

On the continuity of partial actions of Hausdorff groups on metric spaces

Sobre la continuidad de acciones parciales de grupos de Hausdorff en espacios métricos

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ABSTRACT. We provide a sufficient condition for a separately continuous partial action of a Hausdorff group on a metric space to be continuous.

Key words and phrases. partial action, separately continuity, Hausdorff groups.

2010 Mathematics Subject Classification. 54H15, 54E50, 54E35.

RESUMEN. Proporcionamos condiciones suficientes para que una acción parcial separadamente continua de un grupo de Hausdorff en un espacio métrico sea continua.

Palabras y frases clave. acción parcial, continuidad separada, grupos de Hausdorff.

1. Introduction

The notion of partial action of a group is a weakening of the classical concept of group action. It was introduced in [7] and [4], and developed in [1] and [11], in which the authors provided examples in different guises. Partial actions have been an important tool in C^* -algebras and dynamical systems, and in the construction of new cohomological theories [6], [5] and [12]. Every partial action of a group G on a set X can be obtained, roughly speaking, as a restriction of a global action (see [1] and [11]) on some bigger set X_G , called the enveloping space of X . Nevertheless, in the category of topological spaces, when G acts partially on a space X , the superspace X_G does not necessarily inherit its topological properties; for instance the globalization of a partial action of a group on a Hausdorff space is not in general Hausdorff (see e.g. [1, Example 1.4], [1, Proposition 1.2]).

On the other hand, actions of Polish groups have important connections with many areas of mathematics (see [2], [8] and the references therein). Recall that a Polish space is a topological space which is separable and completely metrizable, and a Polish group is a topological group whose topology is Polish. Partial actions of Polish groups have been recently considered in [9, 13, 14].

It is known that an action of a Polish group G on a metric space \mathbb{X} is continuous provided that it is separately continuous (see for instance [8, Theorem 3.1.4]). In this note, under a mild restriction, we generalize that result in two directions: First, we only assume that G is Hausdorff and Baire, and second, G acts partially on \mathbb{X} .

2. The notions

Let G be a topological group with identity element 1 and \mathbb{X} be a topological space. A partially defined map $m : G \times \mathbb{X} \rightarrow \mathbb{X}$ is a map whose domain is a subset of $G \times \mathbb{X}$. We write $\exists g \cdot x$ to mean that (g, x) is in the domain of m . Following [1, 11], m defines a *partial action* of G on \mathbb{X} , if for all $g, h \in G$ and $x \in \mathbb{X}$ the following assertions hold:

$$(PA1) \exists g \cdot x \text{ implies } \exists g^{-1} \cdot (g \cdot x) \text{ and } g^{-1} \cdot (g \cdot x) = x,$$

$$(PA2) \exists g \cdot (h \cdot x) \text{ implies } \exists (gh) \cdot x \text{ and } g \cdot (h \cdot x) = (gh) \cdot x,$$

$$(PA3) \exists 1 \cdot x, \text{ and } 1 \cdot x = x.$$

Let $G * \mathbb{X} = \{(g, x) \in G \times \mathbb{X} \mid \exists g \cdot x\}$ be the domain of m , $\mathbb{X}_{g^{-1}} = \{x \in \mathbb{X} \mid \exists g \cdot x\}$, and $m_g : \mathbb{X}_{g^{-1}} \ni x \mapsto g \cdot x \in \mathbb{X}_g$. Then a partial action $m : G * \mathbb{X} \rightarrow \mathbb{X}$ induces a family of bijections $\{m_g : \mathbb{X}_{g^{-1}} \rightarrow \mathbb{X}_g\}_{g \in G}$. We also denote m as $\{m_g : \mathbb{X}_{g^{-1}} \rightarrow \mathbb{X}_g\}_{g \in G}$. The following result expresses the notion of a partial action in terms of this family of bijections.

Proposition 2.1. [15, Lemma 1.2] *A partial action m of G on \mathbb{X} is a family $m = \{m_g : \mathbb{X}_{g^{-1}} \rightarrow \mathbb{X}_g\}_{g \in G}$, where $\mathbb{X}_g \subseteq \mathbb{X}$, $m_g : \mathbb{X}_{g^{-1}} \rightarrow \mathbb{X}_g$ is bijective, for all $g \in G$, and such that:*

$$(i) \mathbb{X}_1 = \mathbb{X} \text{ and } m_1 = \text{Id}_{\mathbb{X}};$$

$$(ii) m_g(\mathbb{X}_{g^{-1}} \cap \mathbb{X}_h) = \mathbb{X}_g \cap \mathbb{X}_{gh};$$

$$(iii) m_g m_h : \mathbb{X}_{h^{-1}} \cap \mathbb{X}_{h^{-1}g^{-1}} \rightarrow \mathbb{X}_g \cap \mathbb{X}_{gh}, \text{ and } m_g m_h = m_{gh} \text{ in } \mathbb{X}_{h^{-1}} \cap \mathbb{X}_{h^{-1}g^{-1}};$$

for all $g, h \in G$.

Notice that conditions (ii) and (iii) in Lemma 2.1 say that m_{gh} is an extension of $m_g m_h$, for all $g, h \in G$. We consider $G \times \mathbb{X}$ with the product topology and $G * \mathbb{X} \subseteq G \times \mathbb{X}$ inherits the subspace topology.

Definition 2.2. A topological partial action of a group G on a topological space \mathbb{X} is a partial action $m = \{m_g : \mathbb{X}_{g^{-1}} \rightarrow \mathbb{X}_g\}_{g \in G}$ such that each \mathbb{X}_g is

open in \mathbb{X} and each m_g is a homeomorphism, for all $g \in G$. If $m: G * \mathbb{X} \rightarrow \mathbb{X}$ is continuous, we say that the partial action is continuous.

Example 2.3. Induced partial action: Let G be a topological group, \mathbb{Y} a topological space, $u: G \times \mathbb{Y} \rightarrow \mathbb{Y}$ a continuous action of G on \mathbb{Y} and $\mathbb{X} \subseteq \mathbb{Y}$ an open set. For $g \in G$, set $\mathbb{X}_g = \mathbb{X} \cap u_g(\mathbb{X})$ and let $m_g = u_g \upharpoonright \mathbb{X}_{g^{-1}}$ (the restriction of u_g to $\mathbb{X}_{g^{-1}}$). Then $m: G * \mathbb{X} \ni (g, x) \mapsto m_g(x) \in \mathbb{X}$ is a topological partial action of G on \mathbb{X} .

The interested reader may consult others examples of topological partial actions in [1, Example 1.2, Remark 1.1, Example 1.3, Example 1.4] and [11, p. 108].

3. Topological partial actions of Hausdorff groups on metric spaces

Let $m: G * \mathbb{X} \rightarrow \mathbb{X}$ be a topological partial action. For $x \in \mathbb{X}$, let $G^x = \{g \in G \mid (g, x) \in G * \mathbb{X}\}$, notice that by (PA3) $1 \in G^x$. Let $m^x: G^x \ni g \rightarrow m(g, x) \in \mathbb{X}$. Then the map m is called *separately continuous* if the maps m^x are continuous, for all $x \in \mathbb{X}$. Notice that for any $g \in G$, the map m_g is continuous by definition of a topological partial action.

It is known that Polish group actions on metric spaces are continuous, if and only if, they are separately continuous (see, for instance, [10, Theorem 2.3.2] where it is attributed to R. Ellis). We present an extension of this result to partial actions. The proof we present is inspired by the corresponding proof for total actions as given in [8, Theorem 3.1.4] and [10, Theorem 2.3.2].

Theorem 3.1. *Let G be a Hausdorff group, (\mathbb{X}, d) a metric space, and m a topological partial action of G on \mathbb{X} . Suppose that G is Baire and G^x is open in G , for any $x \in \mathbb{X}$. Then m is continuous, if and only if, it is separately continuous.*

Proof. It is clear that continuous partial actions are separately continuous. For the converse, suppose that m is separately continuous and let $(g_0, x_0) \in G * \mathbb{X}$, we check that m is continuous at (g_0, x_0) . Let $l, n \in \mathbb{N}$ and set

$$F_{n,l} = \{g \in G^{x_0} \mid \forall x \in \mathbb{X}_{g^{-1}} (d(x, x_0) < 2^{-n} \Rightarrow d(m(g, x), m(g, x_0)) \leq 2^{-l})\}.$$

We shall check that $F_{n,l}$ is a closed subset of G^{x_0} . Indeed, let $g \in G^{x_0}$ and $\{g_i\}_{i \in I}$ a net with $g_i \rightarrow g$ such that $g_i \in F_{n,l}$, for all $i \in I$. Then

$$(\forall i \in I) \left(\forall x \in \mathbb{X}_{g_i^{-1}} \right) \left(d(x, x_0) < \frac{1}{2^n} \Rightarrow d(m(g_i, x), m(g_i, x_0)) \leq \frac{1}{2^l} \right).$$

Let $x \in \mathbb{X}_{g^{-1}}$, that is $g \in G^x$, since G^x is open we may assume that $\{g_i\}_{i \in I} \subseteq G^x$, and $x \in \mathbb{X}_{g_i^{-1}} \cap \mathbb{X}_{g^{-1}}$, for all $i \in I$. By the continuity of m^x and m^{x_0} , we

have that $m(g_i, x) \rightarrow m(g, x)$ and $m(g_i, x_0) \rightarrow m(g, x_0)$, which gives

$$(\forall x \in \mathbb{X}_{g^{-1}}) \left(d(x, x_0) < \frac{1}{2^n} \Rightarrow d(m(g, x), m(g, x_0)) \leq \frac{1}{2^l} \right),$$

and we conclude that $g \in F_{n,l}$. Now we check that

$$G^{x_0} = \bigcap_l \bigcup_n F_{n,l}. \quad (1)$$

It is clear that $G^{x_0} \supseteq \bigcap_l \bigcup_n F_{n,l}$. Conversely, take $g \in G^{x_0}$ and $l \in \mathbb{N}$. Since m_g

is continuous at x_0 , for $\varepsilon = \frac{1}{2^l}$, there exists $\delta > 0$ such that

$$(\forall x \in \mathbb{X}_{g^{-1}}) \left(d(x, x_0) < \delta \Rightarrow d(m(g, x), m(g, x_0)) \leq \frac{1}{2^l} \right).$$

Let $n \in \mathbb{N}$ with $\frac{1}{2^n} < \delta$, then $g \in F_{n,l}$. Thus $G^{x_0} \subseteq \bigcap_l \bigcup_n F_{n,l}$, and we have shown (1).

Since $F_{l,n}$ is closed in G^{x_0} , for all n, l , then $D = \bigcup_l \bigcup_n (F_{n,l} \setminus \text{int}(F_{n,l}))$ is meager. Since G is Baire and G^{x_0} is a non empty open set, then there is $g_1 \in G^{x_0} \setminus D$. We shall check that m is continuous at (g_1, x_0) . Indeed, let $\{(h_\alpha, y_\alpha)\}_{\alpha \in \Lambda} \subseteq G * \mathbb{X}$ be a net converging to (g_1, x_0) . Take $\varepsilon > 0$ and $l \in \mathbb{N}$ such that $\frac{1}{2^{l-1}} < \varepsilon$. By (1), there exists $n \in \mathbb{N}$ such that $g_1 \in F_{n,l}$. Since $g_1 \notin D$ then $g_1 \in \text{int}(F_{n,l})$. Since $h_\alpha \rightarrow g_1$, then there is $\alpha_1 \in \Lambda$ such that $h_\alpha \in \text{int}(F_{n,l})$, for all $\alpha \geq \alpha_1$. Also, since $y_\alpha \rightarrow x_0$, there exists $\alpha_2 \in \Lambda$ such that $d(y_\alpha, x_0) < \frac{1}{2^n}$, for all $\alpha \geq \alpha_2$. Additionally, by the continuity of m^{x_0} , we have that $m(h_\alpha, x_0) \rightarrow m(g_1, x_0)$. Thus, there exists $\alpha_3 \in \Lambda$ such that $d(m(h_\alpha, x_0), m(g_1, x_0)) < \frac{1}{2^l}$, for all $\alpha \geq \alpha_3$.

Let $\alpha \geq \max\{\alpha_1, \alpha_2, \alpha_3\}$. Then $h_\alpha \in F_{n,l}$ and $d(y_\alpha, x_0) < \frac{1}{2^n}$, hence

$$\begin{aligned} d(m(h_\alpha, y_\alpha), m(g_1, x_0)) &\leq d(m(h_\alpha, y_\alpha), m(h_\alpha, x_0)) + d(m(h_\alpha, x_0), m(g_1, x_0)) \\ &< \frac{1}{2^{l-1}} < \varepsilon. \end{aligned}$$

Thus, m is continuous at (g_1, x_0) .

Since $x_0 \in \mathbb{X}_{g_1^{-1}} \cap \mathbb{X}_{g_0^{-1}}$, we have, by Proposition 2.1(ii), that $g_1 \cdot x_0 \in \mathbb{X}_{g_1} \cap \mathbb{X}_{g_1 g_0^{-1}}$. Thus $(g_0 g_1^{-1}) \cdot (g_1 \cdot x_0)$ is defined and $(g_0 g_1^{-1}) \cdot (g_1 \cdot x_0) = g_0 \cdot x_0$, by (PA2). That is

$$m(g_0, x_0) = m(g_0 g_1^{-1}, m(g_1, x_0)). \quad (2)$$

Finally take a net $\{(h_j, y_j)\}_{j \in J}$ in $G * \mathbb{X}$ converging to (g_0, x_0) . Then $g_1 g_0^{-1} h_j \rightarrow g_1 \in G^{x_0}$, and since G^{x_0} is open we can assume that the net $\{g_1 g_0^{-1} h_j\}_{j \in J}$ is contained in G^{x_0} , thus $x_0 \in \mathbb{X}_{(g_1 g_0^{-1} h_j)^{-1}}$, for all $j \in J$. But $y_j \rightarrow x_0$, then there is $j_0 \in J$ such that $y_j \in \mathbb{X}_{(g_1 g_0^{-1} h_j)^{-1}}$, for all $j \geq j_0$. Now $y_j \in \mathbb{X}_{(g_1 g_0^{-1} h_j)^{-1}} \cap \mathbb{X}_{h_j^{-1}}$, then $h_j \cdot y_j \in \mathbb{X}_{(g_1 g_0^{-1})^{-1}}$ thanks to Proposition 2.1(ii). Now, by (PA2) we get

$$(g_1 g_0^{-1} h_j) \cdot y_j = (g_1 g_0^{-1}) \cdot (h_j \cdot y_j) \in \mathbb{X}_{g_1 g_0^{-1}}.$$

Thus, by (PA1),

$$h_j \cdot y_j = (g_0 g_1^{-1}) \cdot [(g_1 g_0^{-1} h_j) \cdot y_j].$$

Notice that $(g_1 g_0^{-1} h_j, y_j) \rightarrow (g_1, x_0)$. Then by continuity of $m_{g_0 g_1^{-1}}$ and the continuity of m at (g_1, x_0) one gets

$$m(h_j, y_j) = (g_0 g_1^{-1}) \cdot [(g_1 g_0^{-1} h_j) \cdot y_j] \rightarrow (g_0 g_1^{-1}) \cdot (g_1 \cdot x_0) = m(g_0 g_1^{-1}, m(g_1, x_0)),$$

and by (2) we obtain $m(h_j, y_j) \rightarrow m(g_0, x_0)$. \checkmark

Corollary 3.2. *Let G be a topological Hausdorff group and $a: G \times \mathbb{X} \rightarrow \mathbb{X}$ an action of G on a metric space \mathbb{X} . If G is Baire, then a is continuous, if and only if, a is separately continuous.*

Corollary 3.3. *Let G be a countable discrete group and m a topological partial action of G on a metric space \mathbb{X} . Then m is continuous if and only if m is separately continuous.*

Example 3.4. Möbius transformations [3, p. 175] The group $G = \text{GL}(2, \mathbb{R})$ is Polish and acts partially on \mathbb{R} by setting

$$g \cdot x = \frac{ax + b}{cx + d}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Notice that for all $g \in G$, $\mathbb{R}_{g^{-1}} = \{x \in \mathbb{R} \mid cx + d \neq 0\}$ is open and $m = \{m_g: \mathbb{R}_{g^{-1}} \ni x \rightarrow g \cdot x \in \mathbb{R}_g\}_{g \in G}$ is a topological partial action. For $x \in \mathbb{R}$ let $t_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, then for $y \in \mathbb{R}$ one has that $t_{y-x} \cdot x = y$. Since

$$G^0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : d \neq 0 \right\}$$

is open, and

$$G^x = G^{t_x \cdot 0} = G^0 t_x^{-1} = G^0 \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix},$$

then G^x is open. Finally, since

$$m^x : G^x \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ax + b}{cx + d} \in \mathbb{R},$$

is continuous, then m is continuous.

Acknowledgements

The authors thank the referee for his/her suggestions and comments that helped to improve the quality of the paper. In particular, for suggesting the current version of Theorem 3.1, that was initially established only for Polish groups. The second and third authors thank La Vicerrectoría de Investigación y Extensión de la Universidad Industrial de Santander for the financial support for this work, which is a part of the VIE project # 5761.

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(Recibido en agosto de 2017. Aceptado en noviembre de 2017)

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