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Absoluteness theorems for arbitrary Polish spaces

Teoremas de absolutidad para espacios polacos arbitrarios

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Abstract. By coding Polish metric spaces with metrics on countable sets, we propose an interpretation of Polish metric spaces in models of ZFC and extend Mostowski's classical theorem of absoluteness of analytic sets for any Polish metric space in general. In addition, we prove a general version of Shoenfield's absoluteness theorem.

Key words and phrases. Mostowski's Absoluteness Theorem, Shoenfield's Absoluteness Theorem, Polish metric spaces.

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Resumen. Mediante la codificación de espacios polacos con métricas de conjuntos contables, proponemos una interpretación de espacios métricos polacos en modelos de ZFC y extendemos el clásico Teorema de Absolutidad (para conjuntos analíticos) de Mostowski para cualquier espacio métrico polaco en general. Adicionalmente, probamos una versión general del Teorema de Absolutidad de Shoenfield.

Palabras y frases clave. Teorema de Absolutidad de Mostowski, Teorema de Absolutidad de Shoenfield, espacios métricos polacos.

1. Introduction

Mostowski's Absoluteness Theorem (also known as $\boldsymbol{\Sigma}^1_1$ *absoluteness*) states that any analytic subset of the Baire space ω^ω is absolute for transitive models of ZFC. To be more precise, for any tree $T \subseteq (\omega \times \omega)^{<\omega}$ the statement $\exists y \in \omega^\omega((x, y) \in [T])$ is absolute (recall that any closed subset of $(\omega \times \omega)^\omega$ is characterized by such a tree, so they code all the analytic subsets). On the other hand, *Shoenfields's Absoluteness Theorem* (also known as $\mathbf{\Sigma}^1_2$ *absoluteness*) states that any $\mathbf{\Sigma}^1_2$ subset of the Baire space is absolute for transitive models $M \subseteq N$ of ZFC when $\omega_1^N \subseteq M$. Since, $\mathbf{\Pi}^1_n$ sets are the complements of $\mathbf{\Sigma}^1_n$ sets (for $n = 1, 2, \ldots$), the previous theorems are equivalent to $\mathbf{\Pi}^1_1$ absoluteness and $\mathbf{\Pi}^1_2$ absoluteness, respectively.

The $\mathbf{\Sigma}^1_1$ absoluteness theorem was proven by Mostowski [9] in 1959, while $\mathbf{\Sigma}^1_2$ absoluteness was proven by Shoenfield [10] in 1961,[1] though proofs with modern notation can be found in standard references like [4] and [8]. Although they are only proved in the context of ω^ω, the same proof works for the Cantor space 2^ω and for any Polish space of the form $\prod_{n<\omega} S(n)$ with the product topology where each $S(n)$ is a countable discrete space. The same arguments seem to be able to be readapted for spaces like $\mathbb{R}$, $\mathbb{R}^\omega$, or any other standard Polish spaces.

However, it seems that there is no reference of a general version of these absoluteness theorems for arbitrary Polish spaces. It may be that mathematicians trust that they can be reproved similarly for each particular Polish space that comes at hand, so there is no worry to provide general statements. Another reason may be that each well known Polish space has a certain shape that tells how to be interpreted in an arbitrary model of ZFC and standard ways of interpreting may vary depending on each space.

In this paper the main result is the generalization of Mostowski's theorem and Shoenfield's theorem for any arbitrary Polish space or, more concretely, for any *Polish metric space*. To achieve this, we provide a way to code Polish metric spaces by reals, define how to interpret a Polish metric space with respect to a given code, and use Solovay-type codes (cf. [11, Sect. II.1]) of analytic and $\mathbf{\Sigma}^1_2$ subsets to be able to state and prove the general theorems.

The proof of our theorems is also a generalization of the original proofs. For a tree $T \subseteq (\omega \times A)^{<\omega}$ and $x \in \omega^\omega$ denote $T(x) := \{t \in A^{<\omega} : (x{\restriction}|t|, t) \in T\}$. The proof of Mostowski's Absoluteness Theorem indicates that, for any analytic $A \subseteq \omega^\omega$, we can find a tree $T \subseteq (\omega \times \omega)^{<\omega}$ such that, for any $x \in \omega^\omega$, $x \in A$ iff the tree $T(x)$ is ill-founded (compare with the first paragraph of this introduction); the main point of Shoenfield's theorem is that, for any $\mathbf{\Sigma}^1_2$-set $P \subseteq \omega^\omega$, there is a tree $\hat{T} \subseteq (\omega \times \omega \times \omega_1)^{<\omega}$ such that, for any $x \in \omega^\omega$, $x \in P$ iff the tree $\hat{T}(x)$ is ill-founded. In the case of the Cantor space 2^ω, subtrees of $(2 \times \omega)^{<\omega}$ and $(2 \times \omega \times \omega_1)^{<\omega}$ are considered, respectively, in an analogous way. In our generalizations, for any arbitrary Polish space X, we construct definable functions T_c from X into the subtrees of $\omega^{<\omega}$ (see Definition 3.8), and $\hat{T}_{c,\omega_1}$ from X into the subtrees of $(\omega \times \omega_1)^{<\omega}$ (see Definition 4.1), where c is a real that codes the corresponding $\mathbf{\Sigma}^1_1$ and $\mathbf{\Sigma}^1_2$ sets. We show that, for $x \in X$, x is in the

[1]Shoenfield [10] also states that $\mathbf{\Sigma}^1_1$ absoluteness follows by Klenne's characterization of Π^1_1 sets through well-orders (see [6]), plus the absoluteness of the notion of "well-order".

$\mathbf{\Sigma}^1_1$ (resp. $\mathbf{\Sigma}^1_2$) subset coded by c iff $T_c(x)$ (resp. $\hat{T}_{c,\omega_1}(x)$) is ill-founded. Hence, our generalizations of both absoluteness results come from the absoluteness of the functions T_c and $\hat{T}_{c,\omega_1}$, which we prove accordingly.

One motivation of this paper is the research of forcing preservation results of definable cardinal characteristics of the continuum and the study of Suslin ccc posets presented in [1, Sect. 4], in which arbitrary Polish spaces and their interpretations in forcing extensions play a fundamental role. For instance, when considering a Suslin ccc poset under an arbitrary Polish space, the forcing relation of Borel statements can be described by Σ^1_2-sets, so we can conclude its absoluteness by our general theorems.

We discriminate between the terms 'Polish space' and 'Polish metric space'. A *Polish metric space* is a separable complete metric space $\langle X, d\rangle$ and a *Polish space* is a topological space X which is homeomorphic to some Polish metric space (so note that in the first concept the metric is required). The reason we differentiate those terms is that we code using countable metric spaces as in [2] because any Polish metric space is the completion of such a space, so the actual metric in the Polish (metric) space cannot be ignored. Though this coding is also introduced in [3, Ch. 14] (a standard reference in this topic), the discussion there is focused on the Effros Borel space of the universal Urysohn space. Our preference towards the first coding relies on the practicality to interpret Polish spaces in models of ZFC through this coding.

We fix additional notation. Given two metric spaces $\langle X, d\rangle$ and $\langle X', d'\rangle$, say that a function $\iota : \langle X, d\rangle \to \langle X', d'\rangle$ is an *isometry* if, for all $x, y \in X$, $d(x, y) = d(f(x), f(y))$ (we do not demand an isometry to be onto). Additionally, we say that ι is an *isometrical isomorphism* if it is onto, in which case we say that the metric spaces $\langle X, d\rangle$ and $\langle X', d'\rangle$ are *isometrically isomorphic*. By a *transitive model of ZFC* we actually refer to a transitive model of a large enough finite part of ZFC to carry the arguments at hand.

We structure this paper as follows. In Section 2 we review some general aspects about completions of metric spaces, introduce the coding for Polish metric spaces and look at the complexity of statements concerning these codes. In Section 3 we define interpretations of codes of Polish metric spaces in transitive models of ZFC, and we use a well known coding for open and closed sets to code analytic sets and to state and prove Mostowski's theorem in its general version. Finally, in Section 4, we state and prove a general version of Shoenfield's Absoluteness Theorem.

2. Coding Polish metric spaces and functions

The contents of this section reviews some quite known facts about completion of metric spaces and coding of Polish metric spaces. They are presented as a summary of [7].

Let $\langle X, d\rangle$ be a metric space. Recall that $\langle X^*, d^*, \iota\rangle$ is a *completion of* $\langle X, d\rangle$ if $\langle X^*, d^*\rangle$ is a complete metric space and $\iota : \langle X, d\rangle \to \langle X^*, d^*\rangle$ is a *dense isometry*, that is, an isometry such that $\iota[X]$ is dense in X^*. For our main results, the dense isometries associated with the completions will play an important role.

Given a metric space $\langle X, d\rangle$ and an isometry $\iota : \langle X, d\rangle \to \langle X^*, d^*\rangle$ into a complete metric space $\langle X^*, d^*\rangle$, say that $\langle X^*, d^*, \iota\rangle$ *commutes diagrams of isometries from* $\langle X, d\rangle$ if, for any isometry $f : \langle X, d\rangle \to \langle Y, d'\rangle$ into a complete metric space $\langle Y, d'\rangle$, there is a unique continuous function $\hat{f} : \langle X^*, d^*\rangle \to \langle Y, d'\rangle$ such that $f = \hat{f} \circ \iota$. It is well known that $\langle X^*, d^*, \iota\rangle$ is a completion of $\langle X, d\rangle$ iff it commutes diagrams of isometries, even more, such a completion is unique modulo isometries.

Even more, a completion commutes diagrams of much less than isometries. Recall that a function $f : \langle X, d\rangle \to \langle Y, d'\rangle$ between metric spaces is *Cauchy-continuous* if, for any Cauchy sequence $\langle x_n\rangle_{n<\omega}$ in $\langle X, d\rangle$, $\langle f(x_n)\rangle_{n<\omega}$ is a Cauchy sequence in $\langle Y, d'\rangle$. Though Cauchy-continuity is not equivalent to continuity in general, whenever $f : \langle X, d\rangle \to \langle Y, d'\rangle$ is a function between metric spaces with $\langle X, d\rangle$ complete, f is continuous iff it is Cauchy-continuous.

Theorem 2.1. *Let $\langle X_0, d_0, \iota\rangle$ be a completion of the metric space $\langle X, d\rangle$ and let $f : \langle X, d\rangle \to \langle Y, d'\rangle$ be a continuous function into a complete metric space $\langle Y, d'\rangle$.*

(a) There is at most one continuous function $\hat{f} : X_0 \to Y$ such that $f = \hat{f} \circ \iota$.

(b) $\hat{f}$ as in (a) exists iff f is Cauchy-continuous.

(c) If f is Cauchy-continuous, then

- *(c-1) $\hat{f}$ is uniformly continuous iff f is.*
- *(c-2) $\hat{f}$ is an isometry iff f is.*
- *(c-3) $\hat{f}$ is an isometrical isomorphism iff f is a dense isometry.*

(d) If $\langle X_1, d_1, \iota_1\rangle$ commutes diagrams of isometries from $\langle X, d\rangle$, then there is a unique isometrical isomorphism $\iota^* : \langle X_0, d_0\rangle \to \langle X_1, d_1\rangle$ such that $\iota_1 = \iota^* \circ \iota$. In particular, $\langle X_1, d_1, \iota_1\rangle$ is a completion of $\langle X, d\rangle$.

Now we introduce coding for Polish metric spaces. The completion of any countable metric space is a Polish (metric) space and, conversely, any Polish space is the completion of some countable metric space. As countable metric spaces are simple to describe, they can be used to code Polish metric spaces. Concretely, when d is a metric on $\eta := \mathrm{dom}(d)$ and η is a countable ordinal, we say that d *codes the Polish metric space* $\langle X, d_X\rangle$ if $\langle X, d_X, \iota\rangle$ is a completion of $\langle \eta, d\rangle$ for some dense isometry ι. When Z is a Polish space, we say that d *codes* Z if some (or any) completion of $\langle \eta, d\rangle$ is homeomorphic with Z.

Example 2.2. (1) If d is a metric on a natural number n, then d only codes Hausdorff spaces of size n.

(2) The Polish metric space $\langle\mathbb{R}, d_{\mathbb{R}}\rangle$ with the standard metric is coded by $d_{\mathbb{Q}}$ where $d_{\mathbb{Q}}$ is the (unique) metric on ω that makes the canonical bijection $\iota_{\mathbb{Q}}:\omega\to\mathbb{Q}$ an isometry onto $\langle\mathbb{Q}, d_{\mathbb{R}}{\restriction}(\mathbb{Q}\times\mathbb{Q})\rangle$. As a consequence, $d_{\mathbb{Q}}$ codes $\mathbb{R}$ as a Polish space.

(3) For $S:\omega\to(\omega+1)\smallsetminus\{0\}$ recall the complete metric $d_{\prod S}$ on $\prod S=\prod_{n<\omega}S(n)$ given by $d_{\prod S}(x,y)=2^{-\inf\{n<\omega: x(n)\neq y(n)\}}$, which is compatible with the product topology when each $S(n)$ is discrete. Here, $\langle\prod S, d_{\prod S}\rangle$ is coded by $d_{\mathbb{Q}^S}$ where $\eta=|\mathbb{Q}^S|$,[2] $\mathbb{Q}^S$ is the set of eventually zero sequences in $\prod S$ and $d_{\mathbb{Q}^S}$ is the (unique) metric on η such that the canonical bijection $\iota_{\mathbb{Q}^S}:\eta\to\mathbb{Q}^S$ is an isometry onto $\langle\mathbb{Q}^S, d_{\prod S}{\restriction}(\mathbb{Q}^S\times\mathbb{Q}^S)\rangle$.[3]

(4) As a particular case of (3), consider $\bar{\omega}:\omega\to\{\omega\}$ the constant function on ω. The metric $d_{\mathbb{Q}^{\bar{\omega}}}$ on ω will be used as the standard coding of the Baire space.

As indicated in Example 2.2(1), metrics on finite sets are trivial as codes for Polish spaces. Thereafter, we concentrate our work on metrics on ω. Denote by $\mathcal{D}(\omega)$ the set of metrics on ω. As $\mathcal{D}(\omega)\subseteq\mathbb{R}^{\omega\times\omega}$, it is understood that infinite Polish spaces are coded by *reals*. Define the order $\preceq_{\mathrm{di}}$ on $\mathcal{D}(\omega)$ as $d\preceq_{\mathrm{di}} d'$ iff there is a dense isometry $\iota:\langle\omega,d\rangle\to\langle\omega,d'\rangle$ ('di' stands for 'dense isometry'), and define the equivalence relation $\approx_{\mathrm{di}}$ on $\mathcal{D}(\omega)$ as $d\approx_{\mathrm{di}} d'$ iff $\langle\omega,d\rangle$ and $\langle\omega,d'\rangle$ have isometrically isomorphic completions.

Lemma 2.3 ([7, Cor. 3.4])**.** *For any $d,d'\in\mathcal{D}(\omega)$, $d\approx_{\mathrm{di}} d'$ iff there is a $d^*\in\mathcal{D}(\omega)$ such that $d,d'\preceq_{\mathrm{di}} d^*$.*

The following results compile some features of the coding and their complexity.

Theorem 2.4 ([2, Lemma 4], see also [7, Thm. 3.5])**.** *(a) The family $\mathcal{D}(\omega)$ of metrics on ω is $\mathbf{\Pi}^0_1$ in $\mathbb{R}^{\omega\times\omega}$. In particular, $\mathcal{D}(\omega)$ is a Polish space.*

(b) The statement "x is dense in the metric space $\langle\omega,d\rangle$" is $\mathbf{\Sigma}^0_2$ in $2^\omega\times\mathbb{R}^{\omega\times\omega}$.

(c) The statement "$g:\langle\omega,d\rangle\to\langle\omega,d'\rangle$ is an isometry between metric spaces" is $\mathbf{\Pi}^0_1$ in $\omega^\omega\times(\mathbb{R}^{\omega\times\omega})^2$.

(d) The function $\mathrm{Img}:2^\omega\times\omega^\omega\to 2^\omega$ defined as $\mathrm{Img}(x,g)=g[x]$ is continuous.

(e) The relation $\preceq_{\mathrm{di}}$ is $\mathbf{\Sigma}^1_1$ in $(\mathbb{R}^{\omega\times\omega})^2$.

[2] Note that η is finite iff $S(n)=1$ for all but finitely many n.

[3] Consider $\mathbb{Q}^S$ with the anti-lexicographic order. Since it has order type η, $\iota_{\mathbb{Q}^S}$ can be defined as the (unique) order-isomorphism between both sets.

(f) The relation $\approx_{\mathrm{di}}$ *is* $\mathbf{\Sigma}^1_1$ *in* $(\mathbb{R}^{\omega\times\omega})^2$.

Lemma 2.5 ([7, Lemma 3.6]). *Let* $\langle X, d\rangle$ *be a metric space and let* $\langle X^*, d^*, \iota\rangle$ *be its completion.*

(a) If $z \in X^*$ *is isolated, then* $z \in \iota[X]$.

(b) $x \in X$ *is isolated iff* $\iota(x)$ *is isolated in* X^*.

(c) X^* *is perfect iff* X *is perfect.*

Corollary 2.6 ([7, Cor. 3.7]). $\langle \omega, d\rangle$ *codes a perfect Polish space iff* $\langle \omega, d\rangle$ *is perfect. Even more, the set*

$$\mathcal{D}^*(\omega) := \{d \in \mathcal{D}(\omega) : \langle \omega, d\rangle \text{ is perfect}\}$$

is $\mathbf{\Pi}^0_2$ *in* $\mathbb{R}^{\omega\times\omega}$*, so it is a Polish space.*

Recall that every perfect countable metric space is homeomorphic to $\mathbb{Q}$, so all the codes for Perfect Polish spaces are pairwise homeomorphic. This means that homeomorphic countable metric spaces do not lead to homeomorphic completions.

Cantor-Bendixson Theorem (see, e.g., [5, Thm. 6.4]) states that any Polish space has a unique partition on a perfect set (known as the *perfect kernel*) and a countable open set. More generally, any second countable space has a perfect kernel and its complement is open countable (see [5, Sect. 6.C]). However, the perfect kernel of a countable metric space does not represent the perfect kernel of its completion (for an example, see [7, Sect. 3]).

We finish this section with a brief review about codes of homeomorphic Polish spaces. Define the equivalence relation $\approx_{\mathrm{P}}$ on $\mathcal{D}(\omega)$ by $d_0 \approx_{\mathrm{P}} d_1$ iff the completions of $\langle \omega, d_0\rangle$ and $\langle \omega, d_1\rangle$ are homeomorphic. This means that d_0 and d_1 code the same Polish space (modulo homeomorphism). The following result settles the complexity of this equivalence relation.

Theorem 2.7 ([3, Proposition 14.4.2], see also [7, Sect. 4]). *The relation* $\approx_{\mathrm{P}}$ *is* $\mathbf{\Sigma}^1_2$ *in* $(\mathbb{R}^{\omega\times\omega})^2$.

A deeper study about coding Polish spaces and the equivalence relations $\approx_{\mathrm{di}}$ and $\approx_{\mathrm{P}}$ can be found in [3, Ch. 14].

3. Mostowski's Absoluteness Theorem

Throughout this section, we fix $M \subseteq N$ transitive models of ZFC. Given a countable metric space, it is clear that the Polish metric space it codes is interpreted in a model as a completion inside the model. Though these completions are isometrically isomorphic, we do not restrict the concept of interpretation to a single completion. For this purpose, the dense isometry we deal with is also relevant.

Definition 3.1. Let $d \in \mathcal{D}(\omega) \cap M$. Say that $\langle X_d^M, (d^*)^M, \iota_d^M \rangle$ is an *interpretation in M of the Polish metric space coded by d* if it is an object in M and

$$M \models \text{"}\langle X_d^M, (d^*)^M, \iota_d^M \rangle \text{ is a completion of } \langle \omega, d \rangle\text{"}.$$

Notation 3.2. Let $d \in \mathcal{D}(\omega)$. We write $\langle X_d, d^*, \iota_d \rangle$ to refer to an arbitrary completion of $\langle \omega, d \rangle$, unless it is explicitly specified. In the same way, if M is a transitive model of ZFC and $d \in M$, we write $\langle X_d^M, (d^*)^M, \iota_d^M \rangle$ to refer to an arbitrary completion of $\langle \omega, d \rangle$ *inside* M, unless it is explicitly specified.

When $M \subseteq N$, the interpretation of a Polish metric space (through some code) in N is actually a completion of its interpretation in M, as illustrated in the following result.

Lemma 3.3. *Let $d \in \mathcal{D}(\omega) \cap M$. Then, there is a unique $\iota_d^{M,N} \in N$ such that $\iota_d^N = \iota_d^{M,N} \circ \iota_d^M$ and $N \models$ "$\langle X_d^N, (d^*)^N, \iota_d^{M,N} \rangle$ is a completion of $\langle X_d^M, (d^*)^M \rangle$".*

Proof. Work within N. Choose a completion $\langle X^*, d^*, \iota^* \rangle$ of $\langle X_d^M, (d^*)^M \rangle$. By Theorem 2.1 applied to $\iota^* \circ \iota_d^M$, there is an isometrical isomorphism $f : \langle X^*, d^* \rangle \to \langle X_d^N, (d^*)^N \rangle$ such that $f \circ \iota^* \circ \iota_d^M = \iota_d^N$. Put $\iota_d^{M,N} = f \circ \iota^*$. ☑

To state Mostowski's Absoluteness Theorem in full generality, we need to code analytic sets. For a Polish space X, recall that $A \subseteq X$ is *analytic in X* if A is the projection of some closed set C of $X \times \omega^\omega$, that is, $A = \{x \in X : \exists y \in \omega^\omega((x, y) \in C)\}$. As this definition relies on closed sets, we first aim to code closed and open subsets of a Polish (metric) space.

Definition 3.4. Fix $\{(k_n^*, q_n^*) : n < \omega\}$ a bijective enumeration of $\omega \times \mathbb{Q}^+$.

(1) Define $\mathrm{BC}_1 := \{c \in 2^\omega : c(0) = 0\}$.

(2) If $d \in \mathcal{D}(\omega)$ and $c \in \mathrm{BC}_1$, define the *open subset of X_d coded by c (with respect to d, ι_d)* as

$$\mathrm{Op}(c) = \mathrm{Op}_{X_d}^{d,\iota_d}(c) := \bigcup \{B_{\langle X_d, d^* \rangle}(\iota_d(k_n^*), q_n^*) : c(n+1) = 1,\ n < \omega\}.$$

Define $\mathrm{Cl}(c) = \mathrm{Cl}_{X_d}^{d,\iota_d}(c) := X_d \smallsetminus \mathrm{Op}(c)$ the *closed subset of X^* coded by c (with respect to d, ι_d)*.

The previous definition is close to Solovay's original coding of Borel sets (see [11, Sect. II.1]). In the case of Definition 3.4(2), note that any open subset of X_d is coded by some $c \in \mathrm{BC}_1$, likewise for any closed subset. Though we could have used all 2^ω to code open sets, we use BC_1 to allow, as in Solovay's work, further definition of $\mathrm{BC}_\alpha \subseteq 2^\omega$ for $1 < \alpha < \omega_1$ by recursion so that BC_α codes all the $\boldsymbol{\Sigma}_\alpha^0$ and $\boldsymbol{\Pi}_\alpha^0$ subsets of any Polish metric space. Solovay also proved that the set $\mathrm{BC} := \bigcup_{\alpha < \omega_1} \mathrm{BC}_\alpha$ of codes of Borel sets is Π_1^1 (in an effective way).

For every $x \in X_d$, there is a sequence $\langle m_l \rangle_{l<\omega}$ in ω such that $\langle \iota_d(m_l) \rangle_{l<\omega}$ converges to x and $d(m_l, m_{l+1}) < 2^{-(l+2)}$ for all $l < \omega$. Even more, such a sequence is always Cauchy in $\langle \omega, d \rangle$. This fact let us deal with absoluteness of open and closed sets in a very easy way, as we can see in the following results.

Lemma 3.5. *Let $(k, q) \in \omega \times \mathbb{Q}^+$, $d \in \mathcal{D}(\omega)$, and let $\langle m_l \rangle_{l<\omega}$ be a sequence in ω such that $d(m_l, m_{l+1}) < 2^{-(l+2)}$ for all $l < \omega$. If the sequence $\langle \iota_d(m_l) \rangle_{l<\omega}$ converges to a point $x \in X_d$, then $d^*(x, \iota_d(k)) < q$ iff there are an $\varepsilon \in \mathbb{Q}^+$ and an $l < \omega$ such that $d(k, m_l) < q - 2^{-(l+1)} - \varepsilon$ (even more, the latter holds for all but finitely many $l < \omega$).*

Proof. If $d^*(x, \iota_d(k)) < q$, find $l' < \omega$ and $\varepsilon \in \mathbb{Q}^+$ such that $d^*(x, \iota_d(k)) < q - 2^{-(l'+1)} - \varepsilon$. Now, for all but finitely many $l < \omega$, we have $d^*(\iota_d(m_l), \iota_d(k)) < q - 2^{-(l'+1)} - \varepsilon$, so $d(k, m_l) < q - 2^{-(l'+1)} - \varepsilon$. Therefore, for all but finitelly many l (above l'), $d^*(k, m_l) < q - 2^{-(l+1)} - \varepsilon$.

Now assume that there are an $\varepsilon \in \mathbb{Q}^+$ and an $l^* < \omega$ such that $d(k, m_{l^*}) < q - 2^{-(l^*+1)} - \varepsilon$. Then, by induction, it can be proved that $d(k, m_l) < q - 2^{-(l+1)} - \varepsilon$ for all $l \geq l^*$. Therefore, $d^*(\iota_d(k), x) \leq q - \varepsilon$. ☑

Lemma 3.6. *Let $c \in \mathrm{BC}_1 \cap M$ and $d \in \mathcal{D}(\omega) \cap M$.*

(a) If $(k, q) \in \omega \times \mathbb{Q}^+$ then $(\iota_d^{M,N})^{-1}[B^N_{X_d^N}(\iota_d^N(k), q)] = B^M_{X_d^M}(\iota_d^M(k), q)$.

(b) If $c \in BC_1 \cap M$ then $(\iota_d^{M,N})^{-1}[\mathrm{Op}^N(c)] = \mathrm{Op}^M(c)$ and $(\iota_d^{M,N})^{-1}[\mathrm{Cl}^N(c)] = \mathrm{Cl}^M(c)$.

Proof. (a) Let $x \in X_d^M$, so there exists a Cauchy sequence $\langle m_l \rangle_{l<\omega} \in M$ in $\langle \omega, d \rangle$ such that $\langle \iota_d^M(m_l) \rangle_{l<\omega}$ converges to x and $d(m_l, m_{l+1}) < 2^{-(l+2)}$ for all $l < \omega$. Thus, by Lemma 3.5, $x \in B^M_{X_d^M}(\iota_d^M(k), q)$ iff $\iota_d^{M,N}(x) \in B^N_{X_d^N}(\iota_d^N(k), q)$.

(b) Immediate from (a).

☑

Lemma 3.7. *If $c \in \mathrm{BC}_1 \cap M$ and $d \in \mathcal{D}(\omega)$ then, in N, $\mathrm{Cl}^N(c)$ is the closure of $\iota_d^{M,N}[\mathrm{Cl}^M(c)]$ with respect to X_d^N.*

Proof. It is enough to prove that the statement '$\mathrm{Op}(c) \cap \mathrm{Cl}(c') \neq \varnothing$' in $\mathrm{BC}_1 \times \mathrm{BC}_1$ is absolute for transitive models of ZFC. Let T be a set of finite sequences t such that

(i) $|t| \geq 3$, $t(0) \in \mathbb{Q}^+$ and $t(k) \in \omega$ for all $0 < k < |t|$,

(ii) $c(t(1) + 1) = 1$,

(iii) $d(k^*_{t(1)}, t(2)) < q^*_{t(1)} - 2^{-1} - t(0)$,

and, for any $2 \leq k < |t|$,

(iv) $d(t(k), t(k+1)) < 2^{-k}$ and

(v) for every $n < \omega$, if $c'(n+1) = 1$ then $d(k^*_n, t(k)) \geq q^*_n - 2^{-(k-1)}$.

Note that T is a tree in the sense that, whenever $t \in T$ and $3 \leq k < |t|$, $t \restriction k \in T$. Using this tree, it remains to show that '$\mathrm{Op}(c) \cap \mathrm{Cl}(c') \neq \varnothing$' is equivalent to say '$T$ has an infinite branch' (the latter statement is absolute because it means that T, ordered by $\supseteq$, is not well-founded). First assume that $\mathrm{Op}(c) \cap \mathrm{Cl}(c') \neq \varnothing$, which means that there is some $x \in \mathrm{Op}(c)$ such that $x \notin \mathrm{Op}(c')$. Choose a sequence $\langle m_l \rangle_{l<\omega}$ in ω such that $\langle \iota_d(m_l) \rangle_{l<\omega}$ converges to x and $d(m_l, m_{l+1}) < 2^{-(l+2)}$ for all $l < \omega$. As $x \in \mathrm{Op}(c)$, by Lemma 3.5, there are some $j_0 < \omega$, $\varepsilon_0 \in \mathbb{Q}^+$ and $l_0 < \omega$ such that $c(j_0+1) = 1$ and $d(k^*_{j_0}, m_{l_0}) < q^*_{j_0} - 2^{-(l_0+1)} - \varepsilon_0$. Wlog we may assume that $l_0 = 0$. On the other hand, $x \notin \mathrm{Op}(c')$ implies that $d(k^*_n, m_l) \geq q^*_n - 2^{-(l+1)}$ for any $l < \omega$ and any $n < \omega$ such that $c'(n+1) = 1$ (because $d^*(x, m_l) < 2^{-(l+1)}$). Define the sequence b with domain ω as $b(0) := \varepsilon_0$, $b(1) := j_0$ and $b(k) := m_{k-2}$ for every $k \geq 2$. It is clear that b is a branch in T, that is, $b \restriction k \in T$ for any $k \geq 3$.

To see the converse, assume that b is a branch in T. Define $m_l := b(l+2)$ for any $l < \omega$. By (iv), $\langle m_l : l < \omega \rangle$ is a Cauchy-sequence in $\langle \omega, d \rangle$, so it converges in X_d to some x, even more, $d^*(x, \iota_d(m_l)) < 2^{-(l+1)}$. Thus, by (iii), $d(x, \iota_d(k^*_{b(1)})) \leq q^*_{b(1)} - b(0) < q^*_{b(1)}$ so, as $c(b(1)+1) = 1$ by (i), $x \in \mathrm{Op}(c)$. On the other hand, (v) implies that, whenever $c'(n+1) = 1$, $d^*(\iota_d(k^*_n), x) \geq q^*_n$, that is, $x \notin \mathrm{Op}(c')$. ☑

In the previous proof, the statement '$\mathrm{Op}(c) \cap \mathrm{Cl}(c') \neq \varnothing$' is actually $\mathbf{\Sigma}^1_1$ in $\mathrm{BC}_1 \times \mathrm{BC}_1$, so its absoluteness should be implied by Mostowski's Absoluteness Theorem for the Cantor space (a standard well-known case). Actually, the argument above relies on the absoluteness of well-foundedness, which is actually the main point in the proof of Mostowski's Absoluteness Theorem.

So far we know we can use BC_1 to code analytic sets, but still we should deal with a standard way to code products, or at least products with ω^ω. Recall the coding of the Baire space presented in Example 2.2(4). Fix $X_{d_{\mathbb{Q}^{\bar{\omega}}}} = \omega^\omega$, $\iota_{d_{\mathbb{Q}^{\bar{\omega}}}} = \iota_{\mathbb{Q}^{\bar{\omega}}}$ and $d^*_{\mathbb{Q}^{\bar{\omega}}} = d_{\omega^\omega}$. According to Notation 3.2, $X^M_d = \omega^\omega \cap M$ and $\iota^M_{d_{\mathbb{Q}^{\bar{\omega}}}} = \iota_{d_{\mathbb{Q}^{\bar{\omega}}}}$, so $\iota^{M,N}_{d_{\mathbb{Q}^{\bar{\omega}}}} : \omega^\omega \cap M \to \omega^\omega \cap N$ is the inclusion map. Fix the canonical bijection $i^* = (i^*_0, i^*_1) : \omega \to \omega \times \omega$. For $d \in \mathcal{D}(\omega)$ let $d_\pi \in \mathcal{D}(\omega)$ be the unique metric on ω that makes i^* an isometry onto $\langle \omega \times \omega, d \otimes d_{\mathbb{Q}^{\bar{\omega}}} \rangle$ where $(d \otimes d_{\mathbb{Q}^{\bar{\omega}}})((m_0, n_0), (m_1, n_1)) = \max\{d(m_0, m_1), d_{\mathbb{Q}^{\bar{\omega}}}(n_0, n_1)\}$. Given a completion $\langle X_d, d^*, \iota_d \rangle$ of $\langle \omega, d \rangle$, we denote by $\langle X_{d_\pi}, d^*_\pi, \iota_{d_\pi} \rangle$ the completion of $\langle \omega, d_\pi \rangle$ given by $X_{d_\pi} = X_d \times \omega^\omega$, $d^*_\pi((x_0, y_0), (x_1, y_1)) = \max\{d^*(x_0, x_1), d^*_\pi(y_0, y_1)\}$ and

$\iota_{d_\pi}(k) = (\iota_d(i_0^*(k)), \iota_{d_{\mathbb{Q}^{\bar\omega}}}(i_1^*(k)))$. According to Notation 3.2, $X_{d_\pi}^M = X_d^M \times (\omega^\omega \cap M)$ and $\iota_{d_\pi}^M = (\iota_d^M \circ i_0^*, \iota_{d_{\mathbb{Q}^{\bar\omega}}} \circ i_1^*)$, so $\iota_{d_\pi}^{M,N} = (\iota_d^{M,N}, \mathrm{id}_{\omega^\omega \cap M})$ (here id_A denotes the identity function on the set A).

Definition 3.8. For $c \in \mathrm{BC}_1$ and $d \in \mathcal{D}(\omega)$, define

(1) *the analytic subset of X_d coded by c (with respect to d, ι_d)* as $\mathrm{An}(c) = \mathrm{An}_{X_d}^{d,\iota_d}(c) := \{x \in X_d : \exists y \in \omega^\omega((x,y) \in \mathrm{Cl}_{X_{d_\pi}}^{d_\pi,\iota_{d_\pi}}(c))\}$, and

(2) $T_c = T_{c,X_d}^{d,\iota_d} : X_d \to \mathcal{P}(\omega^{<\omega})$ as

$$T_c(x) = \{t \in \omega^{<\omega} : x \in \mathrm{cl}_{X_d}\{w \in X_d : \exists y \in \omega^\omega(t \subseteq y \text{ and } ((w,y) \in \mathrm{Cl}_{X_{d_\pi}}^{d_\pi,\iota_{d_\pi}}(c))\}\}$$

where cl_{X_d} denotes the closure operation in X_d.

Clearly, any analytic subset of X_d is of the form $\mathrm{An}(c)$ for some $c \in \mathrm{BC}_1$. On the other hand, $T_c(x)$ is a tree in $\omega^{<\omega}$ for any $x \in X_d$ and the map T_c is Borel. The relation between $\mathrm{An}(c)$ and T_c is illustrated in the next theorem, which is sort of a generalization of the tree representation of an analytic set.

Theorem 3.9. *Let $d \in \mathcal{D}(\omega)$ and $c \in \mathrm{BC}_1$. For any $x \in X_d$, $x \in \mathrm{An}(c)$ iff $T_c(x)$ is ill-founded.*

Proof. If $x \in \mathrm{An}(c)$ then there is a $y \in \omega^\omega$ such that $(x,y) \in \mathrm{Cl}(c)$, so $y{\restriction}n \in T_c(x)$ for all $n < \omega$. Conversely, assume that $x \in X_d$ and $T_c(x)$ is ill-founded, so there is a $y \in \omega^\omega$ such that $y{\restriction}n \in T_c(x)$ for all $n < \omega$. By Definition of $T_c(x)$, there are $x_n \in X_d$ and $y_n \in \omega^\omega$ such that $d^*(x,x_n) < 2^{-n}$, $y{\restriction}n \subseteq y_n$ and $(x_n,y_n) \in \mathrm{Cl}(c)$. It is clear that the sequence $\langle (x_n,y_n)\rangle_{n<\omega}$ converges to (x,y) in $X_d \times \omega^\omega$ and, as $\mathrm{Cl}(c)$ is closed, $(x,y) \in \mathrm{Cl}(c)$. Thus, $x \in \mathrm{An}(c)$. ☑

To prove our version of Mostowski's theorem, we deal with the absoluteness of T_c in the following result.

Lemma 3.10. *If $d \in \mathcal{D}(\omega) \cap M$ and $c \in \mathrm{BC}_1 \cap M$ then $T_c^N \circ \iota_d^{M,N} = T_c^M$.*

Proof. Fix $x \in X_d^M$ and $t \in \omega^\omega$. Note that $t \in T_c^M(x)$ iff for all $\varepsilon \in \mathbb{Q}^+$

$$\exists w, y \in M(w \in X_d^M,\ y \in \omega^\omega,\ t \subseteq y,\ (d^*)^M(w,x) < \varepsilon \text{ and } (w,y) \in \mathrm{Cl}^M(c)).$$

On the other hand, $t \in T_c^N(\iota_d^{M,N}(x))$ iff for all $\varepsilon \in \mathbb{Q}^+$

$$\exists w', y' \in N(w' \in X_d^N,\ y' \in \omega^\omega,\ t \subseteq y',\ (d^*)^N(w', \iota_d^{M,N}(x)) < \varepsilon \text{ and } (w',y') \in \mathrm{Cl}^N(c)).$$

It is clear that $t \in T_c^M(x)$ implies $t \in T_c^N(\iota_d^{M,N}(x))$. For the converse, if $\varepsilon \in \mathbb{Q}^+$ and there are $w' \in X_d^N$ and $y' \in \omega^\omega \cap N$ satisfying the statement above for $\frac{\varepsilon}{2}$, by Lemma 3.7 we can find a $(w, y) \in \mathrm{Cl}^M(c)$ such that $t \subseteq y$ and $(d^*)^N(\iota_d^{M,N}(w), w') < \frac{\varepsilon}{2}$. Therefore $(d^*)^N(\iota_d^{M,N}(w), \iota_d^{M,N}(x)) < \varepsilon$, so $(d^*)^M(w, x) < \varepsilon$. ☑

Now, we are ready to prove one of the main results of this paper.

Theorem 3.11 (Mostowki's Absoluteness Theorem for Polish spaces)**.** *If* $d \in \mathcal{D}(\omega) \cap M$ *and* $c \in \mathrm{BC}_1$ *then* $(\iota_d^{M,N})^{-1}[\mathrm{An}^N(c)] = \mathrm{An}^M(c)$.

***Proof*.** Fix $x \in X_d^M$. By Theorem 3.9,

(i) $x \in \mathrm{An}^M(c)$ iff $M \models$"$T_c^M(x)$ is ill-founded", and

(ii) $\iota_d^{M,N}(x) \in \mathrm{An}^N(c)$ iff $N \models$"$T_c^N(\iota_d^{M,N}(x))$ is ill-founded".

By Lemma 3.10, $T_c^M(x) = T_c^N(\iota_d^{M,N}(x))$. So, as the ill-foundedness of a tree is absolute, then $x \in \mathrm{An}^M(c)$ iff $\iota_d^{M,N}(x) \in \mathrm{An}^N(c)$. ☑

As a consequence, the expression '$d \in \mathcal{D}(\omega)$' is absolute, as well as the relations $\approx_{\mathrm{di}}$ and $\approx_{\mathrm{cdi}}$. Moreover,

Corollary 3.12. *All the statements (or sets or relations) in Theorem 2.4 and Corollary 2.6 are absolute for transitive models of ZFC.*

Though the previous corollary could be obtained by versions of Mostowski's theorem for $(\mathbb{R}^{\omega\times\omega})^2$ or other similar spaces, Theorem 3.11 validates it immediately with respect to any metric on ω that codes $(\mathbb{R}^{\omega\times\omega})^2$ as a Polish space.

4. Shoenfield's Absoluteness Theorem

As in the previous section, we fix transitive models $M \subseteq N$ of ZFC and use Notation 3.2 to deal with arbitrary completions and interpretations of a countable metric space.

To state the general version of Shoenfield's theorem, we first code $\mathbf{\Sigma}_2^1$ sets of a Polish space. Let $d \in \mathcal{D}(\omega)$. We use the notation presented after Lemma 3.7 to define the metric $d_{\pi\pi} := (d_\pi)_\pi$ to code $X_d \times \omega^\omega \times \omega^\omega$ and $\iota_{d_{\pi\pi}} := \iota_{(d_\pi)_\pi}$ denotes the corresponding dense isometry. Note that $X_{d_{\pi\pi}}^M = X_d^M \times (\omega^\omega \cap M) \times (\omega^\omega \cap M)$ and $\iota_{d_{\pi\pi}}^{M,N} = (\iota_d^{M,N}, \mathrm{id}_{\omega^\omega\cap M}, \mathrm{id}_{\omega^\omega\cap M})$.

Fix a bijective enumeration $\{s_i : i < \omega\}$ of $\omega^{<\omega}$ such that $|s_i| \leq i$ for all $i < \omega$. If A and B are sets (or even definable classes that may not be sets), we abuse of the notation to say that $(s, t) \in (A \times B)^{<\omega}$ means $s \in A^{<\omega}$, $t \in B^{<\omega}$ and $|s| = |t|$. If $T \subseteq (A \times B)^{<\omega}$ is a tree and $y \in A^\omega$, we denote $T(y) := \{t \in B^{<\omega} : (y{\restriction}|t|, t) \in T\}$.

Definition 4.1. Let $d \in \mathcal{D}(\omega)$, $c \in \mathrm{BC}_1$ and let $\boldsymbol{\alpha}$ be either an ordinal or the class of all ordinals **ON**. Define

(1) the $\boldsymbol{\Sigma}^1_2$-*subset of* X_d *coded by* c *(with respect to* d, ι_d*)* as

$$\mathrm{Pca}(c) = \mathrm{Pca}^{\iota_d, d}_{X_d}(c) := \{x \in X_d : \exists y \in \omega^\omega \forall z \in \omega^\omega ((x,y,z) \notin \mathrm{Cl}^{d_{\pi\pi}, \iota_{d_{\pi\pi}}}_{X_{d_{\pi\pi}}}(c))\},$$

(2) the function $\hat{T}_c := \hat{T}^{d,\iota_d}_{c,X_d}$ with domain X_d such that

$$\hat{T}_c(x) := \{(s,t) \in (\omega \times \omega)^{<\omega} : x \in \mathrm{cl}_{X_d}\{u \in X_d : \exists (y,z) \in \omega^\omega \times \omega^\omega (s \subseteq y,\ t \subseteq z \text{ and } (u,y,z) \in \mathrm{Cl}^{d_{\pi\pi}, \iota_{d_{\pi\pi}}}_{X_{d_{\pi\pi}}}(c))\}\},$$

(3) and the function $\hat{T}_{c,\boldsymbol{\alpha}} := \hat{T}^{d,\iota_d}_{c,\boldsymbol{\alpha},X_d}$ with domain X_d such that

$$\hat{T}_{c,\boldsymbol{\alpha}}(x) := \{(t,r) \in (\omega \times \boldsymbol{\alpha})^{<\omega} : \forall i,j < |r| [(s_i \subsetneq s_j \text{ and } (t{\restriction}|s_j|, s_j) \in \hat{T}_c(x)) \Rightarrow r(i) > r(j)]\}.$$

Clearly, any $\boldsymbol{\Sigma}^1_2$-subset of X_d is coded by some member of BC_1. We use the functions $\hat{T}_c$ and $\hat{T}_{c,\boldsymbol{\alpha}}$ to deal with tree representations of $\boldsymbol{\Sigma}^1_2$-subsets of X_d, as verified by the following result.

Theorem 4.2. *Let* $c \in \mathrm{BC}_1$*,* $d \in \mathcal{D}(\omega)$*,* $x \in X_d$ *and let* $\boldsymbol{\alpha}$ *be either an uncountable ordinal or* **ON**. *The following statements are equivalent.*

(1) $x \in \mathrm{Pca}(c)$.

(2) There exists a $y \in \omega^\omega$ *such that* $T^{d_\pi, \iota_{d_\pi}}_{c, X_{d_\pi}}(x,y)$ *is well-founded.*

(3) There exists a $y \in \omega^\omega$ *such that* $(\hat{T}_c(x))(y)$ *is well-founded.*

(4) $\hat{T}_{c,\boldsymbol{\alpha}}(x)$ *is ill-founded.*

Proof. Note that $x \in \mathrm{Pca}(c)$ iff there is some $y \in \omega^\omega$ such that $(x,y) \notin \mathrm{An}^{d_\pi, \iota_{d_\pi}}_{X_{d_\pi}}(c)$. Therefore, (1) and (2) are equivalent by Theorem 3.9.

To see (2) $\Leftrightarrow$ (3) we show that, for any $y \in \omega^\omega$, $T^{d_\pi, \iota_{d_\pi}}_{c, X_{d_\pi}}(x,y)$ is ill-founded iff $(\hat{T}_c(x))(y)$ is ill-founded. Note that $(\hat{T}_c(x))(y)$ is ill-founded iff there is some $z \in \omega^\omega$ such that, for any $n < \omega$ and any $\varepsilon \in \mathbb{Q}^+$,

$$\exists u \in X_d \exists (y', z') \in \omega^\omega \times \omega^\omega [d^*(x,u) < \varepsilon,\ y'{\restriction}n = y{\restriction}n,\ z'{\restriction}n = z{\restriction}n \text{ and } ((u,y',z') \in \mathrm{Cl}^{d_{\pi\pi}, \iota_{d_{\pi\pi}}}_{X_{d_{\pi\pi}}}(c))].$$

On the other hand, $T^{d_\pi,\iota_{d_\pi}}_{c,X_{d_\pi}}(x,y)$ is ill-founded iff there is some $z\in\omega^\omega$ such that, for any $\varepsilon\in\mathbb{Q}^+$ and $n,m<\omega$,

$$\exists u\in X_d\exists(y',z')\in\omega^\omega\times\omega^\omega[d^*(x,u)<\varepsilon,\ y'\restriction m=y\restriction m,\ z'\restriction n=z\restriction n \\ \text{and } ((u,y',z')\in \mathrm{Cl}^{d_{\pi\pi},\iota_{d_{\pi\pi}}}_{X_{d_{\pi\pi}}}(c))].$$

Hence, it is clear that, whenever $T^{d_\pi,\iota_{d_\pi}}_{c,X_{d_\pi}}(x,y)$ is ill-founded, so is $(\hat{T}_c(x))(y)$. To see the converse, for $m,n<\omega$ use $\max\{m,n\}$ in the first statement.

Now we prove (3) $\Leftrightarrow$ (4). Assume (3), so there is a function $h:\omega\to\omega_1$ such that, for any $s_i,s_j\in(\hat{T}_c(x))(y)$, if $s_i\subsetneq s_j$ then $h(i)>h(j)$. Therefore, (y,h) is an infinite branch in $\hat{T}_{c,\omega_1}(x)$ and, as $\hat{T}_{c,\omega_1}(x)\subseteq\hat{T}_{c,\boldsymbol{\alpha}}(x)$ (because $\boldsymbol{\alpha}$ is uncountable), then $\hat{T}_{c,\boldsymbol{\alpha}}(x)$ is ill-founded.

To see the converse, assume that $(y,h)\in\omega^\omega\times\boldsymbol{\alpha}^\omega$ is a branch in $T_{c,\boldsymbol{\alpha}}(x)$ (we are abusing notation here), that is, for every $i,j<\omega$, if $(y\restriction|s_j|,s_j)\in\hat{T}_c(x)$ and $s_i\subsetneq s_j$, then $h(i)>h(j)$. This actually means that $(\hat{T}_c(x))(y)$ is well-founded. ☑

Lemma 4.3. *Let $c\in\mathrm{BC}_1\cap M$, $d\in\mathcal{D}(\omega)\cap M$ and let $\boldsymbol{\alpha}$ be such that $N\models$"either $\boldsymbol{\alpha}$ is an ordinal or $\boldsymbol{\alpha}=\mathbf{ON}$". Then:*

(a) $\hat{T}^M_c=\hat{T}^N_c\circ\iota^{M,N}_d$.

(b) If $\boldsymbol{\alpha}\subseteq M$ then $\hat{T}^M_{c,\boldsymbol{\alpha}}=\hat{T}^N_{c,\boldsymbol{\alpha}}\circ\iota^{M,N}_d$.

Proof. (a) can be proved exactly like Lemma 3.10. (b) is immeadiate from (a) because, as $\boldsymbol{\alpha}\subseteq M$, $M\models$"either $\boldsymbol{\alpha}$ is a ordinal or $\boldsymbol{\alpha}=\mathbf{ON}$". ☑

Theorem 4.4 (Shoenfield's Absoluteness Theorem)**.** *Let $c\in\mathrm{BC}_1\cap M$ and $d\in\mathcal{D}(\omega)\cap M$. If $\omega_1^N\subseteq M$, then $(\iota^{M,N}_d)^{-1}[\mathrm{Pca}^N(c)]=\mathrm{Pca}^M(c)$.*

Proof. Let $x\in X^M_d$. By Theorem 4.2,

(i) $\iota^{M,N}_c(x)\in\mathrm{Pca}^N(c)$ iff $\hat{T}^N_{c,\omega_1^N}(\iota^{M,N}_d(x))$ is ill-founded, and

(ii) $x\in\mathrm{Pca}^M(c)$ iff $\hat{T}^M_{c,\omega_1^N}(x)$ is ill-founded (because $\omega_1^N\subseteq M$).

By Lemma 4.3, $\hat{T}^N_{c,\omega_1^N}(\iota^{M,N}_d(x))=\hat{T}^M_{c,\omega_1^N}(x)$ so, as the ill-foundedness of a tree is absolute, then $x\in\mathrm{Pca}^M(x)$ iff $\iota^{M,N}_c(x)\in\mathrm{Pca}^N(c)$. ☑

As a consequence of Theorem 2.7 we have:

Corollary 4.5. *The relation $\approx_{\mathrm{P}}$ is absolute for transitive models $M\subseteq N$ of ZFC when $\omega_1^N\subseteq M$.*

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