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Hilbert spaces with generic predicates

Espacios de Hilbert con predicados genéricos

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Abstract. We study the model theory of expansions of Hilbert spaces by generic predicates. We first prove the existence of model companions for generic expansions of Hilbert spaces in the form of a distance function to a *random substructure*, then a distance to a random subset. The theory obtained with the random substructure is ω-stable, while the one obtained with the distance to a random subset is TP_2 and $NSOP_1$. That example is the first continuous structure in that class.

Key words and phrases. Continuous logic, Random Predicates, TP_2, Stability theory.

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Resumen. Estudiamos la teoría de modelos de expansiones de espacios de Hilbert mediante predicados genéricos. Primero demostramos la existencia de modelo-compañeras de expansiones genéricas de espacios de Hilbert mediante una función-distancia a una *estructura aleatoria*, y luego una distancia a un subconjunto aleatorio. La teoría obtenida con la subestructura aleatoria es ω-estable; la obtenida mediante la distancia a subconjunto aleatorio es TP_2 y $NSOP_1$. Este ejemplo es la primera estructura de esta clase de complejidad en lógica continua.

Palabras y frases clave. Lógica continua, Predicados aleatorios, TP_2, Estabilidad.

1. Introduction

This paper deals with Hilbert spaces expanded with random predicates in the framework of continuous logic as developed in [2]. The model theory of Hilbert spaces is very well understood, see [2, Chapter 15] or [5]. However, we briefly review some of its properties at the end of this section.

In this paper we build several new expansions, by various kinds of random predicates (random substructure and the distance to a random subset) of Hilbert spaces, and study them within the framework of continuous logic. While our constructions are not exactly metric Fraïssé (failing the hereditary property), some of them are indeed amalgamation classes and we study the model theory of their limits.

Several papers deal with generic expansions of Hilbert spaces. Ben Yaacov, Usvyatsov and Zadka [3] studied the expansion of a Hilbert space with a generic automorphism. The models of this theory are expansions of Hilbert spaces with a unitary map whose spectrum is S^1. A model of this theory can be constructed by amalgamating together the collection of n-dimensional Hilbert spaces with a unitary map whose eigenvalues are the n-th roots of unity as n varies in the positive integers. More work on generic automorphisms can be found in [4], where the first author of this paper studied Hilbert spaces expanded with a random group of automorphisms G.

There are also several papers about expansions of Hilbert spaces with random subspaces. In [5] the first author and Buechler identified the saturated models of the theory of beautiful pairs of a Hilbert space. An analysis of lovely pairs (the generalization of beautiful pairs (belles paires) to simple theories) in the setting of compact abstract theories is carried out in [1]. In the (very short) second section of this paper we build the beautiful pairs of Hilbert spaces as the model companion of the theory of Hilbert spaces with an orthonormal projection. We provide an axiomatization for this class and we show that the projection operator into the subspace is interdefinable with a predicate for the distance to the subspace. We also prove that the theory of beautiful pairs of Hilbert spaces is ω-stable. Many of the properties of beautiful pairs of Hilbert spaces are known from the literature or folklore, so this section is mostly a compilation of results.

In the third section we add a predicate for the distance to a random subset. This construction was inspired by the idea of finding an analogue to the first order generic predicates studied by Chatzidakis and Pillay in [6]. The axiomatization we found for the model companion was inspired in the ideas of [6] together with the following observation: in Hilbert spaces there is a definable function that measures the distance between a point and a model. We prove that the theory of Hilbert spaces with a generic predicate is unstable. We also study a natural notion of independence in a monster model of this theory and establish some of its properties. In particular, we show that natural independence notions associated with our expansions of Hilbert spaces have various desirable properties but not enough to guarantee simplicity, or even the NTP_2 properties. However, the theory ends up having the $NSOP_1$ property.

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1.1. Model theory of Hilbert spaces (quick review)

1.1.1. Hilbert spaces

We follow [2] in our study of the model theory of a real Hilbert space and its expansions. We assume the reader is familiar with the basic concepts of continuous logic as presented in [2]. A Hilbert space $\mathcal{H}$ can be seen as a multi-sorted structure $(B_n(H), 0, +, \langle,\rangle, \{\lambda_r : r \in \mathbb{R}\})_{0<n<\omega}$, where $B_n(H)$ is the ball of radius n, $+$ stands for addition of vectors (defined from $B_n(H) \times B_n(H)$ into $B_{2n}(H)$), $\langle,\rangle \colon B_n(H) \times B_n(H) \to [-n^2, n^2]$ is the inner product, 0 is a constant for the zero vector and $\lambda_r \colon B_n(H) \to B_{n(\lceil |r| \rceil)}(H)$ is multiplication by $r \in \mathbb{R}$.

We denote by L the language of Hilbert spaces and by T the theory of Hilbert spaces.

By a universal domain $\mathcal{H}$ of T we mean a Hilbert space $\mathcal{H}$ which is κ-saturated and κ-strongly homogeneous with respect to types in the language L, where κ is a regular cardinal larger than $2^{\aleph_0}$. Constructing such a structure is straightforward –just consider a Hilbert space with an orthonormal basis of cardinality at least κ.

We will assume that the reader is familiar with the metric versions of *definable closure* and *non-dividing*. The reader can check [2, 5] for the definitions.

Notation 1. Let dcl stand for the definable closure and acl stand for the algebraic closure in the language L.

Fact Let $A \subset \mathcal{H}$ be small. Then $\mathrm{dcl}(A) = \mathrm{acl}(A) =$ the smallest Hilbert subspace of $\mathcal{H}$ containing A.

Proof. See Lemma 3 in [5, p. 80]. ☑

Recall a characterization of non-dividing in pure Hilbert spaces (that will be useful in the more sophisticated constructions in forthcoming sections):

Proposition 1.1. *Let $B, C \subset H$ be small, let $(a_1, \dots, a_n) \in H^n$ and assume that $C = \operatorname{dcl}(C)$, so C is a Hilbert subspace of H. Denote by P_C the projection on C. Then $\operatorname{tp}(a_1, \dots, a_n/C \cup B)$ does not divide over C if and only if for all $i \leq n$ and all $b \in B$, $a_i - P_C(a_i) \perp b - P_C(b)$.*

***Proof*.** Proved as Corollary 2 and Lemma 8 of [5, pp. 81–82]. ☑

For $A, B, C \subset H$ small, we say that A *is independent from B over C* if for all $n \geq 1$ and $\bar{a} \in A^n$, $\operatorname{tp}(\bar{a}/C \cup B)$ does not divide over C and we write is as $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

Under non-dividing independence, types over sets are stationary. In particular, the independence theorem holds over sets, and we may refer to this property as 3-*existence*. It is also important to point out that non-dividing is *trivial*, that is, for all sets B, C and tuples $(a_1, \dots, a_n)$ from H, $\operatorname{tp}(a_1, \dots, a_n/C \cup B)$ does not divide over C if and only if $\operatorname{tp}(a_i/B \cup C)$ does not divide over C for $i \leq n$.

2. Random subspaces and beautiful pairs

We now deal with the easiest situation: a Hilbert space with an orthonormal projection operator onto a *subspace*. Let $L_p = L \cup \{P\}$ where P is a new unary function and we consider structures of the form $(\mathcal{H}, P)$, where $P\colon H \to H$ is a projection into a subspace. Note that $P\colon B_n(H) \to B_n(H)$ and that P is determined by its action on $B_1(H)$. Recall that projections are bounded linear operators, characterized by two properties:

(1) $P^2 = P$,

(2) $P^* = P$.

The second condition means that for any $u, v \in H$, $\langle P(u), v\rangle = \langle u, P(v)\rangle$. A projection also satisfies, for any $u, v \in H$, the condition $\|P(u) - P(v)\| \leq \|u - v\|$. In particular, it is a uniformly continuous map and its modulus of uniform continuity is the function $\Delta_P(\epsilon) = \epsilon$ (the modulus of uniform continuity is a function providing, for each ϵ, the "corresponding δ" in the usual definition of uniform continuity).

We start by showing that the class of Hilbert spaces with projections has the free amalgamation property:

Lemma 2.1. *Let $(\mathcal{H}_0, P_0) \subset (\mathcal{H}_i, P_i)$ where $i = 1, 2$ and $H_1 \mathop{\smile\!\!\!\!\!\!|}_{H_0} H_2$ be (possibly finite dimensional) Hilbert spaces with projections. Then $H_3 = \operatorname{span}\{H_1, H_2\}$ is a Hilbert space and $P_3(v_3) = P_0(v_0) + P_1(v_1) + P_2(v_2)$ is a projection, where $v_3 = v_0 + v_1 + v_2$ and v_0 is the projection of v_3 in H_0, v_1 is the projection of v_3 in $H_1 \cap H_0^{\perp}$, and v_2 is the projection of v_3 in $H_2 \cap H_0^{\perp}$.*

Proof. It is clear that $H_3 = \text{span}\{H_1 \cup H_2\}$ is a Hilbert space containing H_1 and H_2. It remains to prove that P_3 is a linear map and a projection. Let $u_3, v_3 \in H_3$ and write $u_3 = u_0 + u_1 + u_2$, $v_3 = v_0 + v_1 + v_2$ where u_0, v_0, u_1, v_1, u_2, v_2 are the projections of u_3 and v_3 in H_0, $H_1 \cap H_0^\perp$ and $H_2 \cap H_0^\perp$ respectively. Then $u_0 + v_0$, $u_1 + v_1$, $u_2 + v_2$ are the projections of $u_3 + v_3$ into those spaces and $P_3(u_3 + v_3) = P_0(u_0 + v_0) + P_1(u_1 + v_1) + P_2(u_2 + v_2) = P_3(u_3) + P_3(v_3)$. A similar computation shows that $P_3(\lambda u_3) = \lambda P_3(u_3)$ for any $\lambda \in \mathbb{R}$.

To show it is a projection, note that $P_3(P_3(u_3)) = P_3(P_0(v_0) + P_1(v_1) + P_2(v_2)) = P_0^2(v_0) + P_1^2(v_1) + P_2^2(v_2) = P_0(v_0) + P_1(v_1) + P_2(v_2) = P_3(u_3)$. And since $P_0^*(v_0) + P_1^*(v_1) + P_2^*(v_2) = P_0(v_0) + P_1(v_1) + P_2(v_2)$ we get also that $P_3^* = P_3$. ☑

Let T^P be the theory of Hilbert spaces with a linear projection. It is axiomatized by the theory of Hilbert spaces together with axioms stating that P is linear and the axioms (1) and (2) that say that P is a projection.

Consider first the finite dimensional models. Given an n-dimensional Hilbert space $\mathcal{H}_n$, there are only $n + 1$ many pairs $(\mathcal{H}_n, P)$, where P is a projection, modulo isomorphism. They are classified by the dimension of $P(H)$, which ranges from 0 to n.

In order to characterize the existentially closed models of T^P, note the following facts:

(1) Let $(\mathcal{H}, P)$ be existentially closed, and $(\mathcal{H}_n, P_n)$ be an n-dimensional Hilbert space with an orthonormal projection with the property that $P_n(\mathcal{H}_n) = \mathcal{H}_n$. Then $(\mathcal{H}, P) \oplus (\mathcal{H}_n, P_n)$ is an extension of $(\mathcal{H}, P)$ with $\dim([P \oplus P_n](\mathcal{H} \oplus \mathcal{H}_n)) \geq n$. Since n can be chosen as big as we want and $(\mathcal{H}, P)$ is existentially closed, $\dim(P(H)) = \infty$.

(2) Let $(\mathcal{H}, P)$ be existentially closed, and $(\mathcal{H}_n, P_0)$ be an n-dimensional Hilbert space with an orthonormal projection such that $P_0(H_n) = \{0\}$. Then $(\mathcal{H}, P) \oplus (\mathcal{H}_n, P_0)$ is an extension of (H, P) such that $\dim(([P \oplus P_0](H \oplus \mathcal{H}_n))^\perp) \geq n$. Since n can be chosen as big as we want and $(\mathcal{H}, P)$ is existentially closed, $\dim(P(H)^\perp) = \infty$.

Definition 2.2. Let T_ω^P be the theory T^P (stating that P is a linear map and a projection) together with axioms stating that there are infinitely many pairwise orthonormal vectors v satisfying $P(v) = v$ and also infinitely many pairwise orthonormal vectors u satisfying $P(u) = 0$.

Note that the theory T_ω^P has a unique separable model and thus is complete. It corresponds to the theory of beautiful pairs of Hilbert spaces (see [5]). Note that any model (H, P) of T^P can be embedded (after increasing the dimension of $P(H)$ and $P(H)^\perp$ so that they are infinite) into a model of T_ω^P. We will prove bellow that T^P_ω has quantifier elimination and thus is the model companion of T^P. We will know study the theory T^P_ω.

Let $(\mathcal{H}, P) \models T^P_\omega$ and for any $v \in H$ let $d_P(v) = \|v - P(v)\|$. Then $d_P(v)$ measures the distance between v and the subspace $P(H)$. The distance function $d_P(x)$ is definable in $(\mathcal{H}, P)$. We will now prove the converse, that is, that we can definably recover P from d_P.

Lemma 2.3. *Let $(\mathcal{H}, P) \models T^P_\omega$. For any $v \in H$ let $d_P(v) = \|v - P(v)\|$. Then $P(v) \in dcl(v)$ in the structure $(\mathcal{H}, d_P)$.*

Proof. Note that $P(v)$ is the unique element x in $P(H)$ satisfying $\|v - x\| = d_P(v)$. Thus $P(v)$ is the unique realization of the condition

$$\varphi(x) = \max\{d_P(x), |\|v - x\| - d_P(v)|\} = 0.$$

☑

Proposition 2.4. *Let $(\mathcal{H}, P) \models T^P_\omega$. For any $v \in H_\omega$ let $d_P(v) = \|v - P(v)\|$. Then the projection function $P(x)$ is definable in the structure $(\mathcal{H}, d_P)$.*

Proof. Let $(\mathcal{H}, P) \models T^P_\omega$ be κ-saturated for $\kappa > \aleph_0$ and let $d_P(v) = \|v - P(v)\|$. Since d_P is definable in the structure $(\mathcal{H}, P)$, the new structure $(\mathcal{H}, d_P)$ is still κ-saturated. Let $\mathcal{G}_P$ be the graph of the function P. Then by the previous lemma $\mathcal{G}_P$ is type-definable in $(\mathcal{H}, d_P)$ and thus by [2, Proposition 9.24] P is definable in the structure $(\mathcal{H}, d_P)$. ☑

Notation 2. We write tp for L-types, tp_P for L_P-types and qftp_P for quantifier free L_P-types. We write acl_P for the algebraic closure in the language L_P. We follow a similar convention for dcl_P.

Lemma 2.5. *T^P_ω has quantifier elimination.*

Proof. It suffices to show that quantifier free L_P-types determine the L_P-types. Let $(\mathcal{H}, P) \models T^P_\omega$ and let $\bar{a} = (a_1, \ldots, a_n), \bar{b} = (b_1, \ldots, b_n) \in H^n$. Assume that $\text{qftp}_P(\bar{a}) = \text{qftp}_P(\bar{b})$. Then

$$\text{tp}(P(a_1), \ldots, P(a_n)) = \text{tp}(P(b_1), \ldots, P(b_n))$$

and

$$\text{tp}(a_1 - P(a_1), \ldots, a_n - P(a_n)) = \text{tp}(b_1 - P(b_1), \ldots, b_n - P(b_n)).$$

Let $H_0 = P(H)$ and let $H_1 = H_0^\perp$, both are then infinite dimensional Hilbert spaces and $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Let $f_0 \in Aut(\mathcal{H}_0)$ satisfy $f_0(P(a_1), \ldots, P(a_n)) = (P(b_1), \ldots, P(b_n))$ and let $f_1 \in Aut(\mathcal{H}_1)$ be such that $f_1(a_1 - P(a_1), \ldots, a_n - P(a_n)) = (b_1 - P(b_1), \ldots, b_n - P(b_n))$. Let f be the automorphism of $\mathcal{H}$ induced by f_0 and f_1, that is, $f = f_0 \oplus f_1$. Then $f \in Aut(\mathcal{H}, P)$ and $f(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, so $\text{tp}_P(\bar{a}) = \text{tp}_P(\bar{b})$. ☑

Characterization of types: By the previous lemma, the L_P-type of an n-tuple $\bar{a} = (a_1, \dots, a_n)$ inside a structure $(\mathcal{H}, P) \models T_\omega^P$ is determined by the L-type $\mathrm{tp}(P(a_1), \dots, P(a_n), a_1 - P(a_1), \dots, a_n - P(a_n))$ of its projections onto $P(H)$ and $P(H)^\perp$. In particular, we may regard $(\mathcal{H}, P)$ as the direct sum of the two independent pure Hilbert spaces $(P(H), +, 0, \langle , \rangle)$ and $(P(H)^\perp, +, 0, \langle , \rangle)$.

We may therefore characterize definable and algebraic closure, as follows.

Proposition 2.6. *Let $(\mathcal{H}, P) \models T_\omega^P$ and let $A \subset H$. Then $\mathrm{dcl}_P(A) = \mathrm{acl}_P(A) = \mathrm{dcl}(A \cup P(A))$.*

We leave the proof to the reader. Another consequence of the description of types is:

Proposition 2.7. *The theory T_ω^P is ω-stable.*

Proof. Let $(\mathcal{H}, P) \models T_\omega^P$ be separable and let $A \subset H$ be countable. Replacing $(\mathcal{H}, P)$ for $(\mathcal{H}, P) \oplus (\mathcal{H}, P)$ if necessary, we may assume that $P(H) \cap dcl_P(A)^\perp$ is infinite dimensional and that $P(H)^\perp \cap dcl_P(A)^\perp$ is infinite dimensional. Thus every L_p-type over A is realized in the structure $(\mathcal{H}, P)$ and $(S_1(A), d)$ is separable. ☑

Proposition 2.8. *Let $(\mathcal{H}, P) \models T_\omega^P$ be a κ-saturated domain and let $A, B, C \subset H$ be small. Then the type $\mathrm{tp}_P(A/B \cup C)$ does not fork over C if and only if $\mathrm{tp}(A \cup P(A)/B \cup P(B) \cup C \cup P(C))$ does not fork over $C \cup P(C)$.*

Proof. For $A, B, C \subset H$, let $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ be defined as $\mathrm{tp}(A \cup P(A)/B \cup P(B) \cup C \cup P(C))$ does not fork over $C \cup P(C)$. We check that $\mathop{\smile\!\!\!\!\!\!|}^*$ is an independence notion:

(1) Invariance under automorphisms of $(\mathcal{H}, P)$. Clear.

(2) Symmetry: $A \mathop{\smile\!\!\!\!\!\!|}^*_C B \iff B \mathop{\smile\!\!\!\!\!\!|}^*_C A$. It follows from the fact that independence in Hilbert spaces has the same property.

(3) Transitivity: $A \mathop{\smile\!\!\!\!\!\!|}^*_C BD$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}^*_{BC} D$. It follows from the corresponding property for Hilbert spaces.

(4) Finite Character: $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ if and only $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_C B$ for all $\bar{a} \in A$ finite. It follows from finite character for independence in Hilbert spaces.

(5) Local Character: If $\bar{a}$ is any finite tuple, then there is countable $B_0 \subseteq B$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_{B_0} B$. It follows from local character for Hilbert spaces regarding $(\mathcal{H}, P)$ as a direct sum of $P(H)$ and $P(H)^\perp$.

(6) Stationarity. Let $A, A', B, C \subset \mathcal{H}$ be small sets satifying $\mathrm{tp}_P(A/C) = \mathrm{tp}_P(A'/C)$ and $A \downarrow^*_C B$, $A' \downarrow^*_C B$. Then $\mathrm{tp}(A \cup P(A)/C \cup P(C)) = \mathrm{tp}_P(A' \cup P(A')/C \cup P(C))$. On the other hand the independence condition implies $A \cup P(A) \downarrow_{C \cup P(C)} B \cup P(B)$ and $A' \cup P(A') \downarrow_{C \cup P(C)} B \cup P(B)$, so $\mathrm{tp}(A \cup P(A)/B \cup P(B) \cup C \cup P(C)) = \mathrm{tp}_P(A' \cup P(A')/B \cup P(B) \cup C \cup P(C))$ and thus $\mathrm{tp}_P(A/B \cup C) = \mathrm{tp}_P(A'/B \cup C)$.

(7) Extension. It is easy to prove decomposing $\mathcal{H}$ into $P(H)$ and $P(H)^\perp$ and then using the extension property for Hilbert spaces in each subspace.

Since forking is stable theories characterized by these properties (see [2, Theorem 14.14]) we get the desired result. ☑

3. Continuous random predicates

We now come to our main theory and to our first set of results. We study the expansion of a Hilbert space with a distance function to a subset of H. Let d_N be a new unary predicate and let L_N be the language of Hilbert spaces together with d_N. We denote the L_N structures by $(\mathcal{H}, d_N)$, where $d_N : \mathcal{H} \to [0, 1]$ and we want to consider the structures where d_N is a distance to a subset of H. Instead of measuring the actual distance to the subset, we truncate the distance at one. We start by characterizing the functions d_N corresponding to distances.

3.1. The basic theory T_0

We denote by T_0 the theory of Hilbert spaces together with the next two axioms (compare with Theorem 9.11 in [2]):

(1) $\sup_x \min\{1 \dot{-} d_N(x), \inf_y \max\{|d_N(x) - \|x - y\||, d_N(y)\}\} = 0$;

(2) $\sup_x \sup_y [d_N(y) - \|x - y\| - d_N(x)] \leq 0$.

We say a point is *black* if $d_N(x) = 0$ and *white* if $d_N(x) = 1$. All other points are gray, darker if $d(x)$ is close to zero and whiter if $d_N(x)$ is close to one. This terminology follows [9]. From the second axiom we get that d_N is uniformly continuous (with modulus of uniform continuity $\Delta(\epsilon) = \epsilon$). Thus we can apply the tools of continuous model theory to analyze these structures.

Lemma 3.1. *Let $(\mathcal{H}, d) \models T_0$ be $\aleph_0$-saturated and let $N = \{x \in H : d_N(x) = 0\}$. Then for any $x \in H$, $d_N(x) = \mathrm{dist}(x, N)$ (here,* dist *denotes the actual distance as computed in the Hilbert space).*

Proof. Let $v \in H$ and let $w \in N$. Then by the second axiom $d_N(v) \leq \|v - w\|$ and thus $d_N(v) \leq \mathrm{dist}(v, N)$.

Now let $v \in H$. If $d_N(v) = 1$, then $\mathrm{dist}(v, N) \geq 1$ by the previous paragraph, but since we are truncating the computation of distances at 1, this means that

$\operatorname{dist}(v, N) = 1$. Let us now assume that $d_N(v) < 1$. Consider now the set of statements $p(x)$ given by $d_N(x) = 0$, $\|x - v\| = d_N(v)$.

Claim. The type $p(x)$ is approximately satisfiable.

Let $\varepsilon > 0$. We want to show that there is a realization of the statements $d_N(x) \leq \varepsilon$, $d_N(v) \leq \|x - v\| + \varepsilon$. By the first axiom there is w such that $d_N(w) \leq \varepsilon$ and $d_N(v) \leq \|v - w\| + \varepsilon$.

Since $(\mathcal{H}, d)$ is $\aleph_0$-saturated, there is $w \in N$ such that $\|v - w\| = d_N(v)$ as we wanted. ☑

There are several ages that need to be considered. We fix $r \in [0, 1)$ and we consider the class $\mathcal{K}_r$ of all models of T_0 such that $d_N(0) = r$. Note that in all finite dimensional spaces in $\mathcal{K}_r$ we have at least a point v with $d_N(v) = 0$.

Notation 3. If $(\mathcal{H}_i, d_N^i) \models T_0$ for $i \in \{0, 1\}$, we write $(H_0, d_N^0) \subset (H_1, d_N^1)$ if $H_0 \subset H_1$ and $d_N^0 = d_N^1 \restriction_{H_0}$ (for each sort).

We will work in $\mathcal{K}_r$. We start with constructing free amalgamations:

Lemma 3.2. *Let* $(\mathcal{H}_0, d_N^0) \subset (\mathcal{H}_i, d_N^i)$ *where* $i = 1, 2$ *and* $H_1 \perp_{H_0} H_2$ *be Hilbert spaces with distance functions, all of them in* $\mathcal{K}_r$. *Let* $H_3 = \operatorname{span}\{H_1, H_2\}$ *and let*

$$d_N^3(v) = \min\Big\{\sqrt{d_N^1(P_{H_1}(v))^2 + \|P_{H_2\cap H_0^\perp}(v)\|^2},\ \sqrt{d_N^2(P_{H_2}(v))^2 + \|P_{H_1\cap H_0^\perp}(v)\|^2}\Big\}.$$

Then $(\mathcal{H}_i, d_N^i) \subset (\mathcal{H}_3, d_N^3)$ *for* $i = 1, 2$, *and* $(\mathcal{H}_3, d_N^3) \in \mathcal{K}_r$.

Proof. For arbitrary $v \in H_1$, $\sqrt{d_N^1(P_{H_1}(v))^2 + \|P_{H_2\cap H_0^\perp}(v)\|^2} = d_N^1(v)$. Since $(\mathcal{H}_0, d_N^0) \subset (\mathcal{H}_i, d_N^i)$ we also have $\sqrt{d_N^2(P_{H_2}(v))^2 + \|P_{H_1\cap H_0^\perp}(v)\|^2} = \sqrt{d_N^0(P_{H_0}(v))^2 + \|P_{H_1\cap H_0^\perp}(v)\|^2} \geq d_N^1(v)$. Similarly, for any $v \in H_2$, we have $d_N^3(v) = d_N^2(v)$.

Therefore $(\mathcal{H}_3, d_N^3) \supset (\mathcal{H}_i, d_N^i)$ for $i \in \{1, 2\}$. Now we have to prove that the function d_N^3 that we defined is indeed a distance function.

Geometrically, $d_N^3(v)$ takes the minimum of the distances of v to the selected black subsets of H_1 and H_2. That is, the random subset of the amalgamation of (H_1, d_N^1) and (H_2, d_N^2) is the union of the two random subsets. It is easy to check that $(\mathcal{H}_3, d_N^3) \models T_0$. Since each of (H_1, d_N^1), (H_2, d_N^2) belongs to $\mathcal{K}_r$, we have $d_N^1(0) = r = d_N^2(0)$ and thus $d_N^3(0) = r$. ☑

The class $\mathcal{K}_0$ also has the JEP: let $(\mathcal{H}_1, d_N^1), (\mathcal{H}_2, d_N^2)$ belong to $\mathcal{K}_0$ and assume that $\mathcal{H}_1 \perp \mathcal{H}_2$. Let $N_1 = \{v \in H_1 : d_N^1(v) = 0\}$ and let $N_2 = \{v \in H_2 :$

$d_N^2(v)=0\}$. Let $\mathcal{H}_3=\operatorname{span}(\mathcal{H}_1\cup\mathcal{H}_2)$ and let $N_3=N_1\cup N_2\subset H_3$ and finally, let $d_N^3(v)=\operatorname{dist}(v,N_3)$. Then (H_3,d_N^3) is a witnesses of the JEP in $\mathcal{K}_0$.

Lemma 3.3. *There is a model $(\mathcal{H},d_N)\models T_0$ with $(\mathcal{H},d_N)\in\mathcal{K}_0$ such that $\mathcal{H}$ is a $2n$-dimensional Hilbert space and there are orthonormal vectors $v_1,\dots,v_n\in H$, $u_1,\dots,u_n\in H$ such that $d_N((u_i+v_j)/2)=\sqrt{2}/2$ for $i\leq j$, $d_N(0)=0$ and $d_N((u_i+v_j)/2)=0$ for $i>j$.*

Proof. Let H be a Hilbert space of dimension $2n$, and fix some orthonormal basis $\langle v_1,\dots,v_n,u_1,\dots,u_n\rangle$ for H. Let $N=\{(u_i+v_j)/2 : i>j\}\cup\{0\}$ and let $d_N(x)=\operatorname{dist}(x,N)$. Then $d_N(0)=0$ and $d_N((u_i+v_j)/2)=0$ for $i>j$. Since $\|(u_i+v_j)/2-(u_k+v_j)/2\|=\sqrt{2}/2$ for $i\neq k$ and $\|(u_i+v_j)/2-0\|=\sqrt{2}/2$, we get that $d_N(u_i+v_j)=\sqrt{2}/2$ for $i\leq j$. ☑

Thus, existentially closed models of $T_0\cup\{d_N(0)=0\}$ will not be stable.

3.2. The model companion

3.2.1. Basic notations

We now provide the axioms of the model companion of $T_0\cup\{d_N(0)=0\}$.

Call T_N the theory of the structure built out of amalgamating all separable Hilbert spaces together with a distance function belonging to the age $\mathcal{K}_0$. Informally speaking, $T_N=Th(\varinjlim(\mathcal{K}_0))$. We show how to axiomatize T_N.

The idea for the axiomatization of this part follows the lines of Theorem 2.4 of [6]. There are however important differences in the behavior of algebraic closures and independence, due to the metric character of our examples.

3.2.2. An informal description of the axioms

We give a general description of the situation.

Let (M,d_N) in $\mathcal{K}_0$ be an existentially closed structure and take some extension $(M_1,d_N)\supset(M,d_N)$. Let $\bar{x}=(x_1,\dots,x_{n+k})$ be elements in $M_1\setminus M$ and let $z_1,\dots,z_{n+k}$ be their projections on M. Assume that for $i\leq n$ there are $\bar{y}=(y_1,\dots,y_n)$ in $M_1\setminus M$ that satisfy $d_N(x_i)=\|x_i-y_i\|$ and $d_N(y_i)=0$. Also assume that for $i>n$, the witnesses for the distances to the black points belong to M, that is, $d_N^2(x_i)=\|x_i-z_i\|^2+d_N^2(z_i)$ for $i>n$. Also, let us assume that all points in $\bar{x},\bar{y}$ live in a ball of radius L around the origin. Let $\bar{u}=(u_1,\dots,u_n)$ be the projection of $\bar{y}=(y_1,\dots,y_n)$ over M.

Let $\varphi(\bar{x},\bar{y},\bar{z},\bar{u})$ be a formula such that $\varphi(\bar{x},\bar{y},\bar{z},\bar{u})=0$ describes the values of the inner products between all the elements of the tuples, that is, it determines the (Hilbert space) geometric locus of the tuple $(\bar{x},\bar{y},\bar{z},\bar{u})$. The statement $\varphi(\bar{x},\bar{y},\bar{z},\bar{u})=0$ expresses the position of the potentially new points $\bar{x}$, $\bar{y}$ with respect to their projections into a model. Since $d_N(x_i)=\|x_i-y_i\|$

and $d_N(y_i)=0$, we have $\|x_i-y_j\|\geq\|x_i-y_i\|$ for $j\leq n$, $i\leq n$. Also, for $i>n$, $d_N^2(x_i)=\|x_i-z_i\|^2+d_N^2(z_i)$, and get $\|x_i-y_j\|^2\geq\|x_i-z_i\|^2+d_N^2(z_i)$ for $j\leq n$.

Note that as $(M_1,d_N)\supset(M,d_N)$, for all $z\in M$, $d_N^2(z)\leq\|z-y_i\|^2=\|z-u_i\|^2+\|y_i-u_i\|^2$ for $i\leq n$. We may also assume that there is a positive real η_φ such that $\|x_i-z_i\|\geq\eta_\varphi$ for $i\leq n+k$ and $\|y_i-u_i\|\geq\eta_\varphi$ for $i\leq n$.

We want to express that for any parameters $\bar z$, $\bar u$ in the structure if we can find realizations $\bar x$, $\bar y$ of $\varphi(\bar x,\bar y,\bar z,\bar u)=0$ such that for all w and $i\leq n$, $d_N^2(w)\leq\|w-u_i\|^2+\|u_i-y_i\|^2$, $\|x_i-y_i\|^2\leq\|x_i-z_i\|^2+d_N^2(z_i)$ for $i\leq n$, $\|x_i-y_j\|^2\geq\|x_i-z_j\|^2+d_N^2(z_j)$ for $i>n$ and $j\leq n$, then there are tuples $\bar x'$, $\bar y'$ such that $\varphi(\bar x',\bar y',\bar z,\bar u)=0$, $d_N(y_i')=0$, $d_N(x_i')=\|x_i'-y_i'\|$ for $i\leq n$ and $d_N^2(x_j)=\|x_j-z_j\|^2+d_N^2(z_j)$ for $j>n$.

That is, for any $\bar z$, $\bar u$ in the structure, if we can find realizations $\bar x$, $\bar y$ of the Hilbert space locus given by φ, and we prescribe "distances" d_N that do not clash with the d_N information we already had, in such a way that for $i\leq n$, the y_i's are black and are witnesses for the distance to the black set for the x_i's, and for $i>n$ the x_i's do not require new witnesses, then we can actually find arbitrarily close realizations, *with the prescribed distances.*

The only problem with this idea is that we do not have an implication in continuous logic. We replace the expression "$p\to q$" by a sequence of approximations indexed by ε.

3.2.3. The axioms of T_N

Notation 4. Let $\bar z,\bar u$ be tuples in M and let $x\in M_1$. By $P_{\mathrm{span}(\bar z\bar u)}(x)$ we mean the projection of x in the space spanned by $(\bar z,\bar u)$.

For fixed $\varepsilon\in(0,1)$, let $f\colon[0,1]\to[0,1]$ be a continuous function such that whenever $\varphi(\bar t)<f(\varepsilon)$ and $\varphi(\bar t')=0$ ($\bar t$ and $\bar t'$ tuples in our models), then

(a) $\|P_{\mathrm{span}(\bar z\bar u)}(x_i)-z_i\|<\varepsilon$,

(b) $\|P_{\mathrm{span}(\bar z\bar u)}(y_i)-u_i\|<\varepsilon$,

(c) $|\|t_i-t_j\|-\|t_i'-t_j'\||<\varepsilon$ where $\bar t$ is the concatenation of $\bar x,\bar y,\bar z,\bar u$.

Choosing ε small enough, we may assume that

(d) $\|x_i-P_{\mathrm{span}(\bar z\bar u)}(x_i)\|\geq\eta_\varphi/2$ for $i\leq n+k$,

(e) $\|y_i-P_{\mathrm{span}(\bar z\bar u)}(y_i)\|\geq\eta_\varphi/2$ for $i\leq n$.

Let $\delta = 2\sqrt{\varepsilon(L+2)}$ and consider the following axiom $\psi_{\varphi,\varepsilon}$ (which we write as a positive bounded formula for clarity) where the quantifiers range over a ball of radius $L+1$:

$$\forall\bar{z}\forall\bar{u}\Big(\exists\bar{x}\exists\bar{y}\Big[\psi_1(\bar{z},\bar{u},\bar{x},\bar{y})\wedge\exists w\psi_2(w,\bar{u},\bar{y})\wedge\psi_3(\bar{z},\bar{x},\bar{y})\wedge\psi_4(\bar{x},\bar{y})\wedge\psi_5(\bar{z},\bar{x},\bar{y})\Big]$$

$$\Longrightarrow_c$$

$$\exists\bar{x}'\exists\bar{y}'\Big[\theta_1(\bar{z},\bar{u},\bar{x}',\bar{y}')\wedge\theta_2(\bar{y}')\wedge\theta_3(\bar{x}',\bar{y}')\wedge\theta_4(\bar{x}',\bar{z})\Big]\Big),$$

where the components are:

- $\psi_1(\bar{z},\bar{u},\bar{x},\bar{y}) : \varphi(\bar{x},\bar{y},\bar{z},\bar{u}) \geq f(\varepsilon)$,
- $\psi_2(w,\bar{u},\bar{y}) : \bigvee_{i\leq n}(d_N^2(w) \geq \|w-u_i\|^2 + \|y_i-u_i\|^2+\varepsilon^2)$,
- $\psi_3(\bar{z},\bar{x},\bar{y}) : \bigvee_{i>n,j\leq n}(\|x_i-y_j\|^2 \leq \|x_i-z_i\|^2 + d_N^2(z_i)+\varepsilon^2)$,
- $\psi_4(\bar{x},\bar{y}) : \bigvee_{i,j\leq n,j\neq i}(\|x_i-y_j\| \leq \|x_i-y_i\|-\varepsilon)$,
- $\psi_5(\bar{z},\bar{x},\bar{y}) : \bigvee_{i\leq n}(\|x_i-z_i\|^2 + d_N^2(z_i) \leq \|x_i-y_i\|^2-\varepsilon^2)$,
- $\theta_1(\bar{z},\bar{u},\bar{x}',\bar{y}') : \varphi(\bar{x}',\bar{y}',\bar{z},\bar{u}) \leq f(\varepsilon)$,
- $\theta_2(\bar{y}') : \bigwedge_{i\leq n} d_N(y_i') \leq \delta)$,
- $\theta_3(\bar{x}',\bar{y}') : \bigwedge_{i\leq n} |d_N(x_i') - \|x_i'-y_i'\|| \leq 2\delta)$,
- $\theta_4(\bar{x}',\bar{z}) : \bigwedge_{i>n} |d_N^2(x_i) - \|x_i-z_i\|^2 - d_N^2(z_i)| \leq 4\delta L$, and
- $\Longrightarrow_c$ abbreviates implication in continuous logic[1].

The axiom then has the usual form of "existential closure" but in the sense described above. This is exactly what we need for the model companion we are looking for.

Let T_N be the theory T_0 together with this scheme of axioms $\psi_{\varphi,\varepsilon}$ indexed by all Hilbert space geometric locus formulas $\varphi(\bar{x},\bar{y},\bar{z},\bar{u}) = 0$ and $\varepsilon \in (0,1)\cap\mathbb{Q}$. The radius of the ball that contains all elements, L, as well as n and k are determined from the configuration of points described by the formula $\varphi(\bar{x},\bar{y},\bar{z},\bar{u}) = 0$.

[1]The exact formula would be almost unparseable: $\forall\bar{z}\forall\bar{u}\Big(\forall\bar{x}\forall\bar{y}\varphi(\bar{x},\bar{y},\bar{z},\bar{u}) \geq f(\varepsilon) \vee \exists w \bigvee_{i\leq n}(d_N^2(w) \geq \|w-u_i\|^2+\|y_i-u_i\|^2+\varepsilon^2) \vee \bigvee_{i>n,j\leq n}(\|x_i-y_j\|^2 \leq \|x_i-z_i\|^2 + d_N^2(z_i)+\varepsilon^2) \vee$
$\bigvee_{i,j\leq n,j\neq i}(\|x_i-y_j\| \leq \|x_i-y_i\|-\varepsilon) \vee \bigvee_{i\leq n}(\|x_i-z_i\|^2+d_N^2(z_i) \leq \|x_i-y_i\|^2-\varepsilon^2)$
$\vee$
$\vee\exists\bar{x}\exists\bar{y}\big[(\varphi(\bar{x},\bar{y},\bar{z},\bar{u}) \leq f(\varepsilon) \wedge \bigwedge_{i\leq n} d_N(y_i) \leq \delta) \wedge \bigwedge_{i\leq n}|d_N(x_i) - \|x_i-y_i\|| \leq 2\delta) \wedge \bigwedge_{i>n}|d_N^2(x_i) - \|x_i-z_i\|^2 - d_N^2(z_i)| \leq 4\delta L\big]\Big)$

3.2.4. Existentially closed models of T_0

Theorem 3.4. *Assume that $(M, d_N) \models T_0$ is existentially closed. Then we also have $(M, d_N) \models T_N$.*

***Proof*.** Fix $\varepsilon > 0$ and φ as above. Let $\bar{z} \in M^{n+k}$, $\bar{u} \in M^n$ and assume that there are $\bar{x}, \bar{y}$ with $\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{u}) < f(\varepsilon)$ and $d_N^2(w) < \|w-u_i\|^2+\|y_i-u_i\|^2+\varepsilon^2$ for all $w \in M$, $\|x_i-y_i\|^2 < \|x_i-z_i\|^2+d_N^2(z_i)+\varepsilon^2$ for $i \leq n$, $\|x_i-y_j\| > \|x_i-y_i\|-\varepsilon$ for $i, j \leq n$, $i \neq j$, $\|x_i - y_j\|^2 > \|x_i - z_i\|^2 + d_N^2(z_j) + \varepsilon^2$ for $i > n$, $j \leq n$. Let $\varepsilon' < \varepsilon$ be such that $\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{u}) < f(\varepsilon')$ and

(f) $d_N^2(w) < \|w - u_i\|^2 + \|y_i - u_i\|^2 + \varepsilon'^2$ for all $w \in M$,

(g1) $\|x_i - y_i\|^2 > \|x_i - z_i\|^2 + d_N(z_i) + \varepsilon'^2$ for $i \leq n$,

(g2) $\|x_i - y_j\| > \|x_i - y_i\| - \varepsilon'$ for $i, j \leq n$, $i \neq j$,

(h) $\|x_i - y_j\|^2 \geq \|x_i - z_i\|^2 + d_N^2(z_i) + \varepsilon'^2$ for $i > n$, $j \leq n$.

We construct an extension $(H, d_N) \supset (M, d_N)$ where the conclusion of the axiom indexed by ε' holds. Since (M, d_N) is existentially closed and the conclusion of the axiom is true for (H, d_N) replacing ε for $\varepsilon' < \varepsilon$, then the conclusion of the axiom indexed by ε will hold for (M, d_N).

So let $H \supset M$ be such that $\dim(H \cap M^\perp) = \infty$. Let $a_1, \ldots, a_{n+k}$ and $c_1, \ldots, c_n \in H$ be such that $\mathrm{tp}(\bar{a}, \bar{c}/\bar{z}\bar{u}) = \mathrm{tp}(\bar{x}, \bar{y}/\bar{z}\bar{u})$ and $\bar{a}\bar{c} \mathop{\smile\!\!\!\!\!\!|}_{\bar{z}\bar{u}} M$. We can write $a_i = a_i' + z_i'$ and $c_i = c_i' + u_i'$ for some $z_i', u_i' \in M$ and $a_i', c_i' \in M^\perp$. By (d) and (e) $\|a_i'\| \geq \eta/2$ for $i \leq n+k$ and $\|c_i'\| \geq \eta/2$ for $i \leq n$. Now let $\hat{c}_i = c_i' + u_i' + \delta' c_i'/\|c_i'\|$, where $\delta' = \sqrt{2\varepsilon'(L+2)}$.

Let the black points in H be the ones from M plus the points $\hat{c}_1, \ldots, \hat{c}_n$. Now we check that the conclusion of the axiom $\psi_{\varphi,\varepsilon'}$ holds.

(1) $\varphi(\bar{a}, \bar{c}, \bar{z}, \bar{u}) \leq f(\varepsilon')$ since $\mathrm{tp}(\bar{a}, \bar{c}/\bar{z}\bar{u}) = \mathrm{tp}(\bar{x}, \bar{y}/\bar{z}\bar{u})$.

(2) Since $\|c_i - \hat{c}_i\| \leq \delta'$ and $\hat{c}_i$ is black we have $d_N(c_i) \leq \delta'$.

(3) We check that the distance from a_i to the black set is as prescribed for $i \leq n$. $d_N(a_i) \leq \|a_i - \hat{c}_i\| \leq \|a_i - c_i\| + \delta'$ for $i \leq n$.

Also, for $i \neq j, i, j \leq n$, using $(g2)$ we prove $\|a_i - \hat{c}_j\| \geq \|a_i - c_j\| - \delta' \geq \|a_i - c_i\| - \varepsilon' - \delta' \geq \|a_i - c_i\| - 2\delta'$. Finally by (a) $\|a_i - P_M(a_i)\|^2 + d_N^2(P_M(a_i)) \geq (\|a_i - z_i\| - \varepsilon')^2 + (d_N(z_i) - \varepsilon')^2 \geq \|a_i - z_i\|^2 - 2L\varepsilon' + \varepsilon'^2 + d_N^2(z_i) - 2\varepsilon' + \varepsilon'^2$ and by $(g1)$, we get $\|a_i - z_i\|^2 - 2L\varepsilon' + \varepsilon'^2 + d_N^2(z_i) - 2\varepsilon' + \varepsilon'^2 \geq \|a_i - c_i\|^2 - 2L\varepsilon' - 2\varepsilon' \geq \|a_i - c_i\|^2 - 4\delta'^2$.

(4) We check that $d_N(a_i)$ is as desired for $i > n$. Clearly $\|a_j - \hat{c}_i\| \geq \|a_j - c_i\| - \delta'$, so $\|a_j - \hat{c}_i\|^2 \geq \|a_j - c_i\|^2 + \delta'^2 - 2\delta' 2L$ and by (h) we get $\|a_j - c_i\|^2 + \delta'^2 - 4\delta' L \geq \|a_j - z_j\|^2 + d_N^2(z_j) - 4\delta' L - \varepsilon'^2 + \delta'^2 \geq \|a_j - z_j\|^2 + d_N^2(z_j) - 4\delta' L$.

It remains to show that $(M, d_N) \subset (H, d_N)$, i.e., the function d_N on H extends the function d_N on M . Since we added the black points in the ball of radius $L+1$, we only have to check that for any $w \in M$ in the ball of radius $L+2$, $d_N^2(w) \leq \|w - \hat{c}_i\|^2 = \|w - u_i'\|^2 + \|c_i' + \delta'(c_i'/\|c_i'\|)\|^2$.

But by (f) $d_N^2(w) \leq \|w-u_i\|^2 + \|c_i - u_i\|^2 + \varepsilon'^2$, so it suffices to show that

$$\|w-u_i\|^2 + \|c_i - u_i\|^2 + \varepsilon'^2 \leq \|w-u_i'\|^2 + \|c_i'\|^2 + 2\delta'\|c_i'\| + \delta'^2$$

By (a) $\|w-u_i'\|^2 \geq (\|w-u_i\| - \varepsilon')^2$ and it is enough to prove that

$$\|w-u_i\|^2 + \|c_i - u_i\|^2 + \varepsilon'^2 \leq (\|w-u_i\| - \varepsilon')^2 + \|c_i'\|^2 + 2\delta'\|c_i'\| + \delta'^2.$$

But $(\|w-u_i\|-\varepsilon')^2+\|c_i'\|^2+2\delta'\|c_i'\|+\delta'^2 = \|w-u_i\|^2-2\varepsilon'\|w-u_i\|+\varepsilon'^2+\|c_i'\|^2+2\delta'\|c_i'\|+\delta'^2$ and $\|c_i-u_i\|^2 \leq \|c_i-u_i'\|^2+2\varepsilon'\|c_i-u_i'\|+\varepsilon'^2 = \|c_i'\|^2+2\varepsilon'\|c_i'\|+\varepsilon'^2$. Thus, after simplifying, we only need to check $2\varepsilon'\|w-u_i\| + \varepsilon'^2 \leq \delta'^2$ which is true since $2\varepsilon'\|w-u_i\| + \varepsilon'^2 \leq 2\varepsilon'(2L+2) + \varepsilon'^2 \leq 4\varepsilon'(L+2)$. ☑

Theorem 3.5. *Assume that* $(M, d_N) \models T_N$. *Then* (M, d_N) *is existentially closed.*

Proof. Let $(H, d_N) \supset (M, d_N)$ and assume that (H, d_N) is $\aleph_0$-saturated. Let $\psi(\bar{x}, \bar{v})$ be a quantifier free L_N-formula, where $\bar{x} = (x_1, \dots x_{n+k})$ and $\bar{v} = (v_1, \dots v_l)$. Suppose that there are $a_1, \dots, a_{n+k} \in H \setminus M$ and $e_1, \dots e_l \in M$ such that $(H, d_N) \models \psi(\bar{a}, \bar{e}) = 0$. After enlarging the formula ψ if necessary, we may assume that $\psi(\bar{x}, \bar{v}) = 0$ describes the values of $d_N(x_i)$ for $i \leq n+k$, the values of $d_N(v_j)$ for $j \leq l$ and the inner products between those elements. We may assume that for $i \leq n$ there is $\rho > 0$ such that $d_N(a_i) - d(a_i, z) \geq 2\rho$ for all $z \in M$ with $d_N(z) \leq \rho$. Since (H, d_N) is $\aleph_0$-saturated, there are $c_1, \dots c_n \in H$ such that $d_N(a_i) = \|a_i - c_i\|$ and $d_N(c_i) = 0$. Then $d(c_i, M) \geq \rho$. Fix $\varepsilon > 0$, $\varepsilon < \rho, 1$. We may also assume that for $i > n$, $|d_N^2(a_i) - \|a_i - P_M(a_i)\|^2 - d_N^2(P_M(a_i))| \leq \varepsilon/2$. Also, assume that all points mentioned so far live in a ball of radius L around the origin. Let $b_1, \dots, b_{n+k} \in M$ be the projections of $a_1, \dots, a_{n+k}$ onto M and let $d_1, \dots, d_n \in M$ be the projections of $c_1, \dots, c_n$ onto M. Let $\varphi(\bar{x}, \bar{y}, \bar{z}, \bar{u}) = 0$ be an L-statement that describes the inner products between the elements listed and such that $\varphi(\bar{a}, \bar{c}, \bar{b}, \bar{d}) = 0$. Using the axioms we can find $\bar{a}'$, $\bar{c}'$ in M such that $\varphi(\bar{a}', \bar{c}', \bar{b}, \bar{d}) \leq f(\varepsilon)$, $d_N(c_i') \leq \delta$ for $i \leq n$, $|d_N(a_i') - \|a_i' - c_i'\|| \leq \delta$ for $i \leq n$ and $|d_N^2(a_i) - \|a_i - b_i\|^2 - d_N^2(b_i)| \leq 4L\delta$, where $\delta = \sqrt{2\varepsilon(L+2)}$. Since $\varepsilon > 0$ was arbitrary we get $(M, d_N) \models \inf_{x_1} \dots \inf_{x_{n+k}} \psi(\bar{x}, \bar{v}) = 0$. ☑

4. Model theoretic analysis of T_N

We prove two theorems in this section about the theory T_N:

- T_N is not NTP_2 (so T_N is not simple).

- T_N is $NSOP_1$. Therefore, in spite of having a tree property, our theory is still "close to being simple" in the precise sense of not having the SOP_1 tree property.

These results place T_N in a very interesting situation in the stability hierarchy for continuous logic.

Notation 5. We write tp for types of elements in the language L and tp_N for types of elements in the language L_N. Similarly we denote by acl_N the algebraic closure in the language L_N and by acl the algebraic closure for pure Hilbert spaces. Recall that for a set A, $\mathrm{acl}(A) = \mathrm{dcl}(A)$, and this corresponds to the closure of the space spanned by A (Fact 1.1.1).

Observation 4.1. The theory T_N does not have elimination of quantifiers. We use the characterization of quantifier elimination given in Theorem 8.4.1 from [8]. Let H_1 be a two dimensional Hilbert space, let $\{u_1, u_2\}$ be an orthonormal basis for H_1 and let $N_1 = \{0, u_0 + \frac{1}{4}u_1\}$ and let $d_N^1(x) = \min\{1, \mathrm{dist}(x, N_1)\}$. Then $(H_1, d_N^1) \models T_0$. Let $a = u_0$, $b = u_0 - \frac{1}{4}u_1$ and $c = u_0 + \frac{1}{4}u_1$. Note that $d_N^1(b) = \frac{1}{2}$. Let $(H_1', d_N^1) \supset (H_1, d_N^1)$ be existentially closed. Now let H_2 be an infinite dimensional separable Hilbert space and let $\{v_i : i \in \omega\}$ be an orthonormal basis. Let $N_2 = \{x \in H : \|x - v_1\| = \frac{1}{4}, P_{\mathrm{span}(v_1)}(x) = v_1\} \cup \{0\}$ and let $d_N^2(x) = \min\{1, \mathrm{dist}(x, N_2)\}$. Let $(H_2', d_N^2) \supset (H_2, d_N^2)$ be existentially closed. Then $(\mathrm{span}(a), d_N^1 \restriction_{\mathrm{span}(a)}) \cong (\mathrm{span}(v_1), d_N^2 \restriction_{\mathrm{span}(v_1)})$ and they can be identified say by a function F. But (H_1', d_N^1) and (H_2', d_N^2) cannot be amalgamated over this common substructure: if they could, then we would have $\mathrm{dist}(F(b), v_1 + \frac{1}{4}v_i) = \mathrm{dist}(b, v_1 + \frac{1}{4}v_i) < \frac{1}{2}$ for some $i > 1$ and thus $d_N^1(b) < \frac{1}{2}$, a contradiction.

In this case, the main reason for this failure of amalgamation resides in the fact that $(\mathrm{span}(a), d_N^1 \restriction_{\mathrm{span}(a)}) \cong (\mathrm{span}(v_1), d_N^2 \restriction_{\mathrm{span}(v_1)})$ is not a model of T_0: informally, the distance values around v_1 are determined by an "external attractor" (the black point $u_0 + \frac{1}{4}u_1$ or the black ring orthogonal to v_1 at distance $\frac{1}{4}$) that the subspace $(\mathrm{span}(a), d_N^1 \restriction_{\mathrm{span}(a)})$ simply cannot see. This violates Axiom (1) in the description of T_0. This "noise external to the substructure" accounts for the failure of amalgamation, and ultimately for the lack of quantifier elimination.

In [6, Corollary 2.6], the authors show that the algebraic closure of the expansion of a simple structure with a generic subset corresponds to the algebraic in the original language. However, in our setting, the new algebraic closure $\mathrm{acl}_N(X)$ does not agree with the old algebraic closure $\mathrm{acl}(X)$.

Observation 4.2. The previous construction shows that acl_N does not coincide with acl. Indeed, $c \in \mathrm{acl}_N(a) \setminus \mathrm{acl}(a)$ - the set of solutions of the type $\mathrm{tp}_N(c/a)$ is $\{c\}$, but $c \notin dcl(a)$ as $c \notin \mathrm{span}(a)$.

However, models of the basic theory T_0 are L_N-algebraically closed. The proof is similar to [6, Proposition 2.6(3)].

Lemma 4.3. *Let $(M, d_N) \models T_N$ and let $A \subset M$ be such that $A = \mathrm{dcl}(A)$ and $(A, d_N \restriction_A) \models T_0$. Let $a \in M$. Then $a \in \mathrm{acl}_N(A)$ if and only if $a \in A$.*

Proof. Assume $a \notin A$. We will show that $a \notin \mathrm{acl}_N(A)$. Let $a' \models \mathrm{tp}(a/A)$ be such that $a' \downarrow_A M$. Let (M', d_N) be an isomorphic copy of (M, d_N) over A through $f : M \to_A M'$ such that $f(a) = a'$. We may assume that $M' \downarrow_A M$. Since $(A, d_N \restriction_A)$ is an amalgamation base (i.e. for any two extensions $(A^i, d_N \restriction_{A^i}$) of $(A, d_N \restriction_A)$ $(i = 1, 2)$ there exists an amalgam over $(A, d_N \restriction_A))$, $(N, d_N) = (M \oplus_A M', d_N) \models T_0$. Let $(N', d_N) \supset (N, d_N)$ be an existentially closed structure. Then $\mathrm{tp}_N(a/A) = \mathrm{tp}_N(a'/A)$ and therefore $a \notin \mathrm{acl}_N(A)$. ☑

As T_N is model complete, the types in the extended language are determined by their existential formulas, i.e. formulas of the form $\inf_{\bar{y}} \varphi(\bar{y}, \bar{x}) = 0$.

Another difference with the work of Chatzidakis and Pillay is that the analogue to [6, Proposition 2.5] no longer holds. Let a, b, c be as in Observation 4.2; notice that $(\mathrm{span}(a), d_N \restriction_{\mathrm{span}(a)}) \cong (\mathrm{span}(v_1), d_N \restriction_{\mathrm{span}(v_1)})$. However, $(H_1', d_N, a) \not\equiv (H_2', d_N, v_1)$. Instead, we can show the following weaker version of that proposition.

Proposition 4.4. *Let (M, d_N) and (N, d_N) be models of T_N and let A be a common subset of M and N such that $(\mathrm{span}(A), d_N \restriction_{\mathrm{span}(A)}) \models T_0$. Then*

$$(M, d_N) \equiv_A (N, d_N).$$

Proof. Assume that $M \cap N = \mathrm{span}(A)$. Since $(\mathrm{span}(A), d_N \restriction_{\mathrm{span}(A)}) \models T_0$, it is an amalgamation base and therefore we may consider the free amalgam $(M \oplus_{\mathrm{span}(A)} N, d_N)$ of (M, d_N) and (N, d_N) over $(\mathrm{span}(A), d_N \restriction_{\mathrm{span}(A)})$. Let now (E, d_N) be a model of T_N extending $(M \oplus_{\mathrm{span}(A)} N, d_N)$. By the model completeness of T_N, we have that $(M, d_N) \prec (E, d_N)$ and $(N, d_N) \prec (E, d_N)$ and thus $(M, d_N) \equiv_A (N, d_N)$. ☑

4.1. Generic independence

In this section we define an abstract notion of independence and study its properties.

Fix $(\mathcal{U}, d_N) \models T_N$ be a κ-universal domain.

Definition 4.5. Let $A, B, C \subset \mathcal{U}$ be small sets. We say that A is $*$-independent from B over C and write $A \downarrow^*_C B$ if $\mathrm{acl}_N(A \cup C)$ is independent (in the sense of Hilbert spaces) from $\mathrm{acl}_N(C \cup B)$ over $\mathrm{acl}_N(C)$. That is, $A \downarrow^*_C B$ if for all $a \in \mathrm{acl}_N(A \cup C)$, $P_{\overline{B \cup C}}(a) = P_{\overline{C}}(a)$, where $\overline{B \cup C} = \mathrm{acl}_N(C \cup B)$ and $\overline{C} = \mathrm{acl}_N(C)$.

Proposition 4.6. *The relation $\mathop{\smile\!\!\!\!\!\!|}^*$ satisfies the following properties (here A, B, etc., are any small subsets of $\mathcal{U}$):*

(1) Invariance under automorphisms of $\mathcal{U}$.

(2) Symmetry: $A \mathop{\smile\!\!\!\!\!\!|}^*_C B \iff B \mathop{\smile\!\!\!\!\!\!|}^*_C A$.

(3) Transitivity: $A \mathop{\smile\!\!\!\!\!\!|}^*_C BD$ if and only if $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}^*_{BC} D$.

(4) Countable Character: $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ if and only$A_0 \mathop{\smile\!\!\!\!\!\!|}^*_C B$ for all $A_0 \subset A$ countable.

(5) Local Character: If $\bar{a}$ is any finite tuple, then there is countable $B_0 \subseteq B$ such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_{B_0} B$.

(6) Extension property over models of T_0. If $(C, d_N \restriction_C) \models T_0$, then we can find A' such that $\mathrm{tp}_N(A/C) = \mathrm{tp}_N(A'/C)$ and $A' \mathop{\smile\!\!\!\!\!\!|}^*_C B$.

(7) Existence over models: $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_M M$ for any $\bar{a}$.

(8) Monotonicity: $\bar{a}\bar{a}' \mathop{\smile\!\!\!\!\!\!|}^*_M \bar{b}\bar{b}'$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}$.

Proof. (1) This is clear.

(2) This follows from the fact that independence in Hilbert spaces satisfies Symmetry (see Proposition 1.1).

(3) This follows from the fact that independence in Hilbert spaces satisfies Transitivity (see Proposition 1.1).

(4) Clearly $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ implies that $A_0 \mathop{\smile\!\!\!\!\!\!|}^*_C B$ for all $A_0 \subset A$ countable. On the other hand, assume $A_0 \mathop{\smile\!\!\!\!\!\!|}^*_C B$ for all $A_0 \subset A$ countable. Let $a \in \mathrm{acl}_N(AC)$, then $a \in \mathrm{acl}_N(A_0C)$ for some A_0 countable. Since $A_0 \mathop{\smile\!\!\!\!\!\!|}^*_C B$ we have $P_{\overline{B \cup C}}(a) = P_{\overline{C}}(a)$. Thus $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$.

(5) Local Character: let $\bar{a}$ be a finite tuple. Since independence in Hilbert spaces satisfies local character, there is $B_1 \subseteq \mathrm{acl}_N(B)$ countable such that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_{B_1} B$. Now let $B_0 \subseteq B$ be countable such that $\mathrm{acl}_N(B_0) \supset B_1$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}^*_{B_0} B$.

(6) Let C be such that $(C, d_N \restriction_C) \models T_0$. Let $D \supset A \cup C$ be such that $(D, d_N \restriction_D) \models T_0$ and let $E \supset B \cup C$ be such that $(E, d_N \restriction_E) \models T_0$. Changing D for another set D' with $\mathrm{tp}_N(D'/C) = \mathrm{tp}_N(D/C)$, we may assume that the space generated by $D' \cup E$ is the free amalgamation of D' and E over C. By lemma 4.3 D', E are algebraically closed and $D' \mathop{\smile\!\!\!\!\!\!|}^*_C B$.

(7) This follows from the definition of $*$-independence.

(8) This follows from the definition of $*$-independence and transitivity for independence in Hilbert spaces.

☑

Therefore we have a natural independence notion that satisfies many good properties, but not enough to guarantee the simplicity of T_N.

We will show below that the theory T_N has both TP_2 and $NSOP_1$. This places it in an interesting area of the stability hierarchy for continuous model theory: while having the tree property TP_2 and therefore lacking the good properties of NTP_2 theories, it still has a quite well-behaved independence notion $\mathop{\smile\hskip-0.9em\lower0.5ex\hbox{$\mid$}}^*$, good enough to guarantee that it does not have the SOP_1 tree property. Therefore, although the theory is not simple, it is reasonably close to this family of theories.

4.2. T_N has the tree property TP_2

Theorem 4.7. *The theory T_N has the tree property TP_2.*

Proof. We will construct a complete submodel $M \models T_0$ of the monster model, of density character $2^{\aleph_0}$, and a quantifier free formula $\varphi(x; y, z)$ that witnesses TP_2 inside M. Since this model can be embedded in the monster model of T_N preserving the distance to black points, this will show that T_N has TP_2.

We fix some orthonormal basis of M, listed as

$$\{b_i|i < \omega\} \cup \{c_{n,i}|n, i < \omega\} \cup \{a_f|f : \omega \to \omega\}.$$

Also let the "black points" of M consist of the set

$$N = \{a_f + b_n + (1/2)c_{n,f(n)} \mid n < \omega, f : \omega \to \omega\} \cup \{0\}$$

and, as usual, define $d_N(x)$ as the distance from x to N. This is a model of T_0 and thus a submodel of the monster.

Let $\varphi(x, y, z) = \max\{1 - d_N(x + y + (1/2)z), d_N(x + y - (1/2)z)\}$.

Claim 1. For each i, the conditions $\{\varphi(x, b_i, c_{i,j}) = 0 : j \in \omega\}$ are 2-inconsistent.

Assume otherwise, so we can find a (in an extension of M) such that $d_N(a+b_i + (1/2)c_{i,j}) = 0$ and $d_N(a + b_i - (1/2)c_{i,l}) = 1$ for some $j < l$. But then $d(a + b_i + (1/2)c_{i,j}, a + b_i - (1/2)c_{i,l}) = d((1/2)c_{i,j}, -(1/2)c_{i,l}) = \sqrt{2}/2 < 1$. Since $a+b_i+(1/2)c_{i,j}$ is a black point, we get that $d_N(a+b_i-(1/2)c_{i,l}) \leq \sqrt{2}/2$ a contradiction.

Claim 2. For each f the conditions $\{\varphi(x, b_i, c_{i,f(i)}) = 0 : i \in \omega\}$ are consistent.

Indeed fix f and consider a_f, then by construction $d_N(a_f+b_n+(1/2)c_{n,f(n)}) = 0$ and $d(a_f + b_n - (1/2)c_{n,f(n)}, a_f + b_n + (1/2)c_{n,f(n)}) = 1$, so $d_N(a_f + b_n - (1/2)c_{n,f(n)}) \leq 1$.

Now we check the distance to the other points in N. It is easy to see that $d(a_f + b_n - (1/2)c_{n,f(n)}, a_f + b_m + (1/2)c_{m,f(m)}) > 1$ for $m \neq n$, $d(a_f + b_n - (1/2)c_{n,f(n)}, a_g + b_k + (1/2)c_{k,g(k)}) > 1$ for $g \neq f$ and all indexes k. Finally, $d(a_f+b_n-(1/2)c_{n,f(n)}, 0) > 1$. This shows that a_f is a witness for the claim. ☑

This stands in sharp contrast with respect to the result by Chatzidakis and Pillay in the (discrete) first order case. The existence of these incompatible types is rendered possible here by the presence of "euclidean" interactions between the elements of the basis chosen.

So far we have two kinds of expansions of Hilbert spaces by predicates: either they remain stable (as in the case of the distance to a Hilbert subspace as in the previous section) or they are not even TP_2.

4.3. T_N and the property $NSOP_1$

Chernikov and Ramsey have proved that whenever a first order discrete theory has an (abstract) independence relation that satisfies the following properties (for arbitrary models and tuples), then the theory satisfies the $NSOP_1$ property (see [7, Prop. 5.3]).

- *Strong finite character:* whenever $\bar{a}$ depends on $\bar{b}$ over M, there is a formula $\varphi(x, \bar{b}, \bar{m}) \in \mathrm{tp}(\bar{a}/\bar{b}M)$ such that every $\bar{a}' \models \varphi(\bar{x}, \bar{b}, \bar{m})$ depends on $\bar{b}$ over M.
- *Existence over models:* $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M M$ for any $\bar{a}$.
- *Monotonicity:* $\bar{a}\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}\bar{b}'$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}$.
- *Symmetry:* $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M \bar{b} \iff \bar{b} \mathop{\smile\!\!\!\!\!\!|}_M \bar{a}$.
- *Independent amalgamation:* $\bar{c}_0 \mathop{\smile\!\!\!\!\!\!|}_M \bar{c}_1$, $\bar{b}_0 \mathop{\smile\!\!\!\!\!\!|}_M \bar{c}_0$, $\bar{b}_1 \mathop{\smile\!\!\!\!\!\!|}_M \bar{c}_1$, $\bar{b}_0 \equiv_M \bar{b}_1$ implies there exists $\bar{b}$ with $\bar{b} \equiv_{\bar{c}_0 M} \bar{b}_0$, $\bar{b} \equiv_{\bar{c}_1 M} \bar{b}_1$.

We prove next that in T_N, $\mathop{\smile\!\!\!\!\!\!|}^*$ satisfies analogues of these five properties, we may thereby conclude that T_N can be regarded (following an analogy that has been fruitful for other, lower, levels of the stability hierarchy in the literature) as a $NSOP_1$ continuous theory.

In what remains of the paper, we prove that T_N satisfies these properties.

We focus our efforts in *strong finite character* and *independent amalgamation*, the other properties were proved in Proposition 4.6.

We need the following setting:

Let $\mathbb{M}$ be the monster model of T_N and $A \subset M$. Fix $\mathfrak{A}$ with $A \subset \mathfrak{A} \subset \mathbb{M}$ be such that $\mathfrak{A} \models T_0$ and let $\bar{a} = (a_0, \dots, a_n) \in M$. We say that $(\bar{a}, A, \mathfrak{B})$ is minimal if $\mathrm{tp}(\mathfrak{B}/A) = \mathrm{tp}(\mathfrak{A}/A)$ and for all $\bar{b} \in \mathbb{M}$, if $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{a}/A)$ then

$$\|P_{\mathfrak{B}}(b_0)\| + \cdots + \|P_{\mathfrak{B}}(b_n)\| \geq \|P_{\mathfrak{B}}(a_0)\| + \cdots + \|P_{\mathfrak{B}}(a_n)\|.$$

By compactness, for all $p \in S(A)$ there is a minimal $(\bar{a}, A, \mathfrak{B})$ such that $\bar{a} \models p$.

Now let $\mathrm{cl}_0(A)$ be the set of all x such that for some minimal $(\bar{a}, A, \mathfrak{B})$, $x = P_{\mathfrak{B}}(a_0)$ (the first coordinate of $\bar{a}$).

Lemma 4.8. *If* $\mathrm{tp}(\mathfrak{B}/A) = \mathrm{tp}(\mathfrak{A}/A)$ *and* $x \in \mathrm{cl}_0(A)$ *then* $x \in \mathfrak{B}$.

Proof. Toward a contradiction, assume that $x \notin \mathfrak{B}$. Let $\mathfrak{C}$ and $\bar{a} = (a_1, \dots, a_n)$ witness $x \in \mathrm{cl}_0(A)$, i.e. $(\bar{a}, A, \mathfrak{C})$ is minimal and $x = P_{\mathfrak{C}}(a_1)$. We may choose the previous so that $\bar{a} \downarrow_{\mathfrak{C}} \mathfrak{B}$. Then each a_i is orthogonal to $\mathfrak{B}$ over $\mathfrak{C}$, so

$$a_i - P_{\mathfrak{C}}(a_i) \perp \mathfrak{B},$$

then $0 = P_{\mathfrak{B}}(a_i - P_{\mathfrak{C}}(a_i)) = P_{\mathfrak{B}}(a_i) - P_{\mathfrak{B}}(P_{\mathfrak{C}}(a_i))$, thereby

$$P_{\mathfrak{B}}(a_i) = P_{\mathfrak{B}}(P_{\mathfrak{C}}(a_i)).$$

Thus

$$\|P_{\mathfrak{B}}(a_i)\| \leq \|P_{\mathfrak{C}}(a_i)\| \text{ for every } i. \tag{1}$$

And since $x = P_{\mathfrak{C}}(a_1) \notin \mathfrak{B}$, we have

$$\|P_{\mathfrak{B}}(a_1)\| = \|P_{\mathfrak{B}}(P_{\mathfrak{C}}(a_1))\| = \|P_{\mathfrak{B}}(x)\| < \|x\|. \tag{2}$$

Now choose $\bar{b} = (b_1, \dots, b_n)$ such that $\mathrm{tp}(\bar{b}/\mathfrak{B}) = \mathrm{tp}(\bar{a}/\mathfrak{B})$ and $\bar{b} \downarrow_{\mathfrak{B}} \mathfrak{C}$. Then we also have

$$\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{a}/A)$$

and each b_i is orthogonal to $\mathfrak{C}$ over $\mathfrak{B}$. So as above,

$$\|P_{\mathfrak{C}}(b_i)\| = \|P_{\mathfrak{C}}(P_{\mathfrak{B}}(b_i))\| = \|P_{\mathfrak{C}}(P_{\mathfrak{B}}(a_i))\| \leq \|P_{\mathfrak{C}}(a_i)\|,$$

by (1). And by (a reasoning analogous to) (2),

$$\|P_{\mathfrak{C}}(b_1)\| = \|P_{\mathfrak{C}}(P_{\mathfrak{B}}(b_1))\| = \|P_{\mathfrak{C}}(P_{\mathfrak{B}}(a_1))\| \leq \|P_{\mathfrak{B}}(a_i)\| < \|x\| = \|P_{\mathfrak{C}}(a_1)\|.$$

Thus

$$\|P_{\mathfrak{C}}(b_1)\| + \cdots + \|P_{\mathfrak{C}}(b_n)\| < \|P_{\mathfrak{C}}(a_1)\| + \cdots + \|P_{\mathfrak{C}}(a_n)\|.$$

This contradicts the choice of $\bar{a}$ and $\mathfrak{C}$. ☑

A direct consequence of the previous lemma is that $\mathrm{cl}_0(A) \subset \mathrm{bcl}_N(A) = \bigcap_{A \subset \mathfrak{B} \models T_N} \mathfrak{B}$, as $\mathrm{cl}_0(A)$ belongs to every model of the theory T_N.

We now define the *essential closure* ecl. Let $\mathrm{cl}_{\alpha+1}(A) = \mathrm{cl}_0(\mathrm{cl}_\alpha(A))$ for all ordinals α, $\mathrm{cl}_\delta(A) = \bigcup_{\alpha<\delta}(\mathrm{cl}_\alpha(A))$, and $\mathrm{ecl}(A) = \bigcup_{\alpha \in On} \mathrm{cl}_\alpha(A)$.

Lemma 4.9. *For all $\bar{a}, B, A$, if $\mathrm{ecl}(A) = A$ then there is $\bar{b}$ such that $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{a}/A)$ and $\bar{b} \downarrow_A B$.*

***Proof*.** Choose $\mathfrak{A} \models T_0$ such that $A \subset \mathfrak{A}$ and $\bar{c}$ such that $\mathrm{tp}(\bar{c}/A) = \mathrm{tp}(\bar{a}/A)$ and $(\bar{c}, A, \mathfrak{A})$ is minimal. Since $\mathrm{cl}_0(A) = A$, $P_{\mathfrak{A}}(c_i) \in A$ for all $i \leq n$ ($\bar{c} = (c_0, \ldots, c_n)$), i.e. $\bar{c} \downarrow_A \mathfrak{A}$. Now choose $\bar{b}$ such that $\mathrm{tp}(\bar{b}/\mathfrak{A}) = \mathrm{tp}(\bar{c}/\mathfrak{A})$ and $\bar{b} \downarrow_{\mathfrak{A}} B$. Then $\bar{b}$ is as needed. ☑

Corollary 4.10. $\mathrm{ecl}(A) = \mathrm{acl}_N(A)$.

***Proof*.** Clearly $\mathrm{acl}_N(A) \subset \mathrm{bcl}_N(A)$. On the other hand, assume that $x \notin \mathrm{acl}_N(A)$. Let $\mathfrak{B}$ be a model of T_N such that $A \subset \mathfrak{B}$. By Lemma 4.9, we may assume that $x \downarrow_A \mathfrak{B}$. Then $x \notin \mathfrak{B}$, so $x \notin \mathrm{bcl}(A)$, so $x \notin \mathrm{ecl}(A)$. ☑

Proposition 4.11. *Suppose $\bar{b} \not\downarrow^*_A C$, $A \subset B \cap C$ and (wlog) $C = \mathrm{bcl}(C)$. Then there exists a formula $\chi \in \mathrm{tp}_N(\bar{b}/C)$ such that for all $\bar{a} \models \chi$, $\bar{a} \not\downarrow^*_A C$.*

***Proof*.** By compactness, we can find $\varepsilon > 0$ such that (letting $\bar{b} = (b_0, \ldots, b_n)$, $(\bar{a} = (a_0, \ldots, a_n))$,

$$\forall \bar{a} \models \mathrm{tp}_N(\bar{b}/B), \|P_C(a_0)\| + \cdots + \|P_C(a_n)\| \geq \varepsilon + \|P_{\mathrm{bcl}(A)}(a_0)\| + \cdots + \|P_{\mathrm{bcl}(A)}(a_n)\|. \tag{3}$$

Again by compactness we can find $\chi \in \mathrm{tp}_N(\bar{b}/B)$ such that (3) holds when we replace $\mathrm{tp}_N(\bar{b}/B)$ by χ and ε by $\varepsilon/2$, that is:

$$\forall \bar{a} \models \chi, \|P_C(a_0)\| + \cdots + \|P_C(a_n)\| \geq \varepsilon/2 + \|P_{\mathrm{bcl}(A)}(a_0)\| + \cdots + \|P_{\mathrm{bcl}(A)}(a_n)\|. \tag{4}$$

and in particular $\bar{a} \not\downarrow^*_A C$, as we wanted. ☑

Theorem 4.12. *Suppose $\mathrm{ecl}(A) = A$, $A \subset B, C$, $B \downarrow^*_A C$ (i.e. $\mathrm{ecl}(B) \downarrow_A \mathrm{ecl}(C)$), $\bar{a} \downarrow^*_A B$, $\bar{b} \downarrow^*_A C$ and $\mathrm{tp}_N(\bar{a}/A) = \mathrm{tp}_N(\bar{b}/A)$. Then there is $\bar{c}$ such that $\mathrm{tp}_N(\bar{c}/B) = \mathrm{tp}_N(\bar{a}/B)$, $\mathrm{tp}_N(\bar{c}/C) = \mathrm{tp}_N(\bar{b}/C)$ and $\bar{c} \downarrow^*_A BC$.*

***Proof*.** Wlog $\mathrm{ecl}(B) = B$ and $\mathrm{ecl}(C) = C$. By Lemma 4.9 we can find models A_0, A_1, B^* and C^* of T_0 such that $A\bar{a} \subset A_0$, $A\bar{b} \subset A_1$, $B \subset B^*$ and $C \subset C^*$, such that $B^* \downarrow^*_A C^*$, $A_0 \downarrow^*_A B^*$ and $A_1 \downarrow^*_A C^*$. We can also find models of T_0, A_0^*, A_1^* and D^* such that $A_0 B^* \subset A_0^*$, $A_1 C^* \subset A_1^*$ and $B^* C^* \subset D^*$ and wlog we may assume that $\bar{a}$ and $\bar{b}$ are chosen so that $A_0^* \downarrow_{B^*} D^*$, $A_1^* \downarrow_{C^*} D^*$, and that there is an automorphism F of the monster model fixing A pointwise such that $F(\bar{a}) = \bar{b}$, $F(A_0) = A_1$ and $F(A_0^*) \downarrow_{A_1} A_1^*$. Notice that now

$$A_0 \underset{A}{\downarrow} D^* \text{ and } A_1 \underset{A}{\downarrow} D^*.$$

We can now find Hilbert spaces A^*, A_0^{**}, A_1^{**} and E such that

(i) E is generated by $D^*A_0^{**}A_1^{**}$,

(ii) $A \subset A^* \subset A_0^{**} \cap A_1^{**}$, $B^* \subset A_0^{**}$, $C^* \subset A_1^{**}$,

(iii) There are Hilbert space isomorphisms $G : A_0^{**} \to A_0^*$ and $H : A_1^{**} \to A_1^*$ such that

a) $F \circ G \restriction A^* = H \restriction A^*$,

b) $G \restriction B^* = \mathrm{id}_{B^*}$, $H \restriction C^* = \mathrm{id}_{C^*}$,

c) $G \cup \mathrm{id}_{D^*}$ generate an isomorphism

$$\langle A_0^{**}D^* \rangle \to \langle A_0^* D^* \rangle,$$

d) $H \cup \mathrm{id}_{D^*}$ generate an isomorphism

$$\langle A_1^{**}D^* \rangle \to \langle A_1^* D^* \rangle,$$

e) $F \cup G \cup H$ generate an isomorphism

$$\langle A_0^{**}A_1^{**} \rangle \to \langle F(A_0^*)A_1^* \rangle.$$

We can find these because non-dividing independence in Hilbert spaces has 3-existence (the independence theorem holds for types over sets).

Now we choose the "black points" of our model: $a \in E$ is black if one of the following holds:

(i) $a \in A_0^{**}$ and $G(a)$ is black,

(ii) $a \in A_0^{**}$ and $H(a)$ is black,

(iii) $a \in D^*$ and is black.

Then in E we define the "distance to black" function simply as the real distance. Then in D^* there is no change and G and H remain isomorphisms after adding this structure; D^*, A_0^*, A_1^* and $F(A_0^*)$ witness this.

So we can assume that E is a submodel of the monster, and letting $\bar{c} = G^{-1}(a)$, G witnesses that $\mathrm{tp}_N(\bar{c}/B) = \mathrm{tp}_N(\bar{a}/B)$ and H witnesses that $\mathrm{tp}_N(\bar{c}/C) = \mathrm{tp}_N(\bar{b}/C)$. We have already seen that $A^* \mathop{\smile\!\!\!\!\!\!|}_A D^*$ and thus $\bar{c} \mathop{\smile\!\!\!\!\!\!|}^*_A BC$. ☑

We can now conclude: Proposition 4.11 provides the strong finite character property, and Theorem 4.12 gives independent amalgamation: the tuple $\bar{c}$ witnesses the property. The other properties are immediate. Therefore, the theory T_N has the NSOP_1 property.

References

[1] I. Ben-Yaacov, *Lovely pairs of models: the non first order case*, Journal of Symbolic Logic **69** (2004), 641–662.

[2] I. Ben-Yaacov, A. Berenstein, W. C. Henson, and A. Usvyatsov, *Model Theory for metric structures*, in Model Theory with Applications to Algebra and Analysis, Volume 2, Cambridge University Press.

[3] I. Ben-Yaacov, A. Usvyatsov, and M. Zadka, *Generic automorphism of a Hilbert space*, preprint avaliable at http://ptmat.fc.ul.pt/ alexus/papers.html.

[4] A. Berenstein, *Hilbert spaces with generic groups of automorphisms*, Arch. Math. Logic **46** (2007), no. 3, 289–299.

[5] A. Berenstein and S. Buechler, *Simple stable homogeneous expansions of hilbert space*, Ann. Pure Appl. Logic **128** (2004), no. 1-3, 75–101.

[6] Z. Chatzidakis and A. Pillay, *Generic structures and simple theories*, Ann. Pure Appl. Logic **95** (1998), no. 1-3.

[7] A. Chernikov and N. Ramsey, *On model-theoretic tree properties*, Journal of Mathematical Logic **16** (2016), no. 2.

[8] W. Hodges, *Model Theory*, Cambridge University Press, 1993.

[9] B. Poizat, *Le carré de l'égalité*, Journal of Symbolic Logic **64** (1999), 1339–1355.

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