

On the space-time admitting some geometric structures on energy-momentum tensors

Sobre el espacio-tiempo admitiendo algunas estructuras
geométricas en tensores de energía-momento

KANAK KANTI BAISHYA^{1,✉}, AJOY MUKHARJEE²

¹Kurseong College, Kurseong, India

²St. Joseph's College, Darjeeling, India

ABSTRACT. This paper presents a study of a general relativistic perfect fluid space-time admitting various types of curvature restrictions on energy-momentum tensors and brings out the conditions for which fluids of the space-time are sometimes phantom barrier and some other times quintessence barrier. The existence of a space-time where fluids behave as phantom barrier is ensured by an example.

Key words and phrases. General relativistic perfect fluid space-time, Einstein's field equation, energy-momentum tensor, semi-symmetric energy-momentum tensor.

2010 Mathematics Subject Classification. 53C50, 53C80.

RESUMEN. Este artículo presenta un estudio del tiempo-espacio fluido perfecto relativista general admitiendo varios tipos de restricciones de curvatura en los tensores de energía-momento y saca a relucir las condiciones para las cuales los fluidos del espacio-tiempo son a veces barrera fantasma y otras veces barrera de quintaesencia. La existencia de un espacio-tiempo donde los líquidos se comportan como barrera fantasma es garantizado por un ejemplo.

Palabras y frases clave. Espacio-tiempo fluido general relativista perfecto, campo de Einstein, tensor energía-momento, tensor semi-simétrico de energía-momento.

1. Introduction

Recently, in tune with Yano and Sawaki [16], Baishya and Roy Chowdhury [4] introduced and studied quasi-conformal curvature tensors in the frame of $N(k, \mu)$ -manifolds. The generalized quasi-conformal curvature tensor is defined for n dimensional manifolds as

$$\begin{aligned} \mathcal{W}(X, Y)Z = & \frac{n-2}{n} [1 + (n-1)(a-b) - \{1 + (n-1)(a+b)\}c] C(X, Y)Z \\ & + [1 - b + (n-1)a] E(X, Y)Z + (n-1)(b-a) P(X, Y)Z \\ & + \frac{n-2}{n} (c-1) \{1 + (n-1)(a+b)\} \hat{C}(X, Y)Z \end{aligned} \quad (1)$$

for all $X, Y, Z \in \chi(M)$, the set of all vector fields of the manifold M , where scalars a, b, c are real constants. The beauty of such curvature tensors lies on the fact that it has the flavour of

- (i) Riemannian curvature tensors R if the scalar triple $(a, b, c) \equiv (0, 0, 0)$,
- (ii) Conformal curvature tensors C [9] if $(a, b, c) \equiv (-\frac{1}{n-2}, -\frac{1}{n-2}, 1)$,
- (iii) Conharmonic curvature tensors $\hat{C}$ [10] if $(a, b, c) \equiv (-\frac{1}{n-2}, -\frac{1}{n-2}, 0)$,
- (iv) Concircular curvature tensors E [15, p. 84] if $(a, b, c) \equiv (0, 0, 1)$,
- (v) Projective curvature tensors P [15, p. 84] if $(a, b, c) \equiv (-\frac{1}{n-1}, 0, 0)$,
- (vi) m -projective curvature tensors H [12] if $(a, b, c) \equiv (-\frac{1}{2n-2}, -\frac{1}{2n-2}, 0)$.

Note that (1) can also be written as

$$\begin{aligned} \mathcal{W}(X, Y)Z = & R(X, Y)Z + a[S(Y, Z)X - S(X, Z)Y] \\ & + b[g(Y, Z)QX - g(X, Z)QY] \\ & - \frac{cr}{n} \left(\frac{1}{n-1} + a + b \right) [g(Y, Z)X - g(X, Z)Y]. \end{aligned} \quad (2)$$

The space-time under various curvature restrictions is a subject of vast literature, e.g., [1, 5, 7, 8] and the references there in.

In analogy with [14], an energy-momentum tensor T of type $(0, 2)$ is said to be semi-symmetric type if

$$\mathcal{W}(X, Y) \cdot T = 0 \quad (3)$$

holds where $\mathcal{W}(X, Y)$ acts on T as a derivation.

The paper is structured as follows. Section 2 is concerned with general relativistic perfect fluid space-time (briefly *GRPFS*) obeying Einstein's equation with $\mathcal{W}(X, Y) \cdot T = 0$. It is observed that a fluid of such space-time

always behaves as phantom barrier for each of the restrictions $E(X, Y) \cdot T = 0$ and $H(X, Y) \cdot T = 0$ whereas the same behaves either as a phantom barrier or quintessence barrier for each of the restrictions $R(X, Y) \cdot T = 0$ and $\hat{C}(X, Y) \cdot T = 0$. A detailed study of *GRPFS* obeying Einstein's equation admitting $((X \wedge_S Y) \cdot \mathcal{W}) = 0$ and $(X \wedge_T Y) = 0$, where the endomorphism is defined as $(X \wedge_B Y)Z = B(Y, Z)X - B(X, Z)Y$, has been carried out in Section 3 and 4 respectively, with similar types of results as in Section 2. Finally, we give an example in Section 5 of a fluid whose character is phantom barrier.

2. *GRPFS* with semi-symmetric type energy momentum tensors

Einstein's equation can be written as

$$S = kT + \frac{r}{2}g, \quad (4)$$

where k is the gravitational constant and r is the scalar curvature. Let (M^4, g) be a *GRPFS* with (3). Now (3) implies that

$$T(\mathcal{W}(X, Y)U, V) + T(U, \mathcal{W}(X, Y)V) = 0. \quad (5)$$

In view of (4) and (5), we have

$$\begin{aligned} ((\mathcal{W}(X, Y) \cdot S)U, V) &= \frac{r}{2}((\mathcal{W}(X, Y) \cdot g)U, V) \\ &= 0. \end{aligned}$$

In consequence of the above, we have the following:

Proposition 2.1. *A general relativistic space-time with a semi-symmetric type energy-momentum tensor is Ricci semi-symmetric type and vice-versa.*

By [5, Theorem 1, p 1029], we can easily bring out the following:

Proposition 2.2. *A general relativistic space-time with a covariant constant energy-momentum tensor is Ricci semi-symmetric type.*

By virtue of a result of Aikawa and Matsuyama [2] if a tensor field L is recurrent or birecurrent, then $R(X, Y) \cdot L = 0$. Hence we have the following:

Theorem 2.3. *A general relativistic space-time with a recurrent or birecurrent energy-momentum tensor is always Ricci semi-symmetric type.*

Next we consider a perfect fluid space-time whose energy-momentum tensor is semi-symmetric type. An energy-momentum tensor is said to describe a perfect fluid [11] if

$$T(X, Y) = (\sigma + \rho)A(X)A(Y) + \rho g(X, Y), \quad (6)$$

where σ is the energy density, ρ is the isotropic pressure of the fluid and A is a non-zero 1-form such that $g(X, \theta) = A(X)$ for all X , θ being the velocity vector field of the fluid which is a time-like vector that is,

$$g(\theta, \theta) = A(\theta) = -1.$$

By virtue of (4) and (6), we get

$$S(X, Y) = \frac{k(\sigma - \rho)}{2}g(X, Y) + k(\sigma + \rho)A(X)A(Y). \quad (7)$$

In view of (2) and (7), we have

$$\begin{aligned} \mathcal{W}(X, Y)Z = & R(X, Y)Z + k \left[\frac{a+b}{2} - \frac{c(\sigma - 3\rho)}{4} \left(\frac{1}{3} \right. \right. \\ & \left. \left. + a + b \right) \right] [g(Y, Z)X - g(X, Z)Y] \\ & + ak(\sigma + \rho)A(Z)[A(Y)X - A(X)Y] \\ & + bk(\sigma + \rho)[g(Y, Z)A(X) - g(X, Z)A(Y)]\theta. \end{aligned} \quad (8)$$

As consequences of (5) and (6), it follows that

$$\begin{aligned} 0 = & (\sigma + \rho)[A(\mathcal{W}(X, Y)U)A(V) + A(\mathcal{W}(X, Y)V)A(U)] \\ & + \rho[g(\mathcal{W}(X, Y)U, V) + g(\mathcal{W}(X, Y)V, U)] \end{aligned} \quad (9)$$

which yields

$$-\sigma A(\mathcal{W}(X, Y)U) + \rho g(\mathcal{W}(X, Y)\theta, U) = 0 \quad (10)$$

for $V = \theta$ which in turn on contraction gives

$$\begin{aligned} 0 = & k \left[\frac{\sigma + 3\rho}{2} \{ \sigma(1 - a + 3b) + \rho(1 - b + 3a) \} \right. \\ & \left. - (\sigma - 3\rho) \left\{ (a\sigma + \rho b) - \frac{c(\sigma + \rho)(1 + 3a + 3b)}{4} \right\} \right]. \end{aligned} \quad (11)$$

From (11), one can easily bring out the following table by substituting the triple (a, b, c) by $(0, 0, 0)$, $(-\frac{1}{2}, -\frac{1}{2}, 1)$, etc.

Curvature restrictions	Relations between σ & ρ
$R(X, Y) \cdot T = 0$	$\sigma + \rho = 0$ or $\sigma + 3\rho = 0$
$C(X, Y) \cdot T = 0$	σ, ρ are independent
$\hat{C}(X, Y) \cdot T = 0$	$\sigma + \rho = 0$ or $\sigma - 3\rho = 0$
$E(X, Y) \cdot T = 0$	$\sigma + \rho = 0$
$P(X, Y) \cdot T = 0$	$\sigma + \rho = 0$ or $\sigma = 0$
$H(X, Y) \cdot T = 0$	$\sigma + \rho = 0$

Now $\sigma + \rho = 0$ means that the fluid behaves as a cosmological constant [13]. This is also termed as phantom barrier [6]. In cosmology, a choice $\sigma = -\rho$ leads to a rapid expansion of the space-time which is now termed as inflation. Also $\sigma + 3\rho = 0$ or $(\sigma - 3\rho = 0)$ is known as the quintessence barrier. Here the strong energy condition begins to be violated. The present observations indicate that our universe is in quintessence era [3]. Thus from the above discussion we can state the following:

Theorem 2.4. *Let (M^4, g) be a GRPFS obeying Einstein's equation admitting $C(X, Y) \cdot T = 0$. Then the density of the matter and pressure are independent.*

Theorem 2.5. *The behavior of fluids in GRPFS obeying Einstein's equation is always phantom barrier for each of the restrictions $E(X, Y) \cdot T = 0$ and $H(X, Y) \cdot T = 0$.*

Theorem 2.6. *The behavior of a fluid in GRPFS obeying Einstein's equation is either phantom barrier or quintessence barrier for each of the restrictions $R(X, Y) \cdot T = 0$ or $\hat{C}(X, Y) \cdot T = 0$.*

3. GRPFS satisfying $((X \wedge_S Y) \cdot \mathcal{W})(Z, U)V = 0$

Let us consider a perfect fluid space-time satisfying $((X \wedge_S Y) \cdot \mathcal{W})(Z, U)V = 0$,

$$\text{i. e., } 0 = (X \wedge_S Y)\mathcal{W}(Z, U)V + \mathcal{W}((X \wedge_S Y)Z, U)V \\ + \mathcal{W}(Z, (X \wedge_S Y)U)V + \mathcal{W}(Z, U)(X \wedge_S Y)V, \quad (12)$$

where the endomorphism $(X \wedge_S Y)Z$ is defined as

$$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y. \quad (13)$$

In view of (13), (12) becomes

$$0 = S(Y, \mathcal{W}(Z, U)V)X - S(X, \mathcal{W}(Z, U)V)Y \\ + S(Y, Z)\mathcal{W}(X, U)V - S(X, Z)\mathcal{W}(Y, U)V \\ + S(Y, U)\mathcal{W}(Z, X)V - S(X, U)\mathcal{W}(Z, Y)V \\ + S(Y, V)\mathcal{W}(Z, U)X - S(X, V)\mathcal{W}(Z, U)Y. \quad (14)$$

Replacing Y and V by θ , we get

$$0 = (\sigma - \rho)g(X, \mathcal{W}(Z, U)\theta) - (\sigma + 3\rho)A(\mathcal{W}(Z, U)X), \quad (15)$$

which on contraction yields

$$0 = \frac{k(\sigma + 3\rho)}{2} \{[(\sigma + 3\rho)(1 - a + 3b) + (\sigma - \rho)(1 - b + 3a)] \\ - (\sigma - 3\rho)[b(\sigma - \rho) + a(\sigma + 3\rho)] \\ + \frac{c}{2}(\sigma + \rho)(\sigma - 3\rho)(1 + 3a + 3b)\}. \quad (16)$$

Consequently, from (16) one can easily bring out the following:

Curvature restrictions	Relations between σ & ρ
$((X \wedge_S Y) \cdot R)(Z, U)V = 0$	$\sigma + \rho = 0$ or $\sigma + 3\rho = 0$
$((X \wedge_S Y) \cdot C)(Z, U)V = 0$	σ, ρ are independent
$((X \wedge_S Y) \cdot \hat{C})(Z, U)V = 0$	$\sigma + \rho = 0$ or $\sigma - 3\rho = 0$
$((X \wedge_S Y) \cdot E)(Z, U)V = 0$	$\sigma + \rho = 0$
$((X \wedge_S Y) \cdot P)(Z, U)V = 0$	$\sigma + \rho = 0$ or $\sigma + 3\rho = 0$
$((X \wedge_S Y) \cdot H)(Z, U)V = 0$	$\sigma + \rho = 0$

Theorem 3.1. *The behavior of fluids in GRPFS obeying Einstein's equation is always phantom barrier for each of the restrictions $(X \wedge_S Y) \cdot E = 0$ and $(X \wedge_S Y) \cdot H = 0$.*

Theorem 3.2. *The behavior of fluids in GRPFS obeying Einstein's equation is either phantom barrier or quintessence barrier for each of the restrictions $(X \wedge_S Y) \cdot R = 0$, $(X \wedge_S Y) \cdot \hat{C} = 0$ and $(X \wedge_S Y) \cdot P = 0$.*

4. The Perfect fluid space-time satisfying $((X \wedge_T Y) \cdot \mathcal{W}) = 0$

Let us consider the perfect fluid space-time satisfying $((X \wedge_T Y) \cdot \mathcal{W}) = 0$,

$$\begin{aligned} \text{i. e., } 0 = & (X \wedge_T Y)\mathcal{W}(Z, U)V + \mathcal{W}((X \wedge_T Y)Z, U)V \\ & + \mathcal{W}(Z, (X \wedge_T Y)U)V + \mathcal{W}(Z, U)(X \wedge_T Y)V, \end{aligned} \quad (17)$$

where the endomorphism $(X \wedge_T Y)Z$ is defined as

$$(X \wedge_T Y)Z = T(Y, Z)X - T(X, Z)Y. \quad (18)$$

In consequence of (18), (17) becomes

$$\begin{aligned} 0 = & T(Y, \mathcal{W}(Z, U)V)X - T(X, \mathcal{W}(Z, U)V)Y \\ & + T(Y, Z)\mathcal{W}(X, U)V - T(X, Z)\mathcal{W}(Y, U)V \\ & + T(Y, U)\mathcal{W}(Z, X)V - T(X, U)\mathcal{W}(Z, Y)V \\ & + T(Y, V)\mathcal{W}(Z, U)X - T(X, V)\mathcal{W}(Z, U)Y. \end{aligned} \quad (19)$$

In view of (6) and (19), we have

$$\begin{aligned} 0 = & (\sigma + \rho)[A(\mathcal{W}(X, U)V)A(Y)A(Z) - A(\mathcal{W}(Y, U)V)A(X)A(Z) \\ & + A(\mathcal{W}(Z, X)V)A(Y)A(U) - A(\mathcal{W}(Z, Y)V)A(X)A(U) \\ & + A(\mathcal{W}(Z, U)X)A(Y)A(V) - A(\mathcal{W}(Z, U)Y)A(X)A(V)] \\ & + \rho[\bar{\mathcal{W}}(Z, U, V, Y)A(X) - \bar{\mathcal{W}}(Z, U, V, X)A(Y) \\ & + g(Y, Z)A(\mathcal{W}(X, U)V) - g(X, Z)A(\mathcal{W}(Y, U)V) \\ & + g(Y, U)A(\mathcal{W}(Z, X)V) - g(X, U)A(\mathcal{W}(Z, Y)V) \\ & + g(Y, V)A(\mathcal{W}(Z, U)X) - g(X, V)A(\mathcal{W}(Z, U)Y)]. \end{aligned} \quad (20)$$

Replacing Y and V by θ in (20), we get

$$\sigma A(\mathcal{W}(Z, U)X) + \rho g(\mathcal{W}(Z, U)\theta, X) = 0 \quad (21)$$

which on contraction gives

$$0 = -\frac{k(\sigma + 3\rho)}{2} [\{\sigma(1 - a + 3b) - \rho(1 - b + 3a)\}] + k(\sigma - 3\rho)[(a\sigma - b\rho) - \frac{c}{4}(\sigma - \rho)(1 + 3a + 3b)]. \quad (22)$$

From the above one can easily bring out the following:

Curvature Restrictions	Relations between σ & ρ
$((X \wedge_T Y) \cdot R)(Z, U)V = 0$	$\sigma - \rho = 0$ or $\sigma + 3\rho = 0$
$((X \wedge_T Y) \cdot C)(Z, U)V = 0$	σ, ρ are independent
$((X \wedge_T Y) \cdot \hat{C})(Z, U)V = 0$	$\sigma - \rho = 0$ or $\sigma - 3\rho = 0$
$((X \wedge_T Y) \cdot E)(Z, U)V = 0$	$\sigma - \rho = 0$ or $\sigma + \rho = 0$
$((X \wedge_T Y) \cdot P)(Z, U)V = 0$	$\sigma + \rho = 0$ or $\sigma = 0$
$((X \wedge_T Y) \cdot H)(Z, U)V = 0$	$\sigma - \rho = 0$ or $\sigma + \rho = 0$

Theorem 4.1. *The behavior of fluids in GRPFS obeying Einstein's equation is always phantom barrier for each of the restrictions $(X \wedge_T Y) \cdot E = 0$ (for $\sigma \neq \rho$), $(X \wedge_T Y) \cdot H = 0$ for $\sigma \neq \rho$ and $(X \wedge_T Y) \cdot P = 0$ for $\sigma \neq 0$.*

Theorem 4.2. *The behavior of fluids in GRPFS obeying Einstein's equation is always quintessence barrier for each of the restrictions $(X \wedge_T Y) \cdot R = 0$ for $\sigma \neq \rho$ and $(X \wedge_S Y) \cdot \hat{C} = 0$ for $\sigma \neq \rho$.*

5. An example of a fluid whose character is phantom barrier

Example 5.1. Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (1 + 2e^{x^1})[(dx^1)^2 + (dx^3)^2 + (dx^2)^2 - (dx^4)^2],$$

$(i, j = 1, 2, 3, 4)$. The only non-vanishing components of the Christoffel symbols and the Ricci tensors (up to symmetry) are

$$\begin{aligned} \Gamma_{11}^1 &= \Gamma_{44}^1 = \Gamma_{12}^2 = \Gamma_{13}^3 = -\Gamma_{14}^4 = -\Gamma_{22}^1 = -\Gamma_{33}^1 = \frac{e^{x^1}}{1 + 2e^{x^1}}, \\ R_{1221} &= R_{1331} = -R_{1441} = \frac{e^{x^1}}{1 + 2e^{x^1}}, \\ R_{2442} &= R_{3443} = -R_{2332} = -\frac{e^{2x^1}}{1 + 2e^{x^1}}, \\ \frac{1}{3}(1 + 2e^{x^1})R_{11} &= R_{22} = R_{33} = -R_{44} = \frac{e^{x^1}}{1 + 2e^{x^1}}. \end{aligned}$$

The scalar curvature r of the resulting space $(\mathbb{R}^4, g)$ is $r = -\frac{6e^{x^1}(1+e^{x^1})}{(1+2e^{x^1})^3}$.

Now using the above results, we may have

$$\begin{aligned}
 E_{1221} &= E_{1331} = -E_{1441} = \frac{e^{x^1}(3+e^{x^1})}{2+4e^{x^1}}, \\
 E_{2442} &= E_{3443} = -E_{2332} = -\frac{e^{x^1}(1+3e^{x^1})}{(2+4e^{x^1})}, \\
 \hat{C}_{1221} &= \hat{C}_{1331} = -\hat{C}_{1441} = \frac{e^{x^1}(3+e^{x^1})}{2+4e^{x^1}}, \\
 \hat{C}_{2332} &= -\hat{C}_{2442} = -\hat{C}_{3443} = \frac{e^{x^1}(1+3e^{x^1})}{2+4e^{x^1}}, \\
 P_{1212} &= P_{1313} = -P_{1414} = -\frac{2e^{x^1}}{1+2e^{x^1}}, \\
 P_{1221} &= P_{1331} = -P_{1441} = \frac{2e^{x^1}(2+e^{x^1})}{3+6e^{x^1}}, \\
 P_{2323} &= -P_{2424} = -P_{3434} = -\frac{e^{x^1}(1+5e^{x^1})}{3+6e^{x^1}}, \\
 P_{2332} &= -P_{2442} = -P_{3443} = \frac{e^{x^1}(1+5e^{x^1})}{3+6e^{x^1}}, \\
 H_{1221} &= H_{1331} = -H_{1441} = \frac{e^{x^1}(5+e^{x^1})}{3+6e^{x^1}}, \\
 H_{2332} &= -H_{2442} = -H_{3443} = \frac{e^{x^1}(5+e^{x^1})}{3+6e^{x^1}}.
 \end{aligned}$$

Assuming the associate vector field θ in the direction of x^4 , we have

$$T_{11} = \rho = T_{22} = T_{33}, \quad T_{44} = \sigma.$$

As consequences of the above relations, we can easily bring out the following:

$$\begin{aligned}
(R \circ T)_{1414} &= -\frac{(\sigma + \rho)e^{x^1}}{(1 + 2e^{x^1})^2}, \\
(R \circ T)_{2424} &= (R \circ T)_{3434} = -\frac{(\sigma + \rho)e^{2x^1}}{(1 + 2e^{x^1})^2}, \\
(E \circ T)_{1414} &= -\frac{(\sigma + \rho)e^{x^1}(3 + e^{x^1})}{2(1 + 2e^{x^1})^2}, \\
(E \circ T)_{2424} &= (E \circ T)_{3434} = -\frac{(\sigma + \rho)e^{x^1}(1 + 3e^{x^1})}{2(1 + 2e^{x^1})^2}, \\
(\hat{C} \circ T)_{1414} &= -\frac{(\sigma + \rho)e^{x^1}(3 + e^{x^1})}{(1 + 2e^{x^1})^2}, \\
(\hat{C} \circ T)_{2424} &= (\hat{C} \circ T)_{3434} = -\frac{(\sigma + \rho)e^{x^1}(1 + 3e^{x^1})}{2(1 + 2e^{x^1})^2}, \\
(P \circ T)_{1414} &= -\frac{2e^{x^1}(\sigma + \rho)}{(1 + 2e^{x^1})^2}, \quad (P \circ T)_{1441} = \frac{2e^{x^1}(\sigma + \rho)(2 + e^{x^1})}{3(1 + 2e^{x^1})^2}, \\
(P \circ T)_{2424} &= -(P \circ T)_{2442} = (P \circ T)_{3434} = -(P \circ T)_{3443} \\
&= -\frac{(1 + 5e^{x^1})(\sigma + \rho)e^{x^1}}{3(1 + 2e^{x^1})^2}, \\
(H \circ T)_{1414} &= -\frac{e^{x^1}(\sigma + \rho)(5 + e^{x^1})}{3(1 + 2e^{x^1})^2}, \\
(H \circ T)_{2424} &= (H \circ T)_{3434} = -\frac{(1 + 5e^{x^1})(\sigma + \rho)e^{x^1}}{3(1 + 2e^{x^1})^2}.
\end{aligned}$$

This leads to the following

Theorem 5.2. *Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian space endowed with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^i dx^j = (1 + 2e^{x^1})[(dx^1)^2 + (dx^3)^2 + (dx^2)^2 - (dx^4)^2],$$

($i, j = 1, 2, 3, 4$). Then the behavior of fluids in general relativistic perfect fluid space obeying Einstein's equation is always phantom barrier for each of $R(X, Y) \cdot T = 0$, $E(X, Y) \cdot T = 0$, $\hat{C}(X, Y) \cdot T = 0$, $P(X, Y) \cdot T = 0$ and $H(X, Y) \cdot T = 0$.

Acknowledgement. The first author designed and developed the contents of the paper, and the second author only carried out the calculations of Example 5.1 using programs of Wolfram Mathematica 5.1.

References

- [1] Z. Ahsan, *On a geometrical symmetry of the spacetime of geneareal relativity*, Bull. Cal. Math. Soc. **97** (2005), no. 3, 191–200.
- [2] R. Aikawa and Y. Matsuyama, *On the local symmetry of Kaehler hypersurfaces*, Yokohoma Math. J. **51** (2005), 63–73.
- [3] L. Amendola and S. Tsujikawa, *Dark energy: theory and observations*, Cambridge University Press, 2010.
- [4] K. K. Baishya and P. Roy Chowdhury, *On generalized quasi-conformal $n(k, \mu)$ -manifolds*, Commun. Korean Math. Soc. **31** (2016), no. 1, 163–176.
- [5] M. C. Chaki and S. Roy, *Spacetimes with covariant-constant energy-momentum tensor*, Internat. J. Theoret. Phys. **35** (1996), 1027–1032.
- [6] S. Chakraborty, N. Mazumder, and R. Biswas, *Cosmological evolution across phantom crossing and the nature of the horizon*, Astrophys. Space Sci. **334** (2011), 183–186.
- [7] U. C. De and L. Velimirović, *Spacetimes with semisymmetric energy-momentum tensor*, Internat. J. Theoret. Phys. **54** (2015), 17791783.
- [8] K. L. Duggal, *Space time manifolds and contact structures*, Int. J. Math. Math. Sci. **13** (1990), 545–554.
- [9] L. P. Eisenhart, *Riemannian geometry*, Princeton University Press, 1949.
- [10] Y. Ishii, *On conharmonic transformations*, Tensor (N.S.) **7** (1957), 73–80.
- [11] B. O'Neill, *Semi-Riemannian geometry*, Academic Press Inc, New York, 1983.
- [12] G. P. Pokhariyal and R. S. Mishra, *Curvature tensors and their relativistic significance*, Yokohama Math. J. **18** (1970), 105–108.
- [13] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein's field equations*, 2nd ed., Cambridge Monographs on Mathematical Physics, Cambridge Univ. Press, 2003.
- [14] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $r(x, y) \cdot r = 0$. i. the local version*, J. Differential Geom. **17** (1982), 531–582.
- [15] K. Yano and S. Bochner, *Curvature and Betti numbers*, Annals of Mathematics Studies 32, Princeton University Press, 1953.
- [16] K. Yano and S. Sawaki, *Riemannian manifolds admitting a conformal transformation group*, J. Differential Geom. **2** (1968), 161–184.

(Recibido en febrero de 2017. Aceptado en septiembre de 2017)

DEPARTMENT OF MATHEMATICS
KURSEONG COLLEGE
DOWHILL ROAD
KURSEONG
W. BENGAL-734 203
INDIA
e-mail: kanakkanti.kc@gmail.com

DEPARTMENT OF MATHEMATICS
ST. JOSEPH'S COLLEGE
DARJEELING
W. BENGAL-734 104
INDIA
e-mail: ajoyjee@gmail.com