n)}(a)\right|+t\varphi\left(\left|f^{(n)}(a)\right|,\left|f^{(n)}(x)\right|\right)\right)dt$$

$$+\frac{(b-x)^{n+1}}{n!}\int_0^1(1-t)^n\left(\left|f^{(n)}(x)\right|+t\varphi\left(\left|f^{(n)}(x)\right|,\left|f^{(n)}(b)\right|\right)\right)dt$$

$$=\tfrac{(x-a)^{n+1}}{n!}\left(\tfrac{1}{n+1}\left|f^{(n)}(a)\right|+\tfrac{1}{n+2}\varphi\left(\left|f^{(n)}(a)\right|,\left|f^{(n)}(x)\right|\right)\right)$$

$$+\tfrac{(b-x)^{n+1}}{n!}\left(\tfrac{1}{n+1}\left|f^{(n)}(x)\right|+\tfrac{1}{(n+1)(n+2)}\varphi\left(\left|f^{(n)}(x)\right|,\left|f^{(n)}(b)\right|\right)\right),$$

which is the desired result. The proof is completed. ☑

Corollary 2.2. *Let $f: I\to\mathbb{R}$ be n-times differentiable on $[a,b]$ such that $f^{(n)}\in L([a,b])$. If $\left|f^{(n)}\right|$ is convex, we have the following estimate*

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq\left(\frac{(x-a)^{n+1}}{(n+2)!}\left|f^{(n)}(a)\right|+\frac{(b-x)^{n+1}}{(n+2)!}\left|f^{(n)}(b)\right|\right)$$

$$+\ (n+1)\left(\frac{(x-a)^{n+1}}{(n+2)!}+\frac{(b-x)^{n+1}}{(n+2)!}\right)\left|f^{(n)}(x)\right|.$$

Proof. Taking $\varphi(u,v)=v-u$ in Theorem 2.1. ☑

Theorem 2.3. *Let $f: I\to\mathbb{R}$ be n-times differentiable on $[a,b]$ such that $f^{(n)}\in L([a,b])$, and let $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\left|f^{(n)}\right|^q$ is φ-convex, then the following inequality*

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$

$$\leq\frac{(x-a)^{n+1}}{2^{\frac{1}{q}}(np+1)^{\frac{1}{p}}n!}\left(2\left|f^{(n)}(a)\right|^q+\varphi\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(x)\right|^q\right)\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{2^{\frac{1}{q}}(np+1)^{\frac{1}{p}}n!}\left(2\left|f^{(n)}(x)\right|^q+\varphi\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right)^{\frac{1}{q}}\qquad(5)$$

holds for all $x \in [a, b]$.

Proof. From Lemma 1.2, properties of modulus, and Hölder's inequality, we have

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
&\leq \int_a^x \frac{(u-a)^n}{n!} \left| f^{(n)}(u) \right| du + \int_x^b \frac{(b-u)^n}{n!} \left| f^{(n)}(u) \right| du \\
&= \frac{(x-a)^{n+1}}{n!} \int_0^1 t^n \left| f^{(n)}((1-t)a + tx) \right| dt \\
&\quad + \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}((1-t)x + tb) \right| dt \\
&\leq \frac{(x-a)^{n+1}}{n!} \left(\int_0^1 t^{np} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f^{(n)}((1-t)a + tx) \right|^q dt \right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 (1-t)^{np} dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f^{(n)}((1-t)x + tb) \right|^q dt \right)^{\frac{1}{q}} \\
&= \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!} \left(\int_0^1 \left| f^{(n)}((1-t)a + tx) \right|^q dt \right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}} n!} \left(\int_0^1 \left| f^{(n)}((1-t)x + tb) \right|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left| f^{(n)} \right|^q$ is φ-convex, we deduce

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{k=n} \left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} \right] f^{(k)}(x) \right| \\
&\leq \frac{(x-a)^{n+1}}{(np+1)^{\frac{1}{p}} n!} \left(\int_0^1 \left(\left| f^{(n)}(a) \right|^q + t\varphi\left(\left| f^{(n)}(a) \right|^q, \left| f^{(n)}(x) \right|^q \right) \right) dt \right)^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{(b-x)^{n+1}}{(np+1)^{\frac{1}{p}}n!}\left(\int_0^1\left(\left|f^{(n)}(x)\right|^q+t\varphi\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right)dt\right)^{\frac{1}{q}}$$

$$=\frac{(x-a)^{n+1}}{2^{\frac{1}{q}}(np+1)^{\frac{1}{p}}n!}\left(2\left|f^{(n)}(a)\right|^q+\varphi\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(x)\right|^q\right)\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{2^{\frac{1}{q}}(np+1)^{\frac{1}{p}}n!}\left(2\left|f^{(n)}(x)\right|^q+\varphi\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right)^{\frac{1}{q}}.$$

Thus the proof is completed. ☑

Corollary 2.4. *Let $f: I \to \mathbb{R}$ be n-times differentiable on $[a,b]$ such that $f^{(n)} \in L([a,b])$, and let $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\left|f^{(n)}\right|^q$ is convex, then the following inequality holds*

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$
$$\leq\frac{(x-a)^{n+1}}{2^{\frac{1}{q}}(np+1)^{\frac{1}{p}}n!}\left(\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{2^{\frac{1}{q}}(np+1)^{\frac{1}{p}}n!}\left(\left|f^{(n)}(x)\right|^q+\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}. \tag{6}$$

Proof. Taking $\varphi(u,v)=v-u$ in Theorem 2.3. ☑

Corollary 2.5. *Under the same assumptions of Corollary 2.4, we have*

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$
$$\leq\frac{(x-a)^{n+1}}{2^{\frac{1}{q}}(np+1)^{\frac{1}{p}}n!}\left(\left|f^{(n)}(a)\right|+\left|f^{(n)}(x)\right|\right)$$
$$+\frac{(b-x)^{n+1}}{2^{\frac{1}{q}}(np+1)^{\frac{1}{p}}n!}\left(\left|f^{(n)}(x)\right|+\left|f^{(n)}(b)\right|\right).$$

Proof. Taking $\varphi(u,v)=v-u$ in Theorem 2.3, we obtain (6). Then using the following algebraic inequality for all $a,b \geq 0$, and $0 \leq \alpha \leq 1$ we have $(a+b)^\alpha \leq a^\alpha+b^\alpha$, we get the desired result. ☑

Theorem 2.6. *Let $f : I \to \mathbb{R}$ be n-times differentiable on $[a, b]$ such that $f^{(n)} \in L([a, b])$ and let $q > 1$. If $\left|f^{(n)}\right|^q$ is φ-convex, then the following inequality*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+2)!}\left(\frac{1}{n+1}\left|f^{(n)}(a)\right|^q + \frac{1}{n+2}\varphi\left(\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(x)\right|^q\right)\right)^{\frac{1}{q}}$$
$$+ \frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+2)!}\left(\frac{1}{n+1}\left|f^{(n)}(x)\right|^q + \frac{\varphi\left(\left|f^{(n)}(x)\right|^q, \left|f^{(n)}(b)\right|^q\right)}{(n+1)(n+2)}\right)^{\frac{1}{q}}$$

holds for all $x \in [a, b]$.

Proof. From Lemma 1.2, properties of modulus, and power mean inequality, we have

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right] f^{(k)}(x)\right|$$
$$\leq \int_a^x \frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right| du + \int_x^b \frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right| du$$
$$= \frac{(x-a)^{n+1}}{n!}\int_0^1 t^n \left|f^{(n)}((1-t)a + tx)\right| dt$$
$$+ \frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x + tb)\right| dt$$
$$\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^n \left|f^{(n)}((1-t)a + tx)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x + tb)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!}\left(\int_0^1 t^n \left|f^{(n)}((1-t)a + tx)\right|^q dt\right)^{\frac{1}{q}}$$
$$+ \frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!}\left(\int_0^1 (1-t)^n \left|f^{(n)}((1-t)x + tb)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n)}\right|^q$ is φ-convex, we deduce

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq & \frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+1)!} \\
&\times\left(\left|f^{(n)}(a)\right|^q\int_0^1 t^n dt+\varphi\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(x)\right|^q\right)\int_0^1 t^{n+1}dt\right)^{\frac{1}{q}} \\
&+\frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+1)!} \\
&\times\left(\left|f^{(n)}(x)\right|^q\int_0^1(1-t)^n dt+\varphi\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(b)\right|^q\right)\int_0^1 t(1-t)^n dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$\begin{aligned}
=&\frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+2)!}\left(\frac{1}{n+1}\left|f^{(n)}(a)\right|^q+\frac{1}{n+2}\varphi\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(x)\right|^q\right)\right)^{\frac{1}{q}} \\
&+\frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+2)!}\left(\frac{1}{n+1}\left|f^{(n)}(x)\right|^q+\frac{\varphi\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(b)\right|^q\right)}{(n+1)(n+2)}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof is completed. ☑

Corollary 2.7. *Let $f: I\to\mathbb{R}$ be n-times differentiable on $[a,b]$ such that $f^{(n)}\in L([a,b])$ and let $q>1$. If $\left|f^{(n)}\right|^q$ is convex, then the following inequality holds*

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right| \\
\leq&\frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+2)^{\frac{1}{q}}(n+2)!}\left(\frac{1}{(n+1)}\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(x)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+2)^{\frac{1}{q}}(n+2)!}\left(\left|f^{(n)}(x)\right|^q+\frac{1}{n+1}\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.8. *Let $f: I\to\mathbb{R}$ be n-times differentiable on $[a,b]$ such that $f^{(n)}\in L([a,b])$ and let $q>1$. If $\left|f^{(n)}\right|^q$ is convex, then the following inequality* *holds*

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$
$$\leq\frac{(n+1)^{\frac{1}{q}}(x-a)^{n+1}}{(n+2)^{\frac{1}{q}}(n+2)!}\left(\frac{1}{(n+1)^{\frac{1}{q}}}\left|f^{(n)}(a)\right|+\left|f^{(n)}(x)\right|\right)$$
$$+\frac{(n+1)^{\frac{1}{q}}(b-x)^{n+1}}{(n+2)^{\frac{1}{q}}(n+2)!}\left(\left|f^{(n)}(x)\right|+\frac{1}{(n+1)^{\frac{1}{q}}}\left|f^{(n)}(b)\right|\right).$$

Theorem 2.9. *Suppose that all the assumptions of Theorem 2.6 are satisfied, then the following inequality*

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$
$$\leq\frac{(x-a)^{n+1}}{n!}\left(\frac{\left|f^{(n)}(a)\right|^q}{qn+1}+\frac{\varphi\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(x)\right|^q\right)}{qn+2}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{n!}\left(\frac{\left|f^{(n)}(x)\right|^q}{qn+1}+\frac{\varphi\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(b)\right|^q\right)}{(qn+1)(qn+2)}\right)^{\frac{1}{q}}$$

holds for all $x\in[a,b]$.

Proof. From Lemma 1.2, properties of modulus, and power mean inequality, we have

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$
$$\leq\int_a^x\frac{(u-a)^n}{n!}\left|f^{(n)}(u)\right|du+\int_x^b\frac{(b-u)^n}{n!}\left|f^{(n)}(u)\right|du$$
$$=\frac{(x-a)^{n+1}}{n!}\int_0^1 t^n\left|f^{(n)}((1-t)a+tx)\right|dt$$

$$+\frac{(b-x)^{n+1}}{n!}\int_0^1 (1-t)^n \left|f^{(n)}((1-t)\,x+tb)\right| dt$$

$$\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{qn}\left|f^{(n)}((1-t)\,a+tx)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^{qn}\left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}$$

$$=\frac{(x-a)^{n+1}}{n!}\left(\int_0^1 t^{qn}\left|f^{(n)}((1-t)\,a+tx)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1 (1-t)^{qn}\left|f^{(n)}((1-t)\,x+tb)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n)}\right|^q$ is φ-convex, we deduce

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k\,(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}\,(x)\right|$$

$$\leq \frac{(x-a)^{n+1}}{n!}\left(\left|f^{(n)}(a)\right|^q\int_0^1 t^{qn}dt+\varphi\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}\,(x)\right|^q\right)\int_0^1 t^{qn+1}\,dt\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!}\left(\left|f^{(n)}(x)\right|^q\int_0^1 (1-t)^{qn}+\varphi\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}(b)\right|^q\right)\int_0^1 t(1-t)^{qn}dt\right)^{\frac{1}{q}}$$

$$=\frac{(x-a)^{n+1}}{n!}\left(\frac{\left|f^{(n)}(a)\right|^q}{qn+1}+\frac{\varphi\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}\,(x)\right|^q\right)}{qn+2}\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{n+1}}{n!}\left(\frac{\left|f^{(n)}(x)\right|^q}{qn+1}+\frac{\varphi\left(\left|f^{(n)}(x)\right|^q,\left|f^{(n)}\,(b)\right|^q\right)}{(qn+1)\,(qn+2)}\right)^{\frac{1}{q}},$$

which is the desired result. ☑

Corollary 2.10. *Let $f:I\to\mathbb{R}$ be n-times differentiable on $[a,b]$ such that $f^{(n)}\in L\,([a,b])$ and let $q>1$. If $\left|f^{(n)}\right|^q$ is convex, then the following inequality* *holds*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$
$$\leq \frac{(x-a)^{n+1}}{n!}\left(\frac{\left|f^{(n)}(a)\right|^q}{(qn+1)(qn+2)}+\frac{\left|f^{(n)}(x)\right|^q}{qn+2}\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{n!}\left(\frac{\left|f^{(n)}(x)\right|^q}{qn+2}+\frac{\left|f^{(n)}(b)\right|^q}{(qn+1)(qn+2)}\right)^{\frac{1}{q}}.$$

Corollary 2.11. *Let $f: I \to \mathbb{R}$ be n-times differentiable on $[a,b]$ such that $f^{(n)} \in L([a,b])$ and let $q>1$. If $\left|f^{(n)}\right|^q$ is convex, then the following inequality holds*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x)\right|$$
$$\leq \frac{(x-a)^{n+1}}{n!(qn+2)^{\frac{1}{q}}}\left(\frac{\left|f^{(n)}(a)\right|}{(qn+1)^{\frac{1}{q}}}+\left|f^{(n)}(x)\right|\right)$$
$$+\frac{(b-x)^{n+1}}{n!(qn+2)^{\frac{1}{q}}}\left(\left|f^{(n)}(x)\right|+\frac{\left|f^{(n)}(b)\right|}{(qn+1)^{\frac{1}{q}}}\right).$$

3. Applications for some particular mappings

In this section, we give some applications for the special case where the function $\varphi(f(y), f(x)) = f(x) - f(y)$

a) Consider $g:(0,\infty)\to\mathbb{R}$, $g(t)=t^n$, $n\in\mathbb{N}$ with $n\geq 2$. Then $g^{(n)}(t)=n!$ and $g^{(k)}(t)=\frac{n!}{(n-k)!}t^{n-k}$ for $k\leq n$

Using Corollary 2.2, we get

$$\left|\frac{b^{n+1}-a^{n+1}}{n+1}-\sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]\frac{n!}{(n-k)!}x^{n-k}\right|$$
$$\leq \frac{1}{(n+1)}\left((x-a)^{n+1}+(b-x)^{n+1}\right).$$

Moreover, if we choose $x=\frac{a+b}{2}$, we obtain

$$\left|\frac{b^{n+1}-a^{n+1}}{n+1}-\frac{(b-a)(a+b)^n}{2^{n+1}}\sum_{k=0}^{k=n}\left[\frac{1+(-1)^k}{(k+1)!}\right]\left(\frac{b-a}{a+b}\right)^k\frac{n!}{(n-k)!}\right|$$
$$\leq \frac{2}{(n+1)}\left(\frac{b-a}{2}\right)^{n+1}.$$

Particulary, if we choose $a = 0$, we obtain

$$\left|\frac{1}{n+1} - \frac{1}{2^{n+1}}\sum_{k=0}^{k=n}\left[\frac{1+(-1)^k}{(k+1)!}\right]\frac{n!}{(n-k)!}\right| \leq \frac{1}{(n+1)\,2^n}.$$

b) Consider $g : \mathbb{R} \to \mathbb{R}$, $g(t) = e^t$ with $n \in \mathbb{N}$. Then $g^{(n)}(t) = e^t$

Corollary 2.2, we have

$$\begin{aligned}&\left|e^b - e^a - \sum_{k=0}^{k=n}\left[\frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!}\right]e^x\right| \\ \leq\ &\left(\frac{(x-a)^{n+1}}{(n+2)!}e^a + \frac{(b-x)^{n+1}}{(n+2)!}e^b\right) \\ &+\ (n+1)\left(\frac{(x-a)^{n+1}}{(n+2)!} + \frac{(b-x)^{n+1}}{(n+2)!}\right)e^x.\end{aligned}$$

Choosing $a = 0$ and $b = 1$, we have for all $x \in [0, 1]$

$$\begin{aligned}&\left|e - 1 - \sum_{k=0}^{k=n}\left[\frac{(1-x)^{k+1} + (-1)^k x^{k+1}}{(k+1)!}\right]e^x\right| \\ \leq\ &\left(\frac{x^{n+1}}{(n+2)!} + \frac{(1-x)^{n+1}}{(n+2)!}e\right) +\ (n+1)\left(\frac{x^{n+1}}{(n+2)!} + \frac{(1-x)^{n+1}}{(n+2)!}\right)e^x.\end{aligned}$$

Moreover, if we choose $x = \frac{1}{2}$, we get

$$\left|e - 1 - \sum_{k=0}^{k=n}\left[\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right]\sqrt{e}\right| \leq \frac{(1+\sqrt{e})^2 + 2n\sqrt{e}}{2^{n+1}(n+2)!}.$$

References

[1] P. Cerone, S. S. Dragomir, and J. Roumeliotis, *Some Ostrowski type inequalities for n-time differentiable mappings and applications*, Demonstratio Math. **32** (1999), no. 4, 697–712.

[2] M. E. Gordji, M. R. Delavar, and M. De La Sen, *On φ-convex functions*, J. Math. Inequal. **10** (2016), no. 1, 173–183.

[3] J. Pečarić, F. Proschan, and Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, no. 187, Mathematics in Science and Engineering, Academic Press, Inc., Boston, MA, 1992.

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Laboratoire des télécommunications, Faculté des Sciences et de la Technologie,
University of 8 May 1945 Guelma,
P.O. Box 401, 24000 Guelma, Algeria.
Guelma, Algeria
e-mail: badrimeftah@yahoo.fr