Revista Colombiana de Matemáticas
Volumen 52(2018)1, páginas 87-105

Existence of Unique and Global Asymptotically Stable Almost Periodic Solution of a Discrete Predator-Prey System with Beddington-DeAngelis Functional Response and Density Dependent

Existencia de una única solución casi periódica global asintóticamente estable de un sistema Depredador-Presa con respuesta funcional Beddington-DeAngelis y densamente dependiente

Cosme Duque

Universidad de Los Andes, Mérida, Venezuela

Abstract. The main concern of this paper is to study the dynamics of a discrete predator-prey system with Beddington-DeAngelis functional response and density dependent predator, assuming that the coefficients involved in the system are almost periodic. More concretely, under certain conditions, we prove the existence of a unique almost periodic solution which is globally attractive. We exhibit a few numerical examples of the results.

Key words and phrases. Density dependent predator, Beddington-DeAngelis functional response, discrete predator-prey, almost periodic solution.

2010 Mathematics Subject Classification. 34K14, 92D25.

Resumen. El objetivo principal de este artículo es el de estudiar la dinámica de un sistema depredador-presa discreto con respuesta funcional Beddington-DeAngelis y densamente dependiente del depredador, asumiendo que los coeficientes involucrados en el sistema son casi periódicos. De forma más concreta, bajo ciertas condiciones, probaremos la existencia de una única solución casi periódica la cual es globalmente atractiva. Exhibimos algunos ejemplos numéricos de los resultados.

Palabras y frases clave. Densamente dependiente depredador, respuesta funcional Beddington-DeAngelis, depredador-presa discreto, solución casi periódica.

1. Introduction

In 1975, Beddington [3] and DeAngelis [8] independently propose the following predator-prey system

$$\begin{aligned} x'(t) &= x(t)\left(a - bx(t) - \frac{cy(t)}{m_1 + m_2x(t) + m_3y(t)}\right), \\ y'(t) &= y(t)\left(-d + \frac{fx(t)}{m_1 + m_2x(t) + m_3y(t)}\right), \end{aligned} \tag{1}$$

where $x(t)$, $y(t)$ represent the population density of prey and predator at time $t > 0$, respectively, a, b, c, d, f, m_1, m_2, m_2 are positive constants. $a > 0$ is the specific growth rate of prey in the absence of predation and without environment limitation; in the absence of predator the prey population grows logistically with carrying capacity a/b; the predator consumes the prey with functional response of Beddington-DeAngelis type $cxy/(m_1 + m_2x + m_3y)$ and contributes to its growth with rate $fxy/(m_1 + m_2x + m_3y)$. The constant d is the death rate of predator and the term m_3y measures the mutual interference between predators and preys. Predator-prey systems with the Beddington-DeAngelis functional response have been studied extensively in the literature [7, 4, 5, 14, 13, 24].

Recent researches ([15, 19, 18, 17, 16, 26]) confirm that certain environments confine the predator to be density dependent and show that predator dependence is important at not only very high predator densities on per capita predation rate but also at low predator densities. So it is not enough to only require the prey to be density dependent, also we need to take into account realistic levels of predator dependence. With these considerations, Li and Takeuchi [18] considered the system

$$\begin{aligned} x'(t) &= x(t)\left(a - bx(t) - \frac{cy(t)}{m_1 + m_2x(t) + m_3y(t)}\right), \\ y'(t) &= y(t)\left(-d - ey(t) + \frac{fx(t)}{m_1 + m_2x(t) + m_3y(t)}\right), \end{aligned} \tag{2}$$

where e stands for the predator density dependence rate. The authors in [18] show the permanence, local and global asympototic stability of system (2).

On the other hand, the assumption that the environment is constant is rare in real life. Most natural environments are physically highly variable, and in response, birth rates, death rates, and other vital rates of populations, vary greatly in time. When this is taken into account, a model must be non-autonomous and therefore one can take advantage of the properties of those varying parameters. For example, one may assume that the parameters are periodic or almost periodic for seasonal reasons. In this context, Li and Takeuchi

[19] considered the density dependent and nonautonomous predator-prey system with Beddington-Deangelis functional response

$$\begin{aligned} x'(t) &= x(t)\left(a(t) - bx(t) - \frac{c(t)y(t)}{m_1(t) + m_2(t)x(t) + m_3(t)y(t)}\right), \\ y'(t) &= y(t)\left(- d(t) - e(t)y(t) + \frac{f(t)x(t)}{m_1(t) + m_2(t)x(t) + m_3(t)y(t)}\right), \end{aligned} \tag{3}$$

and assumed that $a(t), b(t), c(t), d(t), e(t), f(t), m_1(t), m_2(t), m_3(t)$ are continuous and bounded above and below by positive constant. In [19], the authors address some basic problems for (3), such as positive invariance, permanence, dissipativity, and globally asymptotic stability of system (3). Then, the authors establish sufficient criteria for the existence of a unique positive periodic solution of (3) that is globally asymptotically stable, when all parameters are periodic.

However, many authors [1, 2, 6, 9, 11, 12, 21, 22, 27, 29] have argued that the discrete time models governed by difference equations are more appropiate than the continuous ones when the populations has nonoverlapping generations. In addition, discrete time models can also provide efficient computational models of continuous for numerical simulations. In particular, Zhang and Wang [27] considered the discrete analogous of (3) with $e(t) \equiv 0$ and with the assumption that all coeficients are periodic they studied the existence of positive periodic solutions. This study was extended by Pelen et al. in [23].

When $e(t) \neq 0$, Q. Fang et al. in [10] introduced the discrete analogous of (3), given by the following discrete system:

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{a(k) - b(k)x(k) - \frac{c(k)y(k)}{m_1(k)+m_2(k)x(k)+m_3(k)y(k)}\right\}, \\ y(k+1) &= y(k)\exp\left\{-d(k) - e(k)y(k) + \frac{f(k)x(k)}{m_1(k)+m_2(k)x(k)+m_3(k)y(k)}\right\}. \end{aligned} \tag{4}$$

We refer to the paper of Q. Fang et al. in [10], for the details. In [10] the authors prove the permanence of system (4) and via a Lyapunov function, they obtained sufficient conditions which guarantee the global attactivity of positive solutions of the system (4).

It is pointed out in [20] that various constituent components of the temporally nonuniform environment have incommensurable periods. Hence, in that scenario, it is natural to consider that the coefficients of the system (4) are almost periodic and the model becomes more realistic.

The main goal of this paper is to study system (4), assuming that the coefficients are almost periodic. Concretely, we prove the existence of a unique positive globally attractive almost periodic solution of the discrete predator-prey model with Beddigton-DeAngelis functional response and density dependent (4). This result is contained in Theorem 3.2.

2. Preliminaries

In this section it will be summarized the main facts which will be useful in the sequel.

Definition 2.1. ([28]). A sequence $z : \mathbb{Z} \longrightarrow \mathbb{R}$ is called an almost periodic sequence if the $\varepsilon-$translation set of z,

$$E\{\varepsilon, z\} = \{\tau \in \mathbb{Z} : |z(k+\tau) - z(k)| < \varepsilon, \forall k \in \mathbb{Z}\},$$

is a relatively dense set in $\mathbb{Z}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, z\}$ such that

$$|z(k+\tau) - z(k)| < \varepsilon, \quad \forall k \in \mathbb{Z}.$$

τ is called the $\varepsilon-$translation number or $\varepsilon-$almost period.

Definition 2.2. ([28]) A sequence $z : \mathbb{Z} \longrightarrow \mathbb{R}$ is called an asymptotically almost periodic sequence if

$$z(k) = p(k) + q(k), \quad \forall k \in \mathbb{Z}^+$$

where $\{p(k)\}$ is an almost periodic sequence and $\lim_{k\to\infty} q(k) = 0$.

Lemma 2.3. ([28]) *If $\{z(k)\}$ is an almost periodic sequence, then $\{z(k)\}$ is bounded.*

Lemma 2.4. ([28]) *$\{z(k)\}$ is an almost periodic sequence if and only if, for any sequence $\{m_i\} \subset \mathbb{Z}$, there exists a subsequence $\{m_{i_j}\} \subset \{m_i\}$ such that the sequence $\{z(k+m_{i_j})\}$ converges uniformly for all $k \in \mathbb{Z}$ as $j \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.*

Lemma 2.5. ([28]) *$\{z(k)\}$ is an asymptotically almost periodic sequence if and only if for any sequence $\{m_i\} \subset \mathbb{Z}$ satisfying $m_i > 0$ and $m_i \to \infty$ as $i \to \infty$ there exists a subsequence $\{m_{i_j}\} \subset \{m_i\}$ such that the sequence $\{z(k+m_{i_j})\}$ converges uniformly for all $k \in \mathbb{Z}^+$* as $j \to \infty$.

Lemma 2.6. ([25]) *Suppose that $\{p_1(k)\}$ and $\{p_2(k)\}$ are almost periodic real sequences. Then $\{p_1(k) + p_2(k)\}$ and $\{p_1(k)p_2(k)\}$ are almost periodic; $\{1/p_1(k)\}$ is also almost periodic provided that $p_1(k) \neq 0$ for all $k \in \mathbb{Z}$. Moreover, if $\varepsilon > 0$ is an arbitrary real number, then there exists a relatively dense set that is $\varepsilon-$almost periodic common to $\{p_1(k)\}$ and $\{p_2(k)\}$.*

Definition 2.7. System (4) is said to be permanent if there exist positive constants x_*, x^*, y_*, y^* which are independent of the solutions of the system, such that any positive solution $(x(k), y(k))$ of system (4) satisfies

$$x_* \leq \liminf_{k\longrightarrow\infty} x(k) \leq \limsup_{k\longrightarrow\infty} x(k) \leq x^*,$$

$$y_* \leq \liminf_{k\longrightarrow\infty} y(k) \leq \limsup_{k\longrightarrow\infty} y(k) \leq y^*.$$

System (4) is said to be nonpermanent if there is a positive solution $(x(k), y(k))$ of system (4) satisfying $\min\{\liminf_{k\to\infty} x(k), \liminf_{k\to\infty} y(k)\} = 0$.

Since we are assuming that the coefficients of the system are almost periodic, they are uniformly bounded. Therefore the permanence of the system (4) is an immediate consequence of the following:

Theorem 2.8. ([10]) *Let $(x(k), y(k))$ be a solution of (4) with $x(0) > 0$ and $y(0) > 0$. If*

$$f^u > d^l m_2^l, \quad a^l m_3^l > c^u, \quad -d^u + \frac{f^l x_*}{m_1^u + m_2^u x_* + m_3^u y^*} > 0, \tag{5}$$

then the system (4) is permanent, where

$$x^* = \frac{1}{b^l}\exp\{a^u - 1\}, \quad y^* = \frac{1}{e^l}\exp\left\{\frac{f^u}{m_2^l} - d^l - 1\right\}, \tag{6}$$

and

$$x_* = \frac{1}{b^u}\left[a^l - \frac{c^u}{m_3^l}\right]\exp\left\{a^l - \frac{c^u}{m_3^l} - b^u x^*\right\},$$

$$y_* = \frac{1}{e^u}\left[-d^u + \frac{f^l x_*}{m_1^u + m_2^u x_* + m_3^u y^*}\right]\exp\left\{- d^u + \frac{f^l x_*}{m_1^u + m_2^u x_* + m_3^u y^*} - e^u y^*\right\}. \tag{7}$$

Here $g^l = \min_{k\in\mathbb{N}} g(k)$ and $g^u = \max_{k\in\mathbb{N}} g(k)$ for any bounded sequence $\{g(k)\}$.

Despite that the not permanence of the system (4), it is not quite interesting from theoretical point of view, in applications is important. By using techniques of Lemma 4.2 from [24] we have the following:

Theorem 2.9. *If there exists $K > 0$ such that for all $k > K$,*

$$\sum_{s=0}^{k-1} -d(s) + \frac{f(s)}{m_2(s)} < 0, \tag{8}$$

then the system is not permanent.

Proof. From [23], the equivalent system for system (4) is

$$\begin{aligned} x(k+1) - x(k) =& a(k) - b(k)\exp(x(k)) \\ &- \frac{c(k)\exp(y(k))}{m_1(k) + m_2(k)\exp(x(k)) + m_3(k)\exp(y(k))}, \\ y(k+1) - y(k) =& - d(k) - e(k)\exp(y(k)) \\ &+ \frac{f(k)\exp(x(k))}{m_1(k) + m_2(k)\exp(x(k)) + m_3(k)\exp(y(k))}. \end{aligned} \tag{9}$$

This equivalence is explained in Remark 2 in [23]. By using the second equation of (9), taking the summation of both sides from 0 to $k-1$ and taking the exponential of both sides, one can obtain

$$\exp(y(k)) \leq \exp(y(0)) \exp\left(\sum_{s=0}^{k-1} -d(s) + \frac{f(s)}{m_2(s)}\right).$$

Since $\displaystyle\sum_{s=0}^{k-1} -d(s) + \frac{f(s)}{m_2(s)} < 0$, then $\displaystyle\lim_{k\to\infty} \exp(y(k)) = 0$. Hence the system cannot be permanent. ☑

The following corollary can be deduced from Theorem 2.9

Corollary 2.10. *If $f^u < d^l m_2^l$ then the system (4) is not permanent.*

Example 2.11. If we take $a(k) = 1$, $b(k) = 5 + 0.5\cos(k\frac{\sqrt{3}\pi}{3})$, $c(k) = 0.1$ $d(k) = 1.5 + 0.05\sin(k\frac{\sqrt{3}\pi}{3})$, $e(k) = 2$, $f(k) = 12$, $m_1(k) = 0.125 + 0.05\sin(k\frac{\sqrt{3}\pi}{3})$, $m_2(k) = 10 + 0.05\cos(k\frac{\sqrt{3}\pi}{3})$, $m_3(k) = 0.5 + 0.05\sin(k\frac{\sqrt{3}\pi}{3})$, then $f^u - d^l m_2^l \approx -2.427$ which implies, for the Corollary 2.10, that the system (4) is not permanent. This can be appreciated in Figure 1.

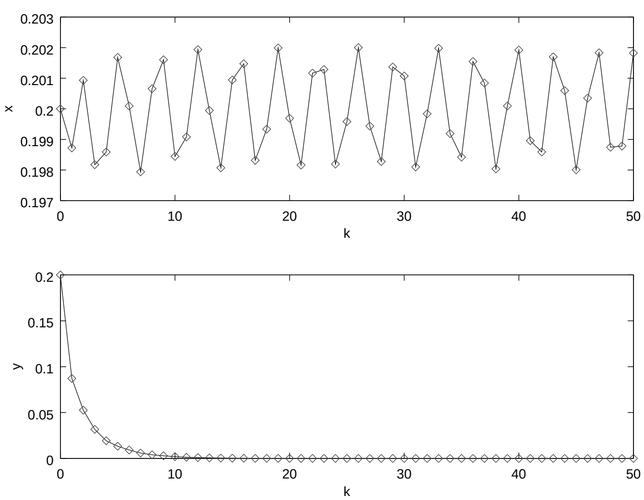

FIGURE 1. Solution of system (4) with $(x(0), y(0)) = (0.2, 0.2)$

3. Existence and global stability of almost periodic solutions

This section is devoted in proving the main result of this paper, which concerns the existence of a unique gobal uniformly asymptotically stable almost periodic solution of system (4).

The next result tells us that there exists a bounded solution of system (4).

Theorem 3.1. *Assume that* (5) *holds, then the system* (4) *has a solution* $(x(k), y(k))$ *satisfying* $x_* \leq x(k) \leq x^*$ *and* $y_* \leq y(k) \leq y^*$ *for* $k \in \mathbb{Z}$.

Proof. Since the coeficients $a(k), b(k), c(k), d(k), e(k), f(k)$ and $m_i(k)$, $i = 1, 2, 3$ are almost periodic, there exists an integer valued sequence $\{\delta_p\}$ with $\delta_p \to \infty$ as $p \to \infty$ such that

$$\begin{array}{lll} a(k+\delta_p) \to a(k), & b(k+\delta_p) \to b(k), & c(k+\delta_p) \to b(k), \\ d(k+\delta_p) \to d(k), & e(k+\delta_p) \to e(k), & f(k+\delta_p) \to f(k), \\ m_i(k+\delta_p) \to m_i(k), & i = 1, 2, 3, \text{ as } p \to \infty. & \end{array}$$

Let ε be an arbitrary small positive number. From Theorem 2.8, there exists a positive integer k_0 such that

$$x_* - \varepsilon \leq x(k) \leq x^* + \varepsilon, \quad y_* - \varepsilon \leq y(k) \leq y^* + \varepsilon, \quad \text{for } k \geq k_0.$$

Write $x_p(k) = x(k + \delta_p)$, $y_p(k) = y(k + \delta_p)$ for $k \geq k_0 - \delta_p$ and $p = 1, 2, \ldots$. For any positive integer q, it is easy to see that there exist sequences $\{x_p(k) : p \geq q\}$ and $\{y_p(k) : p \geq q\}$ such that the sequences $\{x_p(k)\}$ and $\{y_p(k)\}$ have subsequences, denoted by $\{x_p(k)\}$ and $\{y_p(k)\}$ again, converging on any finite interval of $\mathbb{Z}$ as $p \to \infty$, respectively. Thus we have sequences $\{\overline{x}(k)\}$ and $\{\overline{y}(k)\}$ such that

$$x_p(k) \to \overline{x}(k), \quad y_p(k) \to \overline{y}(k), \quad \text{for } k \in \mathbb{Z}, \quad \text{as } p \to \infty.$$

Combined with

$$\begin{aligned} x_p(k+1) = x_p(k) \exp\Bigg\{ & a(k+\delta_p) - b(k+\delta_p)x_p(k) \\ & - \frac{c(k+\delta_p)y_p(k)}{m_1(k+\delta_p) + m_2(k+\delta_p)x_p(k) + m_3(k+\delta_p)y_p(k)} \Bigg\}, \end{aligned}$$

$$\begin{aligned} y_p(k+1) = y_p(k) \exp\Bigg\{ & -d(k+\delta_p) - e(k+\delta_p)y_p(k) \\ & + \frac{f(k+\delta_p)x_p(k)}{m_1(k+\delta_p) + m_2(k+\delta_p)x_p(k) + m_3(k+\delta_p)y_p(k)} \Bigg\}, \end{aligned}$$

gives

$$\begin{aligned}\overline{x}(k+1)&=\overline{x}(k)\exp\left\{a(k)-b(k)\overline{x}(k)-\frac{c(k)\overline{y}(k)}{m_1(k)+m_2(k)\overline{x}(k)+m_3(k)\overline{y}(k)}\right\},\\ \overline{y}(k+1)&=\overline{y}(k)\exp\left\{-d(k)-e(k)\overline{y}(k)+\frac{f(k)\overline{x}(k)}{m_1(k)+m_2(k)\overline{x}(k)+m_3(k)\overline{y}(k)}\right\}.\end{aligned} \tag{10}$$

$(\overline{x}(k),\overline{y}(k))$ is a solution of system (4) and $x_*-\varepsilon\leq\overline{x}(k)\leq x^*+\varepsilon$, $y_*-\varepsilon\leq\overline{y}(k)\leq y^*+\varepsilon$ for $k\in\mathbb{Z}$. By the arbitrariness of ε we have that $x_*\leq\overline{x}(k)\leq x^*$ and $y_*\leq\overline{y}(k)\leq y^*$. ☑

For convenience, we introduce the following notation:

$g_p(k)=g(k+\tau_p)$, $\eta(k,x,y)=m_1(k)+m_2(k)x(k)+m_3(k)y(k)$, $\eta^u(k,x,y)=m_1^u+m_2^ux(k)+m_3^uy(k)$ and $\eta^l(k,x,y)=m_1^l+m_2^lx(k)+m_3^ly(k)$.

Theorem 3.2. *Suppose that* $f^u>d^lm_2^l$, $a^lm_3^l>c^u$, $-d^u+f^lx_*/\eta^u(k,x_*,y^*)>0$ *and*

$$\begin{aligned}\lambda_1=&\ \max\left\{\left|1+\frac{c^lm_2^ly_*x_*}{\eta^u(k,x^*,y^*)^2}-b^ux^*\right|,\left|1+\frac{c^um_2^uy^*x^*}{(m_1^l)^2}-b^lx_*\right|\right\}\\ &+\frac{c^uy^*}{\eta^l(k,x_*,y_*)}<1,\\ \lambda_2=&\ \max\left\{\left|1-\left[e^u+\frac{f^um_3^ux^*}{(m_1^l)^2}\right]y^*\right|,\left|1-\left[e^l+\frac{f^lm_3^lx_*}{\eta^u(k,x^*,y^*)^2}\right]y_*\right|\right\}\\ &+\frac{f^ux^*}{\eta^l(k,x_*,y_*)}<1.\end{aligned} \tag{11}$$

Then there exists a unique almost periodic sequence solution of system (4) *which is globally attractive.*

Proof. It follows from Theorem 3.1 that there exists a solution $(x(t),y(t))$ such that $x_*\leq x(k)\leq x^*$ and $y_*\leq y(k)\leq y^*$ for $k\in\mathbb{Z}^+$.

Let $\{\tau_p\}$ be any integer valued sequence such that $\tau_p\to\infty$ as $p\to\infty$. By using the Mean Value Theorem, for $p\neq q$, we get

$$\begin{aligned}x_p(k)-x_q(k)=&\ \exp\{\ln x_p(k)\}-\exp\{\ln x_q(k)\}\\ =&\ \xi_x(k,\tau_p,\tau_q)[\ln x_p(k)-\ln x_q(k)],\\ y_p(k)-y_q(k)=&\ \exp\{\ln y_p(k)\}-\exp\{\ln y_q(k)\}\\ =&\ \xi_y(k,\tau_p,\tau_q)[\ln y_p(k)-\ln y_q(k)],\end{aligned} \tag{12}$$

where $\xi_x(k,\tau_p,\tau_q)$ lies between $x_p(k)$ and $x_q(k)$, and $\xi_y(k,\tau_p,\tau_q)$ lies between $y_p(k)$ and $y_q(k)$, then

$$\begin{aligned} &|x_p(k)-x_q(k)| \le x^*|\ln x_p(k)-\ln x_q(k)|,\\ &|y_p(k)-y_q(k)| \le y^*|\ln y_p(k)-\ln y_q(k)|, \quad k\in\mathbb{Z}^+. \end{aligned} \tag{13}$$

Let $u(k,\tau_p,\tau_q)=|\ln x_p(k)-\ln x_q(k)|$ and $v(k,\tau_p,\tau_q)=|\ln y_p(k)-\ln y_q(k)|$ for $k\in\mathbb{Z}^+$, $\tau_p>0$ and $\tau_q>0$.

Thus

$$\begin{aligned} u(k+1,\tau_p,\tau_q) &= \big|\ln x_p(k+1)-\ln x_q(k+1)\big|\\ &= \Bigg|[\ln x_p(k)-\ln x_q(k)]+[a_p(k)-a_q(k)]-[b_p(k)x_p(k)-b_q(k)x_q(k)]\\ &\quad -\Bigg[\frac{c_p(k)y_p(k)}{\eta_p(k,x,y)}-\frac{c_q(k)y_q(k)}{\eta_q(k,x,y)}\Bigg]\Bigg|\\ &\le \Bigg|[\ln x_p(k)-\ln x_q(k)]+\Bigg[-b_p(k)+\frac{c_p(k)m_{2p}(k)y_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}\Bigg][x_p(k)-x_q(k)]\Bigg|\\ &\quad +\Bigg|\Bigg[\frac{c_p(k)m_{3p}(k)y_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}-\frac{c_p(k)}{\eta_q(k,x,y)}\Bigg][y_p(k)-y_q(k)]\Bigg|+\big|a_p(k)-a_q(k)\big|\\ &\quad +\big|x_q(k)[b_p(k)-b_q(k)]\big|+\Bigg|\frac{c_p(k)y_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}[m_{1p}(k)-m_{1q}(k)]\Bigg|\\ &\quad +\Bigg|\frac{c_p(k)y_p(k)x_q(k)}{\eta_p(k,x,x)\eta_q(k,x,y)}[m_{2p}(k)-m_{2q}(k)]\Bigg|\\ &\quad +\Bigg|\frac{c_p(k)y_p(k)y_q(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}[m_{3p}(k)-m_{3q}(k)]\Bigg|\\ &\quad +\Bigg|\frac{y_q(k)}{\eta_q(k,x,y)}[c_p(k)-c_q(k)]\Bigg|. \end{aligned}$$

Analogously,

$$\begin{aligned} v(k+1,\tau_p,\tau_q) &= \big|\ln y_p(k+1)-\ln y_q(k+1)\big|\\ &= \Bigg|[\ln y_p(k)-\ln y_q(k)]-[d_p(k)-d_q(k)]-[e_p(k)y_p(k)-e_q(k)y_q(k)]\\ &\quad +\Bigg[\frac{f_p(k)x_p(k)}{\eta_p(k,x,y)}-\frac{f_q(k)x_q(k)}{\eta_q(k,x,y)}\Bigg]\Bigg|\\ &\le \Bigg|[\ln y_p(k)-\ln y_q(k)]+\Bigg[-e_p(k)-\frac{f_p(k)m_{3p}(k)x_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}\Bigg][y_p(k)-y_q(k)]\Bigg|\\ &\quad +\Bigg|\Bigg[\frac{f_p(k)}{\eta_q(k,x,y)}-\frac{f_p(k)m_{2p}(k)x_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}\Bigg][x_p(k)-x_q(k)]\Bigg|+\big|d_p(k)-d_q(k)\big|\\ &\quad +\big|y_q(k)[e_p(k)-e_q(k)]\big|+\Bigg|\frac{f_p(k)x_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}[m_{1p}(k)-m_{1q}(k)]\Bigg| \end{aligned}$$

$$+\left|\frac{f_p(k)x_p(k)x_q(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}[m_{2p}(k)-m_{2p}(k)]\right|$$
$$+\left|\frac{f_p(k)x_p(k)y_q(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}[m_{3p}(k)-m_{3q}(k)]\right|+\left|\frac{x_q(k)}{\eta_q(k,x,y)}[f_p(k)-f_q(k)]\right|.$$

Let $\varepsilon>0$, since the sequences $a(k),b(k),c(k),d(k),e(k),f(k)$ and $m_i(k)$, $i=1,2,3$ are almost periodic and $\{x(k)\}$, $\{y(k)\}$ are bounded, it follows of lemmas 2.3, 2.4 and 2.6 that there exists a positive integer $l_0=l_0(\varepsilon)$ such that, for any $\tau_q\geq\tau_p\geq l_0$, and $k\in\mathbb{Z}^+$ (if necessary, it can choose subsequences of $\{\tau_p\}$ and $\{\tau_q\}$)

$$\left|a_p(k)-a_q(k)\right|<\frac{\varepsilon}{6},\left|x_p(k)[b_p(k)-b_q(k)]\right|<\frac{\varepsilon}{6},\left|\frac{c_p(k)y_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}[m_{3p}(k)-m_{3q}(k)]\right|<\frac{\varepsilon}{6},$$

$$\left|\frac{c_p(k)y_p(k)x_q(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}[m_{2p}(k)-m_{2q}(k)]\right|<\frac{\varepsilon}{6},\left|\frac{c_p(k)y_p(k)y_q(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}[m_{3p}(k)-m_{3q}(k)]\right|<\frac{\varepsilon}{6},$$

$$\left|\frac{y_q(k)}{\eta_q(k,x,y)}[c_p(k)-c_q(k)]\right|<\frac{\varepsilon}{6},\left|d_p(k)-d_q(k)\right|<\frac{\varepsilon}{6},\left|y_q(k)[e_p(k)-e_q(k)]\right|<\frac{\varepsilon}{6},$$

$$\left|\frac{f_p(k)x_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}[m_{1p}(k)-m_{1q}(k)]\right|<\frac{\varepsilon}{6},\left|\frac{f_p(k)x_p(k)x_q(k)}{\eta_p(k,x,y)\eta_p(k,x,y)}[m_{2p}(k)-m_{2q}(k)]\right|<\frac{\varepsilon}{6},$$

$$\left|\frac{f_p(k)x_p(k)y_q(k)}{\eta_p(k,x,y)\eta_p(k,x,y)}[m_{3p}(k)-m_{3q}(k)]\right|<\frac{\varepsilon}{6},\left|\frac{x_q(k)}{\eta_q(k,x,y)}[f_p(k)-f_q(k)]\right|<\frac{\varepsilon}{6}.$$

This implies that

$$\begin{aligned}u(k+1,\tau_p,\tau_q)\leq&\Bigg|[\ln x_p(k)-\ln x_q(k)]\\&+\left[\frac{c_p(k)m_{2p}(k)y_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}-b_p(k)\right][x_p(k)-x_q(k)]\Bigg|\\&+\left|\left[\frac{c_p(k)m_{3p}(k)y_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}-\frac{c_p(k)}{\eta_q(k,x,y)}\right][y_p(k)-y_q(k)]\right|+\varepsilon\end{aligned}$$

and

$$\begin{aligned}v(k+1,\tau_p,\tau_q)\leq&\Bigg|[\ln y_p(k)-\ln y_q(k)]\\&-\left[\frac{f_p(k)m_{3p}(k)x_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}+e_p(k)\right][y_p(k)-y_q(k)]\Bigg|\\&+\left|\left[\frac{f_p(k)}{\eta_q(k,x,y)}-\frac{f_p(k)m_{2p}(k)x_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}\right][x_p(k)-x_q(k)]\right|+\varepsilon.\end{aligned}$$

Using (12) we obtain

$$u(k+1,\tau_p,\tau_q) \leq \left|1+\left[\frac{c_p(k)m_{2p}(k)y_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)} - b_p(k)\right]\xi_x(k,\tau_p,\tau_q)\right| u(k,\tau_p,\tau_q)$$
$$+\left|\left[\frac{c_p(k)m_{3p}(k)y_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)} - \frac{c_p(k)}{\eta_q(k,x,y)}\right]\xi_y(k,\tau_p,\tau_q)\right| v(k,\tau_p,\tau_q)+\varepsilon,$$

$$v(k+1,\tau_p,\tau_q) \leq \left|1-\left[\frac{f_p(k)m_{3p}(k)x_p(k)}{\eta_p(k)\eta_q(k)} + e_p(k)\right]\xi_y(k,\tau_p,\tau_q)\right| v(k,\tau_p,\tau_q)$$
$$+\left|\left[\frac{f_p(k)}{\eta_q(k)} - \frac{f_p(k)m_{2p}(k)x_p(k)}{\eta_p(k,x,y)\eta_q(k,x,y)}\right]\xi_x(k,\tau_p,\tau_q)\right| u(k,\tau_p,\tau_q)+\varepsilon.$$

It follows from (11) that, for $k \in \mathbb{Z}^+$ and $\tau_q \geq \tau_p \geq l_0$,

$$u(k+1,\tau_p,\tau_q) \leq \lambda_1 \max\{u(k,\tau_p,\tau_q), v(k,\tau_p,\tau_q)\}+\varepsilon,$$

$$v(k+1,\tau_p,\tau_q) \leq \lambda_2 \max\{u(k,\tau_p,\tau_q), v(k,\tau_p,\tau_q)\}+\varepsilon,$$

which implies that

$$\max\{u(k+1,\tau_p,\tau_q), v(k+1,\tau_p,\tau_q)\} \leq \lambda \max\{u(k,\tau_p,\tau_q), v(k,\tau_p,\tau_q)\}+\varepsilon,$$

where $\lambda = \max\{\lambda_1, \lambda_2\}$. Hence

$$\max\{u(k,\tau_p,\tau_q), v(k,\tau_p,\tau_q)\} \leq \quad \lambda^{k+\tau_p} \max\{u(0,0,\tau_q-\tau_p), v(0,0,\tau_q-\tau_p)\}$$
$$+\varepsilon\frac{1-\lambda^{k+\tau_p}}{1-\lambda}.$$

Since $\lambda < 1$, for arbitrary $\epsilon > 0$, there exists $l_1 = l_1(\epsilon) > l_0$ such that, for any $\tau_q \geq \tau_p \geq l_0$,

$$\max\{u(k,\tau_p,\tau_q), v(k,\tau_p,\tau_q)\} \leq \frac{\epsilon}{\max\{x^*, y^*\}} \quad \text{for } k \in \mathbb{Z}^+. \tag{14}$$

In view of (13) and (14) it follows that

$$|x(k+\tau_p) - x(k+\tau_q)| < \epsilon \text{ and } |y(k+\tau_p) - y(k+\tau_q)| < \epsilon,$$

for $\tau_q \geq \tau_p \geq l_1$ and $k \in \mathbb{Z}^+$. By Lemma (2.5), the sequences $\{x(k)\}$, $\{y(k)\}$ are asymptotically almost periodic. Therefore $\{x(k)\}$ and $\{y(k)\}$ can be expresed as

$$x(k) = w_1(k) + z_1(k) \text{ and } y(k) = w_2(k) + z_2(k), \tag{15}$$

where $w_1(k)$ and $w_2(k)$ are almost periodics in $k \in \mathbb{Z}$, $z_1(k) \to 0$ and $z_2(k) \to 0$ as $k \to \infty$.

Now, we will show that $\{w_1(k)\}$ and $\{w_2(k)\}$ are almost periodic solutions of (4).

Let

$$\begin{aligned} f_1(k) = &\quad a(k) + b(k)[w_1(k) + z_1(k)] - \frac{c(k)[w_2(k) + z_2(k)]}{\eta(k, w_1 + z_1, w_2 + z_2)} \\ g_1(k) = &\quad a(k) + b(k)w_1(k) - \frac{c(k)w_2(k)}{\eta(k, w_1, w_2)} \\ f_2(k) = &\quad -d(k) - e(k)[w_2(k) + z_2(k)] + \frac{f(k)[w_1(k) + z_1(k)]}{\eta(k, w_1 + z_1, w_2 + z_2)} \\ g_2(k) = &\quad d(k) - e(k)w_2(k) + \frac{f(k)w_1(k)}{\eta(k, w_1, w_2)}. \end{aligned}$$

By using (4), (15) and the Mean Value Theorem we obtain that

$$\begin{aligned} x(k+1) = &\quad w_1(k+1) + z_1(k+1) = (w_1(k) + z_1(k)) \exp\{f_1(k)\} \\ = &\quad w_1(k) \exp\{\gamma_1(k)\}[f_1(k) - g_1(k)] + w_1(k) \exp\{g_1(k)\} \\ &\quad + z_1(k) \exp\{f_1(k)\}, \\ y(k+1) = &\quad w_2(k+1) + z_2(k+1) = (w_2(k) + z_2(k)) \exp\{f_2(k)\} \\ = &\quad w_2(k) \exp\{\gamma_2(k)\}[f_2(k) - g_2(k)] + w_2(k) \exp\{g_2(k)\} \\ &\quad + z_2(k) \exp\{f_2(k)\}, \end{aligned}$$

where $\gamma_i(k) = \theta_i(k) f_i(k) + (1 - \theta_i(k)) g_i(k)$ for some $\theta_i(k) \in [0, 1]$, $i = 1, 2$.

Thus

$$\begin{aligned} w_1(k+1) - w_1(k) \exp\{g_1(k)\} = &\quad w_1(k) \exp\{\gamma_1(k)\}[f_1(k) - g_1(k)] \\ &\quad + z_1(k) \exp\{f_1(k)\}, \\ w_2(k+1) - w_2(k) \exp\{g_2(k)\} = &\quad w_2(k) \exp\{\gamma_2(k)\}[f_2(k) - g_2(k)] \\ &\quad + z_2(k) \exp\{f_2(k)\}. \end{aligned}$$

Since $a(k), b(k), c(k), d(k), e(k), f(k), m_i(k)$, for $i = 1, 2, 3$ are bounded,

$$\begin{aligned} f_1(k) - g_1(k) = &\ b(k) z_1(k) \\ &\ -c(k) \left\{ \frac{(m_1(k) + m_2(k)) z_2(k) - w_2(k) m_2(k) z_1(k)}{\eta(k, w_1, w_2) \eta(k, w_1 + z_1, w_2 + z_2)} \right\}, \\ f_2(k) - g_2(k) = &\ -e z_2(k) \\ &\ + f(k) \left\{ \frac{(m_1(k) + m_3(k) w_2(k)) z_1(k) - w_1(k) m_3(k) z_2(k)}{\eta(k, w_1, w_2) \eta(k, w_1 + z_1, w_2 + z_2)} \right\}, \end{aligned}$$

and the fact that $z_1(k), z_2(k) \to 0$ as $k \to \infty$, we obtain

$$w_i(k+1) - w_i(k) \exp\{g_i(k)\} \to 0, \quad i = 1, 2,$$

as $k \to \infty$.

Now, $w_i(k+1) - w_i(k)\exp\{g_1(k)\} \equiv 0$, $i = 1,2$, in fact, suppose that there exist k_1 and k_2 such that $w_i(k_i+1) - w_i(k_i)\exp\{g_i(k_i)\} \neq 0$, $i = 1,2$. By the almost periodicity of $a(k)$, $b(k)$, $c(k)$, $d(k)$, $e(k)$, $f(k)$, $m_i(k)$, for $i = 1,2,3$, and $w_i(k)$, $i = 1,2$ there exists an integer valued sequence $\{\delta_p\}$ such that $\delta_p \to \infty$ as $p \to \infty$ and

$$\begin{array}{llll} a(k+\delta_p) \to a(k), & b(k+\delta_p) \to b(k), & c(k+\delta_p) \to b(k), & d(k+\delta_p) \to d(k), \\ e(k+\delta_p) \to e(k), & f(k+\delta_p) \to f(k), & \multicolumn{2}{l}{m_i(k+\delta_p) \to m_i(k),\ i = 1,2,3,} \\ \multicolumn{4}{l}{w_i(k+\delta_p) \to w_i(k),\ i = 1,2.} \end{array}$$

uniformly for all $k \in \mathbb{Z}$. Then,

$$w_i(k_i+\delta_p+1) - w_i(k_i+\delta_p)\exp\{g_i(k_i+\delta_p)\} \to w_i(k_i+1) - w_i(k_i)\exp\{g_i(k_i)\}$$

as $p \to \infty$ and $i = 1,2$, which is a contradiction. Hence $(w_1(k), w_2(k))$ is an almost periodic sequence solution of (4).

Now, to finish we will prove that $\lim_{k\to\infty} |x(k) - w_1(k)| = 0$ and $\lim_{k\to\infty} |y(k) - w_2(k)| = 0$, where $(x(k), y(k))$ is any positive solution of (4). In order to accomplish this, let us define

$$u(k) = \ln\left(\frac{x(k)}{w_1(k)}\right) \text{ and } v(k) = \ln\left(\frac{y(k)}{w_2(k)}\right);$$

then,

$$x(k) = w_1(k)\exp\{u(k)\} \text{ and } y(k) = w_2(k)\exp\{v(k)\}.$$

So that (4) is equivalent to

$$\begin{aligned} u(k+1) = \quad & u(k) + b(k)w_1(k)[1 - \exp\{u(k)\}] \\ & + c(k)w_2(k)\left[\frac{1}{\eta(k, w_1, w_2)} - \frac{\exp\{v(k)\}}{\eta(k, x, y)}\right], \\ v(k+1) = \quad & v(k) + e(k)w_2(k)[1 - \exp\{v(k)\}] \\ & - f(k)w_1(k)\left[\frac{1}{\eta(k, w_1, w_2)} - \frac{\exp\{u(k)\}}{\eta(k, x, y)}\right]. \end{aligned}$$

Therefore,

$$
\begin{aligned}
u(k+1) = & \Bigg[1 - b(k)w_1(k)\exp\{\theta_1(k)u(k)\} \\
& + \frac{c(k)m_2(k)w_1(k)w_2(k)}{\eta(k,w_1,w_2)\eta(k,x,y)}\exp\{\theta_1(k)u(k)\}\Bigg]u(k) \\
& - \frac{c(k)(m_1(k)+m_2(k)w_1(k))w_2(k)}{\eta(k,w_1,w_2)\eta(k,x,y)}\exp\{\theta_2(k)v(k)\}v(k), \\
v(k+1) = & \Bigg[1 - e(k)w_2(k)\exp\{\theta_2(k)v(k)\} \\
& - \frac{f(k)m_3(k)w_1(k)w_2(k)}{\eta(k,w_1,w_2)\eta(k,x,y)}\exp\{\theta_2(k)v(k)\}\Bigg]v(k) \\
& + \frac{f(k)(m_1(k)+m_3(k)w_2(k))w_1(k)}{\eta(k,w_1,w_2)\eta(k,x,y)}\exp\{\theta_1(k)u(k)\}u(k),
\end{aligned}
\tag{16}
$$

where $\theta_1(k), \theta_2(k) \in [0,1]$. If $\lim_{k\to\infty} u(k) = 0$ and $\lim_{k\to\infty} v(k) = 0$ then the proof will be complete.

From (11), let $\varepsilon > 0$ such that

$$
\begin{aligned}
\lambda_1^\varepsilon = \quad & \max\Bigg\{\left|1 + \frac{c^l m_2^l (y_* - \varepsilon)(x_* - \varepsilon)}{\eta^u(k, x^* + \varepsilon, y^* + \varepsilon)^2} - b^u(x^* + \varepsilon)\right|, \\
& \left|1 + \frac{c^u m_2^u (y^* + \varepsilon)(x^* + \varepsilon)}{(m_1^l)^2} - b^l(x_* - \varepsilon)\right|\Bigg\} + \frac{c^u(y^* + \varepsilon)}{\eta^l(k, x_* - \varepsilon, y_* - \varepsilon)} < 1, \\
\lambda_2^\varepsilon = \quad & \max\Bigg\{\left|1 - \left[e^u + \frac{f^u m_3^u (x^* + \varepsilon)}{(m_1^l)^2}\right](y^* + \varepsilon)\right|, \\
& \left|1 - \left[e^l + \frac{f^l m_3^l (x_* - \varepsilon)}{\eta^u(k, x^* + \varepsilon, y^* + \varepsilon)^2}\right](y_* - \varepsilon)\right|\Bigg\} + \frac{f^u(x^* + \varepsilon)}{\eta^l(k, x_* - \varepsilon, y_* - \varepsilon)} < 1.
\end{aligned}
$$

By using the Theorem 2.8 we have that there exist $k_0 \in \mathbb{N}$ such that

$$
x_* - \varepsilon \le x(k), w_1(k) \le x^* + \varepsilon, \quad y_* - \varepsilon \le y(k), w_2(k) \le y^* + \varepsilon,
$$

for $k \ge k_0$.

Since $\theta_1(k), \theta_2(k) \in [0,1]$, then $w_1(k)\exp\{\theta_1(k)u(k)\}$, $w_2(k)\exp\{\theta_2(k)v(k)\}$ lies between $w_1(k)$ and $x(k)$, and, $w_2(k)$ and $y(k)$ respectively. From (16), it follows that

$$|u(k+1)| \leq \max\left\{\left|1+\frac{c^l m_2^l(y_*-\varepsilon)(x_*-\varepsilon)}{\eta^u(k,x^*+\varepsilon,y^*+\varepsilon)^2}-b^u(x^*+\varepsilon)\right|,\right.$$
$$\left.\left|1+\frac{c^u m_2^u(y^*+\varepsilon)(x^*+\varepsilon)}{(m_1^l)^2}-b^l(x_*-\varepsilon)\right|\right\}|u(k)|+\frac{c^u(y^*+\varepsilon)}{\eta^l(k,x_*-\varepsilon,y_*-\varepsilon)}|v(k)|,$$

$$|v(k+1)| \leq \max\left\{\left|1-\left[e^u+\frac{f^u m_3^u(x^*+\varepsilon)}{(m_1^l)^2}\right](y^*+\varepsilon)\right|,\right.$$
$$\left.\left|1-\left[e^l+\frac{f^l m_3^l(x_*-\varepsilon)}{(\eta^u(k,x^*+\varepsilon,y^*+\varepsilon)^2}\right](y_*-\varepsilon)\right|\right\}|v(k)|+\frac{f^u(x^*+\varepsilon))}{\eta^l(k,x_*-\varepsilon,y_*-\varepsilon)}|u(k)|,$$

for $k \geq k_0$. Hence,

$$\max\{|u(k+1)|,|v(k+1)|\} \leq \lambda_\varepsilon \max\{|u(k)|,|v(k)|\}, \quad k \geq k_0,$$

where $\lambda_\varepsilon = \max\{\lambda_1^\varepsilon, \lambda_2^\varepsilon\} < 1$. This implies that

$$\max\{|u(k+1)|,|v(k+1)|\} \leq \lambda_\varepsilon^{k-k_0} \max\{|u(k_0)|,|v(k_0)|\}, \quad k \geq k_0.$$

So, $\lim_{k\to\infty} u(k) = 0$ and $\lim_{k\to\infty} v(k) = 0$.

Since $(w_1(k), w_2(k))$ is the global attractor of all positive solutions of (4) then $(w_1(k), w_2(k))$ is the unique almost periodic sequence solution of system (4). This concludes the proof. ☑

Example 3.3. Let $a(k) = 0.9$, $b(k) = 3 + 0.05\cos(k\sqrt{3}/4)$, $c(k) = 0.3$, $d(k) = 0.01 + 0.01\cos(k\sqrt{3}/4)$, $e(k) = 6$, $f(k) = 2.3$, $m_1(k) = 10 + 0.1\sin(k\sqrt{3}/4)$, $m_2(k) = 65 + 0.02\cos(k\sqrt{3}/4)$, $m_3(k) = 60 + 0.01\cos(k\sqrt{3}/4)$.

We can calculate

$$f^u - d^l m_2^l \approx 2.297, \quad a^l m_3^l - c^u \approx 53.691, \quad -d^u + \frac{f l x_*}{m_1^u + m_2 x_* + m_3 y^*} \approx 0.001$$

and

$$\lambda_1 \approx 0.173, \quad \lambda_2 \approx 0.999.$$

Hence coeficient functions satisfy the conditions of Theorem 3.2, therefore system (4) has a unique almost periodic sequence solution which is globally attractive. This can be appreciated in Figure 2.

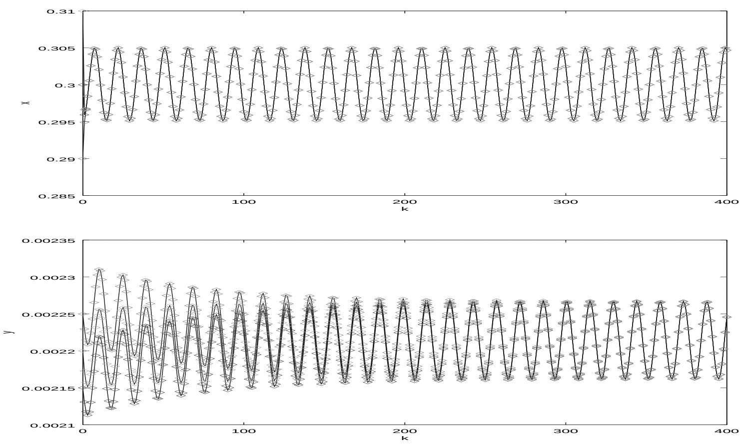

FIGURE 2. Solutions of system (4) with $(x(0), y(0)) = (0.29, 0.00215)$, $(x(0), y(0)) = (0.31, 0.0025)$ and $(x(0), y(0)) = (0.3, 0.0022)$. The solutions tend to the almost periodic solution.

4. Conclusion

In this paper, we studied the discrete predator-prey model (4) of Beddington-DeAngelis type functional response with density dependent predator. We found sufficient conditions where the predator cannot survive (Theorem 2.9, Corollary 2.10). We were able to prove, under some reasonable conditions, the existence of a unique globally attractive almost periodic solution when the coeficients are almost periodic (Theorem 3.2). Finally, the claims in theorems were illustrated using numerical simulations.

Acknowledgements

The author like to thank the referee for her careful reading of the original manuscript and for his many valuable comments and suggestions that improved the presentation of this paper.

References

[1] R. P. Agarwall, *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel Dekker, Inc., New York.

[2] R. P. Agarwall and P.J.Y. Wong, *Advance Topics in Difference Equations*, Kluwer, Dordrecht.

[3] J. R. Beddington, *Mutual interference between parasities of predators and its effect on searching efficiency*, J. Animal. Ecol **44** (1975), 331–340.

[4] R. S. Cantrel and C. Cosner, *On the dynamics of predator-prey models with Beddington-DeAngelis functional response*, J. Math. Anal. Appl. **257** (2001), 206–222.

[5] ______, *Effects of domain size on the persistence of populations in a diffusive food chain model with DeAngelis-Beddington functional response*, Nat. Resour. Modelling **14** (2011), 335–367.

[6] Y. Chen and Z. Zhou, *Stable periodic solution of a discrete periodic Lotka-Volterra competition system*, J. Math. Anal. Appl. **277** (2003), 358–366.

[7] C. Cosner, D. L. DeAngelis, J. S. Ault, and D. B. Olson, *Effects of spatial grouping on the functional response of predator*, Theoret. Population Biol. **56** (1999), 65–75.

[8] D. L. DeAngelis, R. A. Goldstein, and R. V. Neill, *A model for trophic interaction*, Ecology **56** (1975), 881–892.

[9] M. Fan and K. Wang, *Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system*, Math. Comput. Modelling **35** (2002), no. 9,10.

[10] Q. Fang, X. Li, and M. Cao, *Dynamics of a discrete predator-prey system with Beddington-DeAngelis function response*, Appl. Math. **3** (2012), 389–394.

[11] H. I. Freedman, *Deterministic Mathematics Models in Population Ecology*, Marcel Dekker, Inc., New York.

[12] H. F. Huo and W. T. Li, *Stable periodic solution of the discrete periodic Leslie-Gower predator-prey model*, Math. Comput. Modelling **40** (2004), 261–269.

[13] T. W. Hwang, *Uniqueness of limit cycles of the predator-prey system with Beddington-DeAngelis functional response*, J. Math. Anal. Appl. **281** (2003), 395–401.

[14] ______, *Global analysis of th predator-prey system with Beddington-DeAngelis functional response*, J. Math. Anal. Appl. **290** (2004), 113–122.

[15] P. Kratina, M. Vos, A. Bateman, and B. R. Anholt, *Functional response modified by predator density*, Oecologia **159** (2009), 425–433.

[16] H. Li and Z. Lu, *Stability of ratio-dependent delayed predator-prey system with density regulation*, J. Biomath **20** (2005), 264–272.

[17] H. Li and Y. Takeuchi, *Stability for ratio-dependent predator-prey system with density dependent*, Proceedings of the 70th Conference on Biological Dynamics System and Stability of Differential Equations. World Academic Union **I** (2010), 144–147.

[18] ______, *Dynamics of the density dependent predator-prey system with Beddington-DeAngelis functional response*, J. Math. Anal. Appl. **374** (2011), 644–654.

[19] ______, *Dynamics of the density dependent and nonautonomous predator-prey system with Beddington-DeAngelis functional response*, Dyn. Sys. Ser. B **20** (2015), no. 4, 1117–1134.

[20] Y. Li, T. Zhang, and Y. Ye, *On the existence and stability of a unique almost periodic sequence solution in discrete predator-prey models with time delays*, Appl. Math. Modelling **35** (2011), 5448–5459.

[21] Z. Li and F. Chen, *Almost periodic solutions of a discrete almost periodic logistic equation*, Math. and Comp. Modelling **50** (2009), 254–259.

[22] J. D. Murray, *Mathematical Biology*, Springer-Verlag, New York.

[23] N. M. Pelen, A. F. Güvenilir, and B. Kaymakçalan, *Necessary and sufficient condition for the periodic solution of predator-prey system with Beddington-DeAngelis type functional response*, Advances in Difference Equations **15** (2016), 1–19.

[24] ______, *Some results on predator-prey dynamic systems with Beddington-DeAngelis type functional response on the time scale calculus*, Dynamic Systems and Applications **26** (2017), 167–182.

[25] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific Series on Nonlinear Science. World Scientific, Singapore.

[26] J. A. Vucetich, R. O. Peterson, and C. L. Schaeffer, *The effect of prey and predator densities on wolf predation.*

[27] J. Zhang and J. Wang, *Periodic solutions for discrete predator-prey systems with the Beddington-DeAngelis functional response.*

[28] S. N. Zhang and G. Zheng, *Almost periodic solutions of delay difference systems.*

[29] Z. Zhou and X. Zou, *Stable periodic solutions in a discrete periodic logistic equations.*

(Recibido en junio de 2017. Aceptado en diciembre de 2017)

Departamento de Matemáticas
Universidad de Los Andes
Facultad de Ciencias
La Hechicera
Mérida, Venezuela
e-mail: duquec@ula.ve