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On Symmetric $(1,1)$-Coherent Pairs and Sobolev Orthogonal polynomials: an algorithm to compute Fourier coefficients

Sobre $(1,1)$ pares coherentes simétricos y polinomios ortogonales Sobolev: un algoritmo para calcular coeficientes de Fourier

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Abstract. In the pioneering paper [13], the concept of Coherent Pair was introduced by Iserles et al. In particular, an algorithm to compute Fourier Coefficients in expansions of Sobolev orthogonal polynomials defined from coherent pairs of measures supported on an infinite subset of the real line is described. In this paper we extend such an algorithm in the framework of the so called Symmetric $(1,1)-$Coherent Pairs presented in [8].

Key words and phrases. Orthogonal polynomials, Symmetric $(1,1)-$coherent pairs, Sobolev-Fourier series.

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Resumen. En el artículo pionero [13], fue introducido el concepto de Par Coherente por Iserles et al. En particular, allí es descrito un algoritmo para calcular coeficientes de Fourier de expansiones de polinomios ortogonales de tipo Sobolev definidos a partir de pares de medidas coherentes soportadas en un subconjunto infinito de la recta real. En esta contribución extendemos tal algoritmo en el contexto de los llamados Pares Simétricos $(1,1)-$Coherentes presentados en [8].

Palabras y frases clave. Polinomios ortogonales, Pares Simétricos $(1,1)-$Coherentes, Series Sobolev-Fourier.

1. Introduction

Let $\{\mu_0,\mu_1\}$ be a pair of positive Borel measures supported on an infinite subset E on the real line. Let $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ be the corresponding sequences of monic orthogonal polynomials, (SMOP in short). The pair $\{\mu_0,\mu_1\}$ is said to be *coherent* if there exist real numbers $a_n\neq 0$, $n\geq 1$, called *coherent coefficients*, such that

$$R_{n+1}(x)=\frac{P'_{n+2}(x)}{n+2}+a_n\frac{P'_{n+1}(x)}{n+1}, n\geq 0.$$

This concept is introduced in [13], where sequences of *Sobolev polynomials,* i.e. orthogonal with respect to the *Sobolev inner product*

$$\langle p,q\rangle_S=\int_{\mathbb{R}}p(x)q(x)d\mu_0+\lambda\int_{\mathbb{R}}p'(x)q'(x)d\mu_1,\ \ \lambda>0,\ \ p,q\in\mathbf{P},\tag{1}$$

are studied. Here $\mathbf{P}$ denotes the linear space of polynomials with real coefficients. In the last three decades special attention has been paid to the so-called *general Sobolev Orthogonality* defined by the inner product

$$\langle f,g\rangle_S=\sum_{i=0}^{m}\int_{\mathbb{R}}f^{(i)}(x)g^{(i)}(x)d\mu_i(x),\tag{2}$$

where every $\mu_j, j=0,1,\ldots,m$, is a positive Borel measure supported on an infinite subset of the real line. Such an inner product is known in the literature as a Sobolev inner product. In 1947 the foundations of the theory of Sobolev Orthogonality were stated in the pioneering work [15] by D. C. Lewis, where finite Fourier expansions in terms of Sobolev polynomials are the solution of certain extremal problem related to smooth polynomial approximation. In early 60s, P. Althammer presented his first work, see [1], based on the seminal paper of Lewis, and rewrote the Lewis's problem as follows. Given the inner product

$$\langle f,g\rangle_S=\sum_{i=0}^{m}\int_{a}^{b}f^{(i)}(x)g^{(i)}(x)w_i(x)dx,\tag{3}$$

where the $w_i's$ are weight functions in $[a,b]$, and a function, f, defined in $[a,b]$, to determine

$$\min_{Q\in\mathbf{P}_n}\|f-Q\|_S\,,$$

where $\mathbf{P}_n$ represents the linear space of polynomials of degree less than or equal to n and $\|.\|_S$ is the norm induced by $\langle,\rangle_S$. If $\{S_n\}_{n\geq 0}$ is the sequence of orthonormal polynomials with respect to (3), the polynomial Q^*, where the minimum is achieved, will be a linear combination of Sobolev orthogonal polynomials, namely,

$$Q^*(x) = \sum_{k=0}^{n} a_k S_k(x), \text{ with } a_k = \langle f, S_k \rangle_S .$$

[17] and [19] constitute nice surveys on the historical development and state of the art of Sobolev orthogonality. In addition, in [13] a relation between the sequence of monic Sobolev polynomials $\left\{S_n^\lambda\right\}_{n\geq 0}$ (orthogonal with respect to (1)) and $\{P_n\}_{n\geq 0}$, the sequence of monic polynomials orthogonal with respect to $d\mu_0$ is given. Namely,

$$S_{n+2}^\lambda(x) + \eta_n(\lambda) S_n^\lambda(x) = P_{n+2}(x) + a_n P_n(x), n \geq 0,$$

where the values $\eta_n(\lambda)$ are called *Sobolev coefficients*. Let consider the Sobolev space

$$W^{1,2}(E,\mu_0,\mu_1) = \left\{g : E \to \mathbb{R} \mid g \in L^2(E;\mu_0), \quad g' \in L^2(E;\mu_1)\right\}.$$

For a function f in such a space, an efficient algorithm to compute Fourier coefficients when f is expanded by using the orthogonal basis $\left\{S_n^\lambda\right\}_{n\geq 0}$ is described. Such an algorithm does not need the explicit expression of Sobolev orthogonal polynomials. In [6], the most general case of coherence for standard orthogonal polynomials is known in the literature and presented as follows.

Definition 1.1. A pair of positive Borel measures $\{\mu_0, \mu_1\}$ is said to be a $(M,N)-$coherent pair of order (m,k) if the corresponding SMOPs satisfy

$$\sum_{i=0}^{M} a_{i,n} P_{n+m-i}^{[m]}(x) = \sum_{i=0}^{N} b_{i,n} Q_{n+k-i}^{[k]}(x), \tag{4}$$

where $m,k,M,N \in \mathbb{N}\cup\{0\}$, $P_n^{[i]}(x) := \dfrac{P_{n+i}^{(i)}(x)}{(n+1)_i}$, and $\{a_{i,n}\}_{n\geq 0}$, $\{b_{j,n}\}_{n\geq 0}$, $0\leq i \leq M$, $0\leq j \leq N$ are sequences of real numbers with $a_{0,n} = b_{0,n} = 1$. Here $(n)_k, k = 0,1,\ldots,$ denotes the Pochhammer symbol, i.e. $(n)_k = n(n+1)\cdots(n+k-1)$, $k \geq 1$ and $(n)_0 = 1$.

Thus, in [5] the algorithm proposed in [13] for $(1,0)-$coherent pairs of order $(1,0)$ is generalized in a natural way for $(M,N)-$coherent pairs of measures of order $(m,0)$. On the other hand, when the measures μ_0 and μ_1 are symmetric, i.e. invariant under the transformation $x \longmapsto -x$, and the respective SMOPs, $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ satisfy

$$R_{n+2}(x) = \frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1}, n \geq 0.$$

The pair $\{\mu_0, \mu_1\}$ is said to be *symmetric coherent*, a concept that has been also introduced in [13]. A generalization is presented in [8] with the so called *symmetric* $(1,1)-$*coherent pairs of measures*, i.e. when the respective SMOPs satisfy

$$R_{n+2}(x) + b_n R_n(x) = \frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1}, n \geq 0.$$

Therein, connection properties between the coherent and recurrence coefficients, among others, are obtained, as well as a special emphasis in the case μ_0 classical, (Hermite, Gegenbauer) is pointed out. In particular, the symmetric $(1,1)-$coherent pair $\{e^{-x^2}dx, \frac{x^2+a}{x^2+b}e^{-x^2}dx\}$, $a, b > 0$, is obtained. Taking into account the above pair, in [9] asymptotic properties of Sobolev polynomials associated with the above (1,1) Hermite symmetric coherent pair are studied. Finally, in [10] a classification of symmetric $(1,1)-$coherent pairs is presented.

The aim of this contribution is to study the natural generalization of the algorithm displayed in [13], in the symmetric $(1,1)-$coherent framework. So, the structure of this manuscript is as follows. In Section 2 the basic background on orthogonal polynomials associated with a linear functional is presented. In Section 3 the algebraic relation between the Sobolev polynomials and polynomials orthogonal with respect to μ_0 is deeply analyzed. In Section 4 the algorithm to compute Fourier coefficients is described. Finally, some numerical examples are studied.

2. Preliminaries

Let $\mathbf{P}$ be the linear space of polynomials with real coefficients. $\mathbf{P}_n$ will denote the linear subspace of polynomials of degree at most n. Let u be a linear functional in the algebraic dual space of $\mathbf{P}$. It will be denoted $\mathbf{P}'$. $\langle u, p\rangle$ is the action of the linear functional u on the polynomial $p \in \mathbf{P}$. Let $\{u_n\}_{n\geq 0}$ be a sequence of real numbers. u is a *moment functional* associated with the *moment sequence* $\{u_n\}_{n\geq 0}$ if u is linear and $u_n = \langle u, x^n\rangle, n \geq 0$. A sequence of polynomials $\{P_n\}_{n\geq 0}$, with $\deg P_n = n$, determines an unique sequence of linear functionals $\{\mathbf{p}_n\}_{n\geq 0}$, called *dual basis* associated with $\{P_n\}_{n\geq 0}$, in such a way that $\langle \mathbf{p}_n, P_m\rangle = \delta_{n,m}$, where $\delta_{n,m}$ denotes the Kronecker delta symbol. As a consequence, every $u \in \mathbf{P}'$ can be expressed in terms of the basis $\{\mathbf{p}_n\}_{n\geq 0}$ as follows:

$$u = \sum_{k\geq 0} \langle u, P_k\rangle \mathbf{p}_k.$$

On the other hand, if $q \in \mathbf{P}$ and $u \in \mathbf{P}'$, then we define $qu \in \mathbf{P}'$, the *left multiplication*, as

$$\langle qu, p\rangle := \langle u, qp\rangle, \quad p \in \mathbf{P}. \tag{5}$$

The linear functional $\delta(x-c)$ such that $\langle \delta(x-c), p\rangle := p(c)$, $p \in \mathbf{P}$, $c \in \mathbb{C}$, is said to be the *Dirac delta* linear functional at c.

Given $u \in \mathbf{P}'$, let $\sigma \in \mathbf{P}$ be a polynomial of degree n and denote by $x_k \in \mathbb{C}$, $1 \leq k \leq r$, their zeros with multiplicities n_k, respectively, i.e. $\sum_{k=1}^{r} n_k = n$. Then for every $p \in \mathbf{P}$, we define the linear functional $\sigma^{-1}(x)u \in \mathbf{P}'$ as follows:

$$\left\langle \sigma^{-1}(x)u, p(x) \right\rangle := \left\langle u, \frac{p(x) - L_\sigma(x;p)}{\sigma(x)} \right\rangle, \tag{6}$$

where $L_\sigma(x;p)$ is the interpolatory polynomial

$$L_\sigma(x;p) = \sum_{i=1}^{r} \sum_{j=0}^{n_i-1} p^{(j)}(x_i) L_{i,j}(x) \tag{7}$$

and $L_{i,j}(x)$ is the polynomial of degree at most $n-1$ such that $L_{i,j}^{(k)}(x_l) = \delta_{i,l}\delta_{k,j}$, $i,l = 1, \cdots, r$, $p^{(j)}$ the $j-th$ derivative of p and $0 \leq k, j \leq n_i - 1$.

On the other hand, given $q \in \mathbf{P}$ we will denote by $uq \in \mathcal{P}$, the *right-multiplication* of $u \in \mathbf{P}'$ by q, the polynomial

$$(uq)(t) := \left\langle u, \frac{tq(t) - xq(x)}{t - x} \right\rangle,$$

where u acts on the variable x.

The $p-th$ derivative of the functional u, $p \in \mathbb{Z}^+ \cup \{0\}$, denoted by $D^p u$, is a linear functional such that

$$\langle D^p u, q(x) \rangle := (-1)^p \left\langle u, q^{(p)}(x) \right\rangle, \quad q \in \mathcal{P}. \tag{8}$$

For a more detailed description of this algebraic approach to linear functionals, see [18].

2.1. Quasi-definite and Positive-definite linear functionals

Let u be a linear functional and $\{u_n\}_{n \geq 0}$ be the corresponding moment sequence. We define the *Hankel determinant of order* $n+1$

$$\Delta_n^u = \begin{vmatrix} u_0 & u_1 & \cdots & u_n \\ u_1 & u_2 & \cdots & u_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_n & u_{n+1} & \cdots & u_{2n} \end{vmatrix}, \quad n \geq 0. \tag{9}$$

u is said to be **quasi-definite** or **regular**(see [4]) if the leading principal submatrices of the *Hankel matrix* $(u_{i+j})_{i,j=0}^{\infty}$ are non-singular, i.e. $\Delta_n^u \neq 0$ for $n \geq 0$. u is called **positive-definite** if $\langle u, \pi(x) \rangle > 0$ for every non identically zero and non-negative real polynomial π. When there is not risk of confusion we will write Δ_n instead of Δ_n^u.

The positive definiteness of a linear functional can be characterized through the associated moment sequence. Namely,

Theorem 2.1. *([4]). u is positive definite if and only if $\Delta_n > 0$ for $n \geq 0$.*

If u is positive-definite, then there exists a positive Borel measure μ supported on an infinite set $E \subseteq \mathbb{R}$ such that u has an integral representation

$$\langle u, p \rangle = \int_E p(x) d\mu(x), \quad p \in \mathbf{P}.$$

Given a quasi-definite linear functional u on the space $\mathbf{P}$, a bilinear form $\langle , \rangle_u :$ $\mathbf{P} \times \mathbf{P} \to \mathbb{R}$ is defined as $\langle p, q \rangle_u := \langle u, pq \rangle$. If u is positive definite, then the bilinear form is an inner product on $\mathbf{P}$ and, as usual, the induced norm will be represented as

$$\|p\|_u = \langle p, p \rangle_u^{1/2} = \left\langle u, p^2 \right\rangle^{1/2} = \left(\int_E p^2(x) d\mu(x) \right)^{1/2},$$

where μ is the positive Borel measure, supported on E, associated with u.

2.2. Orthogonal polynomials

Definition 2.2. A polynomial sequence $\{P_n\}_{n\geq 0}$ is said to be an orthogonal polynomial sequence, OPS in short, with respect to a linear functional u if for $n, m \geq 0$,

i) P_n is a polynomial of degree n.

ii) $\langle u, P_n P_m \rangle = 0$, for $n \neq m$.

iii) $\left\langle u, P_n^2 \right\rangle \neq 0, n \geq 0$.

If the leading coefficient of P_n is 1 for every $n \geq 0$, then $\{P_n\}_{n\geq 0}$ is said to be a *monic orthogonal polynomial sequence, (SMOP in short)*. The next result gives us conditions for the existence of an OPS associated with a linear functional.

Proposition 2.3. *([4]). Let u be a linear functional. u is quasi-definite if and only if there exists an OPS $\{P_n\}_{n\geq 0}$ with respect to u.*

Under the conditions of above proposition, if $\{u_n\}_{n\geq 0}$ is the moment sequence associated with u, then every monic polynomial P_n can be written as

$$P_n(x) = \frac{1}{\Delta_{n-1}} \begin{vmatrix} u_0 & u_1 & \cdots & u_n \\ u_1 & u_2 & \cdots & u_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & u_n & \cdots & u_{2n-1} \\ 1 & x & & x^n \end{vmatrix}, \quad n \geq 1, \ P_0(x) = 1.$$

Given a quasi-definite linear functional there exists an infinite number of OPS associated with u. Indeed, if $\{P_n\}_{n\geq 0}$ is an OPS associated with u, then

$\{k_nP_n\}_{n\geq 0}$ is also an OPS associated with u for non-zero constants k_n. Thus $\{P_n\}_{n\geq 0}$ is uniquely determined if the leading coefficients are fixed. Conversely, if $\{P_n\}_{n\geq 0}$ is an OPS associated with u, for any $k\neq 0$, then $\{P_n\}_{n\geq 0}$ is also an OPS associated with the linear functional ku. In order to the quasi-definite linear functional and the OPS are uniquely determined, a normalization will be required. In this way, in the sequel we will assume that $\langle u, 1\rangle = 1$ as well as the respective OPS is monic, unless stated otherwise.

The next theorem describes an important characterization of the orthogonality of a sequence of monic polynomials in terms of a recurrence relation satisfied by them. In the literature it is partially accepted that the original version of this result is due to J. Favard[11], but essentially it means the spectral resolution of the multiplication operator.

Theorem 2.4 (Favard's theorem)**.** *([4]). Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials. $\{P_n\}_{n\geq 0}$ is a MOPS with respect to a quasi-definite linear functional u if and only if there exist sequences of real numbers $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$, with $\gamma_n\neq 0, n\geq 1$, such that*

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n\geq 1, \tag{10}$$

$$P_0(x) = 1,\ P_1(x) = x - \beta_0.$$

On the other hand,

$$\beta_n = \frac{\langle u, xP_n^2\rangle}{\langle u, P_n^2\rangle},\ n\geq 0, \quad \gamma_n = \frac{\langle u, xP_nP_{n-1}\rangle}{\langle u, P_{n-1}^2\rangle} = \frac{\langle u, P_n^2\rangle}{\langle u, P_{n-1}^2\rangle},\ n\geq 1.$$

The relation (10) is the so-called Three-Term Recurrence Relation, (TTRR in short). A nice survey about the Favard's theorem, its origins and further development is given in [16]. The TTRR is equivalent to the well known and useful *Christoffel-Darboux Identity.*

Theorem 2.5. *([3], [4]). A SMOP $\{P_n\}_{n\geq 0}$ associated with a quasi-definite linear functional u satisfies (10) if and only if*

$$\sum_{k=0}^{n}\frac{P_k(x)P_k(y)}{\langle u, P_k^2\rangle} = \frac{1}{\langle u, P_n^2\rangle}\frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x-y}.$$

2.3. Symmetric Linear Functionals

A linear functional $u\in\mathbf{P}'$ is said to be symmetric if $u_{2n+1} = \langle u, x^{2n+1}\rangle = 0, n\in\mathbb{N}$. (See [4] for more characterizations of symmetric quasi-definite linear functionals). If $u\in\mathbf{P}'$ is symmetric and quasi-definite and $\{P_n\}_{n\geq 0}$ is its corresponding SMOP, we can define $\widetilde{u}\in\mathbf{P}'$ by

$$\langle\widetilde{u}, x^n\rangle = \langle u, x^{2n}\rangle, \quad n\in\mathbb{N}. \tag{11}$$

In such situation, there exist monic polynomials A_n and $\widetilde{A}_n$, $n \geq 0$, such that

$$P_{2n}(x) = A_n(x^2) \quad \text{and} \quad P_{2n+1}(x) = x\widetilde{A}_n(x^2). \tag{12}$$

As a consequence of the above definition, if u is a symmetric and quasi-definite linear functional, then $\{A_n\}_{n\geq 0}$ and $\{\widetilde{A}_n\}_{n\geq 0}$ are the SMOP corresponding to $\widetilde{u}$ and $x\widetilde{u}$, respectively. When u is symmetric and positive definite and it has an integral representation in terms of the even weight function w on $[-\zeta, \zeta]$, then

$$\langle u, p(x)\rangle = \int_{-\zeta}^{\zeta} p(x)w(x)dx$$

yields

$$\langle \widetilde{u}, p(x)\rangle = \int_{0}^{\zeta^2} p(x)x^{-1/2}w(x^{1/2})dx,$$

assuming the integrals converge.

3. Sobolev polynomials and Symmetric $(1,1)-$ Coherent Pairs

We begin with the definition of Symmetric $(1,1)-$Coherent Pair introduced in [8]. From now on in this manuscript we assume that any linear functional u is normalized by the condition $\langle u, 1\rangle = 1$.

Definition 3.1. Let u and v denote two symmetric quasi-definite linear functionals and $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ will denote their respective SMOP. Assume that there exist sequences of non-zero real numbers $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, with $a_n b_n \neq 0$, such that

$$\frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1} = R_{n+2}(x) + b_n R_n(x), \quad n \geq 0, \tag{13}$$

holds. Then the pair $\{u, v\}$ is said to be a *Symmetric* $(1,1)-$*Coherent Pair*. Furthermore, if u and v are positive-definite and μ_0 and μ_1 are the respective positive Borel measures, then $\{\mu_0, \mu_1\}$ is said to be a *Symmetric* $(1,1)-$*Coherent Pair of measures.*

With the condition $a_n b_n \neq 0,\ n \geq 0$, we are assuming that the relation (13) is non-degenerated. Moreover if $a_i \neq b_i,\ i = 0, 1$, we get

Proposition 3.2. *([7]). Let* $\{u, v\}$ *be a Symmetric* $(1,1)-$*coherent pair satisfying (13). The following statements are equivalent.*

i) $a_i \neq b_i,\ i = 0, 1.$

ii) $R_n(x) \neq \dfrac{P'_{n+1}(x)}{n+1}$, *for* $n \geq 2$.

Let $\{u,v\}$ be a symmetric $(1,1)-$coherent pair with $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ as their respective SMOP such that (13) holds. We assume that u and v are positive-definite with μ_0 and μ_1 as the corresponding positive Borel measures. Let $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ be, respectively, their moment sequences. Then we consider the Sobolev inner product

$$\langle p,q\rangle_S = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\mu_1(x), \quad \lambda > 0. \tag{14}$$

Let $\left\{S_n^\lambda\right\}_{n\geq 0}$ be the sequence of monic Sobolev polynomials orthogonal with respect to (14). The above inner product also will be written as

$$\langle p,q\rangle_s = \langle p,q\rangle_{\mu_0} + \lambda \langle p',q'\rangle_{\mu_1} = \langle u,pq\rangle + \lambda \langle v,p'q'\rangle .$$

For $n\geq 1$, we consider the expansion $S_n^\lambda(x) = x^n + \sum_{j=0}^{n-1} c_{n,j}^\lambda x^j$, and let $\Delta_{S,n} = \det\left[\mu_{i,j}\right]_{i,j=0}^n$ be the determinant of the leading principal submatrix of size $(n+1)\times(n+1)$ associated with the moments $\mu_{i,j} := \left\langle x^i, x^j\right\rangle_S$. According to (14) if $i+j=0,1,$ then $\mu_{i,j} = u_{i+j}$, and if $i+j\geq 2$ we have

$$\begin{aligned}\mu_{i,j} = \left\langle x^i, x^j\right\rangle_S &= \int_{\mathbb{R}} x^{i+j}d\mu_0(x) + ij\lambda\int_{\mathbb{R}} x^{i+j-2}d\mu_1(x)\\ &= u_{i+j} + ij\lambda v_{i+j-2}.\end{aligned}$$

Moreover, if $i+j$ is odd or $ij=0$, then $\mu_{i,j} = u_{i+j}$. It is well known that

$$S_n^\lambda(x) = \frac{1}{\Delta_{S,n-1}}\begin{vmatrix} 1 & u_1 & u_2 & \cdots & u_n\\ u_1 & \mu_{1,1} & \mu_{1,2} & \cdots & \mu_{1,n}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ u_{n-1} & \mu_{n-1,1} & \mu_{n-1,2} & \cdots & \mu_{n-1,n}\\ 1 & x & x^2 & \cdots & x^n\end{vmatrix}, \quad n\geq 1.$$

Furthermore, $c_{n,j}^\lambda = \dfrac{(-1)^{n+2+j}\Delta_{S,n-1}^j}{\Delta_{S,n-1}}$, where $\Delta_{S,n-1}^j$ is obtained deleting the $j-th$ column and the $(n+1)-th$ row of the matrix $\left[\mu_{i,j}\right]_{i,j=0}^n$.

***Remark* 3.3.** Notice that $P_k(x) = S_k^\lambda(x)$ for $k=0,1,2$.

Using properties of the determinants and after cumbersome calculations we obtain

$$\begin{aligned}\Delta_{S,n-1} &= \begin{vmatrix} 1 & u_1 & \cdots & u_{n-1} \\ u_1 & \mu_{1,1} & \cdots & \mu_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & \mu_{n-1,1} & \cdots & \mu_{n-1,n-1} \end{vmatrix} \\ &= \begin{vmatrix} 1 & u_1 & \cdots & u_{n-1} \\ u_1 & u_2+\lambda v_0 & \cdots & u_n+(n-1)\lambda v_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n-1} & u_n+(n-1)\lambda v_{n-2} & \cdots & u_{2n-2}+(n-1)^2\lambda v_{2n-4} \end{vmatrix} \\ &= ((n-1)!)^2\,\Delta^v_{n-2}\lambda^{n-1}+\cdots+\Delta^u_{n-1}.\end{aligned}$$

From the above, every coefficient of S_n^λ is a rational function in λ where the degree numerator is at most the degree of the denominator. Then it makes sense to define the sequence $\{W_n\}_{n\geq 0}$

$$W_n(x) := \lim_{\lambda\to\infty} S_n^\lambda(x),$$

where, as a consequence of the symmetry, if n is even (resp. odd), then W_n is an even function (resp. odd).

On the other hand, if $\deg(q)\leq n$, then

$$\left\langle S^\lambda_{n+1}, q\right\rangle_S = \int_{\mathbb{R}} S^\lambda_{n+1}(x)q(x)d\mu_0(x) + \lambda\int_{\mathbb{R}}\left(S^\lambda_{n+1}\right)'(x)q'(x)d\mu_1(x) = 0.$$

When $\lambda\to\infty$, $\int_{\mathbb{R}} W'_{n+1}(x)q'(x)d\mu_1(x)=0$, i.e.

$$W'_{n+1}(x) = (n+1)R_n(x), n\geq 0. \tag{15}$$

Moreover, for $n\geq 0$, $\left\langle S^\lambda_{n+1}, 1\right\rangle_S = \int_{\mathbb{R}} S^\lambda_{n+1}(x)d\mu_0(x) = 0$. If $\lambda\to\infty$, then we get

$$\int_{\mathbb{R}} W_{n+1}(x)d\mu_0(x) = 0. \tag{16}$$

From (13) and by using (15) we have

$$\frac{W_{n+3}(x)}{n+3} + b_n\frac{W_{n+1}(x)}{n+1} = \frac{P_{n+3}(x)}{n+3} + a_n\frac{P_{n+1}(x)}{n+1} + k_n, \qquad n\geq 0.$$

Integration of the above expression with respect to the measure μ_0, and using (16) yields $k_n=0$, i.e.

$$W_{n+3}(x) + \widetilde{b}_n W_{n+1}(x) = P_{n+3}(x) + \widetilde{a}_n P_{n+1}(x), \qquad n\geq 0, \tag{17}$$

where

$$\widetilde{a}_n = a_n \frac{n+3}{n+1}, \qquad \widetilde{b}_n = b_n \frac{n+3}{n+1}, \qquad n \geq 0.$$

Now let consider the expansion of W_n by using the basis $\left\{S_n^\lambda\right\}_{n\geq 0}$

$$W_n(x) = S_n^\lambda(x) + \sum_{j=0}^{n-1} \sigma_{n,j} S_j^\lambda(x). \tag{18}$$

Notice that

$$\sigma_{n,j} = \frac{\left\langle W_n, S_j^\lambda \right\rangle_S}{\left\| S_j^\lambda \right\|_S^2} = \frac{\int_{\mathbb{R}} W_n(x) S_j^\lambda(x) d\mu_0(x) + \lambda \int_{\mathbb{R}} W_n'(x) \left(S_j^\lambda\right)'(x) d\mu_1(x)}{\left\| S_j^\lambda \right\|_S^2},$$

and $\left\|S_j^\lambda\right\|_S^2 := \left\langle S_j^\lambda, S_j^\lambda \right\rangle_S$. In the same way, $W_{n+3}(x) = S_{n+3}^\lambda(x) + \sum_{j=0}^{n+2} \sigma_{n+3,j}$ $S_j^\lambda(x)$, and multiplying by $\widetilde{b}_n$ in (18) we get

$$\begin{aligned} W_{n+3}(x) + \widetilde{b}_n W_{n+1}(x) = {} & S_{n+3}^\lambda(x) + \sigma_{n+3,n+2} S_{n+2}^\lambda(x) + \left(\sigma_{n+3,n+1} + \widetilde{b}_n\right) S_{n+1}^\lambda(x) \\ & + \sum_{j=0}^{n} \left(\sigma_{n+3,j} + \widetilde{b}_n \sigma_{n+1,j}\right) S_j^\lambda(x). \end{aligned}$$

Taking into account the polynomials W_{n+3} and W_{n+1} are either even or odd functions, then $\sigma_{n+3,n+2} = 0$ holds. Thus

$$W_{n+3}(x) + \widetilde{b}_n W_{n+1}(x) = S_{n+3}^\lambda(x) + \sum_{j=0}^{n+1} \eta_{n,j}(\lambda) S_j^\lambda(x),$$

where every coefficient $\eta_{n,j}(\lambda)$, $j \leq n$, can be written as

$$\begin{aligned} & \eta_{n,j}(\lambda) \\ & = \sigma_{n+3,j} + \widetilde{b}_n \sigma_{n+1,j} \\ & = \frac{\int_{\mathbb{R}} \left(W_{n+3}(x) + \widetilde{b}_n W_{n+1}(x)\right) S_j^\lambda(x) d\mu_0(x) + \lambda \int_{\mathbb{R}} \left(W_{n+3}'(x) + \widetilde{b}_n W_{n+1}'(x)\right) \left(S_j^\lambda\right)'(x) d\mu_1(x)}{\left\| S_j^\lambda \right\|_S^2}, \end{aligned}$$

and $\sigma_{n+1,n+1} := 1$.

By using (17) we obtain

$$\int_{\mathbb{R}} \left(W_{n+3}(x) + \widetilde{b}_n W_{n+1}(x)\right) S_j^\lambda(x) d\mu_0(x) = \int_{\mathbb{R}} (P_{n+3}(x) + \widetilde{a}_n P_{n+1}(x)) S_j^\lambda(x) d\mu_0(x) = 0,$$

for $j = 0, 1, \ldots, n$, and the relation (15) yields

$$\int_{\mathbb{R}} \left(W_{n+3}'(x) + \widetilde{b}_n W_{n+1}'(x)\right) \left(S_j^\lambda\right)'(x) d\mu_1(x) = 0.$$

As a consequence,

$$W_{n+3}(x)+\widetilde{b}_n W_{n+1}(x)=S^{\lambda}_{n+3}(x)+\eta_{n,n+1}(\lambda)S^{\lambda}_{n+1}(x). \tag{19}$$

Equivalently,

$$S^{\lambda}_{n+3}(x)+\eta_{n,n+1}(\lambda)S^{\lambda}_{n+1}(x)=P_{n+3}(x)+\widetilde{a}_n P_{n+1}(x), \quad n\geq 0. \tag{20}$$

Taking derivatives

$$\frac{\left(S^{\lambda}_{n+3}\right)'(x)}{n+3}+\eta_{n,n+1}(\lambda)\frac{\left(S^{\lambda}_{n+1}\right)'(x)}{n+3}=P_{n+2}(x)+a_nP_n(x), n\geq 0. \tag{21}$$

Notice that

$$\begin{aligned}\eta_{n,n+1}(\lambda)&=\sigma^{\lambda}_{n+3,n+1}+\widetilde{b}_n\\ &=\frac{\int_{\mathbb{R}}W_{n+3}(x)S^{\lambda}_{n+1}(x)d\mu_0(x)+\lambda\int_{\mathbb{R}}W'_{n+3}(x)\left(S^{\lambda}_{n+1}\right)'(x)d\mu_1(x)}{\left\|S^{\lambda}_{n+1}\right\|^2_S}+\widetilde{b}_n,\end{aligned}$$

and, again, using (15) we obtain

$$\int_{\mathbb{R}}W'_{n+3}(x)\left(S^{\lambda}_{n+1}\right)'(x)d\mu_1(x)=0.$$

Thus

$$\eta_n(\lambda):=\eta_{n,n+1}(\lambda)=\frac{\int_{\mathbb{R}}W_{n+3}(x)S^{\lambda}_{n+1}(x)d\mu_0(x)}{\left\|S^{\lambda}_{n+1}\right\|^2_S}+\widetilde{b}_n. \tag{22}$$

We summarize the above results in the next

Theorem 3.4. *Let $\{\mu_0,\mu_1\}$ be a symmetric $(1,1)-$coherent pair of measures with $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ as their respective SMOPs satisfying (13) and let $\{S^{\lambda}_n\}_{n\geq 0}$ be the Sobolev polynomials orthogonal with respect to (14). Then, for $n\geq 0$*

$$S^{\lambda}_{n+3}(x)+\eta_n(\lambda)S^{\lambda}_{n+1}(x)=P_{n+3}(x)+\widetilde{a}_nP_{n+1}(x), \tag{23}$$

holds with

$$\eta_n(\lambda)=\frac{\int_{\mathbb{R}}W_{n+3}(x)S^{\lambda}_{n+1}(x)d\mu_0(x)}{\left\|S^{\lambda}_{n+1}\right\|^2_S}+\widetilde{b}_n. \tag{24}$$

The coefficients $\eta_n(\lambda)$ will be called Sobolev Coefficients.

Lemma 3.5. *For $n\geq 0$*

$$\eta_n(\lambda)=\frac{\widetilde{b}_n(n+1)^2\left\langle v,R^2_n\right\rangle\lambda+\widetilde{a}_n\left\langle u,P^2_{n+1}\right\rangle}{\left\|S^{\lambda}_{n+1}\right\|^2_S}. \tag{25}$$

Proof. From (13), multiplying by R_n and using the measure μ_1, we get

$$\begin{aligned} b_n \left\langle v, R_n^2 \right\rangle &= \left\langle \frac{P'_{n+3}(x)}{n+3}, R_n(x) \right\rangle_{\mu_1} + a_n \left\langle \frac{P'_{n+1}(x)}{n+1}, R_n(x) \right\rangle_{\mu_1} \\ &= \left\langle \frac{W'_{n+3}(x)}{n+3} + b_n \frac{W'_{n+1}(x)}{n+1}, R_n(x) \right\rangle_{\mu_1} \\ &= \frac{1}{\lambda} \left(\left\langle \frac{S^\lambda_{n+3}(x)}{n+3} + \eta_n(\lambda) \frac{S^\lambda_{n+1}(x)}{n+3}, \frac{W_{n+1}(x)}{n+1} \right\rangle_S \right. \\ &\qquad \left. - \left\langle \frac{P_{n+3}(x)}{n+3} + a_n \frac{P_{n+1}(x)}{n+1}, \frac{W_{n+1}(x)}{n+1} \right\rangle_{\mu_0} \right) \\ &= \frac{1}{\lambda} \left(\frac{\eta_n(\lambda)}{(n+1)(n+3)} \left\| S_{n+1} \right\|_S^2 - \frac{a_n}{(n+1)^2} \left\langle u, P_{n+1}^2 \right\rangle \right). \end{aligned}$$

Then

$$\lambda b_n (n+1)(n+3) \left\langle v, R_n^2 \right\rangle + \frac{a_n}{(n+1)} (n+3) \left\langle u, P_{n+1}^2 \right\rangle = \eta_n(\lambda) \left\| S_{n+1} \right\|_S^2,$$

and the result follows. ☑

On the other hand, we use (13), (17), (20) and the notation $\langle u, p(x)q(x) \rangle := \langle p(x), q(x) \rangle_{\mu_0}$, i.e. we express u in terms of the associated bilinear form in order to obtain

$$\begin{aligned} &\left\langle W_{n+3}(x), S^\lambda_{n+1}(x) \right\rangle_{\mu_0} \\ &= \widetilde{a}_n \left\langle u, P_{n+1}^2 \right\rangle - \widetilde{b}_n \left\langle u, P_{n+1}^2 \right\rangle - \widetilde{b}_n \widetilde{a}_{n-2} \left\langle W_{n+1}(x), P_{n-1}(x) \right\rangle_{\mu_0} \\ &+ \widetilde{b}_n \eta_{n-2}(\lambda) \left\langle W_{n+1}(x), S^\lambda_{n-1}(x) \right\rangle_{\mu_0} \end{aligned}$$

and

$$\begin{aligned} &\left\langle W_{n+1}(x), P_{n-1}(x) \right\rangle_{\mu_0} \\ &= \left\langle P_{n+1}(x) + \widetilde{a}_{n-2} P_{n-1}(x) - \widetilde{b}_{n-2} W_{n-1}(x), R_{n-1}(x) \right\rangle_{\mu_0} \\ &= \left(\widetilde{b}_{n-2} - \widetilde{a}_{n-2} \right) \left\langle u, P_{n-1}^2 \right\rangle. \end{aligned}$$

Then, from the above relations, for $n \geq 2$ we get

$$\begin{aligned} \left\langle W_{n+3}(x), S^\lambda_{n+1}(x) \right\rangle_{\mu_0} = &\left(\widetilde{a}_n - \widetilde{b}_n \right) \left\langle u, P_{n+1}^2 \right\rangle - \widetilde{b}_n \widetilde{a}_{n-2} \left(\widetilde{a}_{n-2} - \widetilde{b}_{n-2} \right) \left\langle u, P_{n-1}^2 \right\rangle \\ &+ \widetilde{b}_n \eta_{n-2}(\lambda) \left\langle W_{n+1}(x), S^\lambda_{n-1}(x) \right\rangle_{\mu_0}. \end{aligned}$$

If we denote $I_k(\lambda) := \left\langle W_k(x), S_{k-2}^{\lambda}(x)\right\rangle_{\mu_0}$, the above relation can be written as

$$I_{n+3}(\lambda)=\left(\widetilde{a}_n-\widetilde{b}_n\right)\left\langle u,P_{n+1}^2\right\rangle-\widetilde{b}_n\widetilde{a}_{n-2}\left(\widetilde{a}_{n-2}-\widetilde{b}_{n-2}\right)\left\langle u,P_{n-1}^2\right\rangle+\widetilde{b}_n\eta_{n-2}(\lambda)I_{n+1}(\lambda). \tag{26}$$

Moreover, from (25) we get

$$I_{n+3}(\lambda)=\widetilde{b}_n(n+1)^2\lambda\left\langle v,R_n^2\right\rangle+\widetilde{a}_n\left\langle u,P_{n+1}^2\right\rangle-\widetilde{b}_n\left\|S_{n+1}^{\lambda}\right\|_S^2. \tag{27}$$

With $I_{n+1}(\lambda)$ from (24) and from (26) and (27) we get the next.

Lemma 3.6. *For $n \geq 2$,*

$$\begin{aligned}\left\|S_{n+1}^{\lambda}\right\|_S^2 &= (n+1)^2\lambda\left\langle v,R_n^2\right\rangle+\left\langle u,P_{n+1}^2\right\rangle+\widetilde{a}_{n-2}\left(\widetilde{a}_{n-2}-\widetilde{b}_{n-2}\right)\left\langle u,P_{n-1}^2\right\rangle\\ &\quad-\eta_{n-2}\left(\eta_{n-2}-\widetilde{b}_{n-2}\right)\left\|S_{n-1}^{\lambda}\right\|_S^2.\end{aligned} \tag{28}$$

The above formula is useful in order to compute the norms $\left\|S_n^{\lambda}\right\|_S^2$ if the Sobolev coefficients are known. We are going to describe the initial conditions which are needed. First, from their definitions it is easy to see that

$$\left\|S_1^{\lambda}\right\|_S^2=\lambda+\left\langle u,P_1^2\right\rangle, \qquad \eta_0(\lambda)=\frac{\widetilde{b}_0\lambda+\widetilde{a}_0\left\langle u,P_1^2\right\rangle}{\lambda+\left\langle u,P_1^2\right\rangle}. \tag{29}$$

On the other hand, for $n=1$,

$$\eta_1(\lambda)=\frac{4\widetilde{b}_1\left\langle v,R_1^2\right\rangle\lambda+\widetilde{a}_1\left\langle u,P_2^2\right\rangle}{\left\|S_2^{\lambda}\right\|_S^2}, \tag{30}$$

and since $P_2(x)=x^2-\left\langle u,P_1^2\right\rangle$, we get

$$\begin{aligned}\left\|S_2^{\lambda}\right\|_S^2 &= \left\langle S_2^{\lambda},S_2^{\lambda}\right\rangle_S\\ &= 4\lambda\left\langle v,R_1^2\right\rangle+\left\langle u,P_2^2\right\rangle.\end{aligned}$$

As a consequence,

$$\left\|S_2^{\lambda}\right\|_S^2=4\lambda\left\langle v,R_2^2\right\rangle+\left\langle u,P_2^2\right\rangle+\left(\left\langle u,P_1^2\right\rangle-u_2\right)^2, \quad \eta_1(\lambda)=\frac{4\widetilde{b}_1\left\langle v,R_1^2\right\rangle\lambda+\widetilde{a}_1\left\langle u,P_2^2\right\rangle}{4\left\langle v,R_1^2\right\rangle\lambda+\left\langle u,P_2^2\right\rangle}. \tag{31}$$

Thus, using (29) and for $n=2$ in (28) we obtain $\left\|S_3^{\lambda}\right\|_S^2$. Then, from (25), we find $\eta_2(\lambda)$. In an analogue way, for $n=4,6,8,10,\ldots$ in (28) we obtain $\left\|S_{2k+1}^{\lambda}\right\|_S^2$ and $\eta_{2k}(\lambda)$, for every $k\in\mathbb{N}$. Similarly, from(31) we can obtain recurrently $\left\|S_{2k+2}^{\lambda}\right\|_S^2$ and $\eta_{2k+1}(\lambda)$, for every $k\in\mathbb{N}$. In the next section we study a recurrence formula for the coefficients $\eta_n(\lambda)$.

3.1. The Sobolev Coefficients

We define $T_{n+1}(x) = W_{n+1}(x) + \widetilde{b}_{n-2}W_{n-1}(x)$. Through straightforward calculations it is not difficult to prove that

$$\eta_n(\lambda) = \frac{\langle T_{n+3}(x), T_{n+1}(x)\rangle_S}{\langle T_{n+1}(x), T_{n+1}(x)\rangle_S - \eta_{n-2}(\lambda)\langle T_{n-1}(x), T_{n+1}(x)\rangle_S}. \tag{32}$$

This expression is well defined since the denominator is non zero. Indeed,

$$\begin{aligned}
&\langle T_{n+1}(x), T_{n+1}(x)\rangle_S - \eta_{n-2}(\lambda)\langle T_{n-1}(x), T_{n+1}(x)\rangle_S \\
&= \left\langle T_{n+1}(x) - \eta_{n-2}(\lambda)T_{n-1}(x), S^\lambda_{n+1}(x) + \eta_{n-2}(\lambda)S^\lambda_{n-1}(x)\right\rangle_S \\
&= \left\langle S^\lambda_{n+1}(x), S^\lambda_{n+1}(x)\right\rangle_S \neq 0.
\end{aligned}$$

We will express each term in (32) in a simpler form.

$$\begin{aligned}
&\langle T_{n+3}(x), T_{n+3}(x)\rangle_S \\
&= \langle P_{n+3}(x) + \widetilde{a}_n P_{n+1}(x), P_{n+3}(x) + \widetilde{a}_n P_{n+1}(x)\rangle_{\mu_0} \\
&\quad + \lambda\left\langle W'_{n+3}(x) + \widetilde{b}_n W'_{n+1}(x), W'_{n+3}(x) + \widetilde{b}_n W'_{n+1}(x)\right\rangle_{\mu_1} \\
&= \left\langle u, P^2_{n+3}\right\rangle + \widetilde{a}^2_n\left\langle u, P^2_{n+1}\right\rangle \\
&\quad +\lambda\Big\langle (n+3)R_{n+2}(x) + \widetilde{b}_n(n+1)R_n(x), (n+3)R_{n+2}(x) + \widetilde{b}_n(n+1)R_n(x)\Big\rangle_{\mu_1} \\
&= p_{n+3} + \widetilde{a}^2_n p_{n+1} + \lambda\Big((n+3)^2 r_{n+2} + \widetilde{b}^2_n(n+1)^2 r_n\Big).
\end{aligned}$$

Here we have used the notation $r_n := \left\langle v, R^2_n\right\rangle$ and $p_n := \left\langle u, P^2_n\right\rangle$. Also, in a similar way

$$\langle T_{n+1}(x), T_{n+3}(x)\rangle_S = \widetilde{a}_n p_{n+1} + \lambda\widetilde{b}_n(n+1)^2 r_n$$

and, replacing in (32), we get for $n \geq 1$,

$$\begin{aligned}
&\eta_n(\lambda) \\
&= \frac{\widetilde{a}_n p_{n+1} + \lambda\widetilde{b}_n(n+1)^2 r_n}{p_{n+1} + \lambda(n+1)^2 r_n + \widetilde{a}^2_{n-2}p_{n-1} + \lambda\widetilde{b}^2_{n-2}(n-1)^2 r_{n-2} - \eta_{n-2}(\lambda)\Big[\widetilde{a}_{n-2}p_{n-1} + \lambda\widetilde{b}_{n-2}(n-1)^2 r_{n-2}\Big]},
\end{aligned}$$

where $a_{-n} = b_{-n} = 0, n \in \mathbb{N}$.

***Remark* 3.7.** In connection with the Sobolev inner products, a particular case of symmetric $(1, 1)-$coherent pair was studied in [2], when u is classical, and where it is possible to obtain an expression for the Sobolev coefficients $\{\eta_n(\lambda)\}_{n\geq 0}$ as the above one. In adittion, a relation of the type (23) is obtained, which is a necessary and sufficient condition in order to obtain the respective symmetric $(1, 1)-$coherence relation.

Furthermore, if we define for $n \geq 1$,

$A_n = \widetilde{b}_n(n+1)^2 r_n, \quad B_n = \widetilde{a}_n p_{n+1}, \quad C_n = (n+1)^2 r_n + \widetilde{a}_{n-2}^2 p_{n-1}, \quad D_n = p_{n+1}$,

$C_1 = 4r_1$, then we can write

$$\eta_n(\lambda) = \frac{A_n\lambda + B_n}{C_n\lambda + D_n - \eta_{n-2}(\lambda)\left[A_{n-2}\lambda + B_{n-2}\right]}.$$

With this notation we can prove the next

Theorem 3.8. *There exist sequences of polynomials* $\{Q_n(\lambda)\}_{n\geq 0}$ *and* $\left\{\widetilde{Q}_n(\lambda)\right\}_{n\geq 0}$, *with* $\deg(Q_n) = \deg\left(\widetilde{Q}_n\right) = n$ *for every* n, *such that the following three term recurrence relations hold.*

$$Q_{n+1}(\lambda) = (C_{2n}\lambda + D_{2n})\, Q_n(\lambda) - (A_{2n-2}\lambda + B_{2n-2})^2\, Q_{n-1}(\lambda), \tag{33}$$

$$\widetilde{Q}_{n+1}(\lambda) = (C_{2n+1}\lambda + D_{2n+1})\, \widetilde{Q}_n(\lambda) - (A_{2n-1}\lambda + B_{2n-1})^2\, \widetilde{Q}_{n-1}(\lambda), \tag{34}$$

with the initial conditions $Q_0(\lambda) = \widetilde{Q}_0(\lambda) = 1$, $Q_1(\lambda) = \lambda + \left\langle u, P_1^2\right\rangle$, *and* $\widetilde{Q}_1(\lambda) = 4\left\langle v, R_1^2\right\rangle\lambda + \left\langle u, P_2^2\right\rangle$. *Furthermore, the Sobolev coefficients are rational functions in terms of such polynomials, namely*

$$\eta_{2n}(\lambda) = (A_{2n}\lambda + B_{2n})\,\frac{Q_n(\lambda)}{Q_{n+1}(\lambda)} \tag{35}$$

and

$$\eta_{2n+1}(\lambda) = (A_{2n+1}\lambda + B_{2n+1})\,\frac{\widetilde{Q}_n(\lambda)}{\widetilde{Q}_{n+1}(\lambda)}. \tag{36}$$

Proof. The initial conditions are obtained according to the definition of $\eta_0(\lambda)$ and $\eta_1(\lambda)$. Suppose that $\eta_{2n-2}(\lambda) = (A_{2n-2}\lambda + B_{2n-2})\,\dfrac{Q_{n-1}(\lambda)}{Q_n(\lambda)}$, then

$$\begin{aligned}\eta_{2n}(\lambda) &= \frac{A_{2n}\lambda + B_{2n}}{C_{2n}\lambda + D_{2n} - (A_{2n-2}\lambda + B_{2n-2})^2\,\dfrac{Q_{n-1}(\lambda)}{Q_n(\lambda)}} \\ &= \frac{(A_{2n}\lambda + B_{2n})\,Q_n(\lambda)}{(C_{2n}\lambda + D_{2n})\,Q_n(\lambda) - (A_{2n-2}\lambda + B_{2n-2})^2\,Q_{n-1}(\lambda)}.\end{aligned}$$

Thus (33) holds with $Q_{n+1}(\lambda)$ as the denominator. In an analogous way, $\eta_{2n-1}(\lambda) = (A_{2n-1}\lambda + B_{2n-1})\,\dfrac{\widetilde{Q}_{n-1}(\lambda)}{\widetilde{Q}_n(\lambda)}$ and we get

$$\eta_{2n+1}(\lambda) = \frac{(A_{2n+1}\lambda + B_{2n+1})\,\widetilde{Q}_n(\lambda)}{(C_{2n+1}\lambda + D_{2n+1})\,\widetilde{Q}_n(\lambda) - (A_{2n-1}\lambda + B_{2n-1})^2\,\widetilde{Q}_{n-1}(\lambda)}.$$

If we denote the denominator by $\widetilde{Q}_{n+1}(\lambda)$, then we get (34). ☑

***Remark* 3.9.** Notice that $B_n = \widetilde{b}_n r_{n+1} \neq 0$. Moreover, if $\widetilde{a}_n = 0$, then $A_n = 0$, for every n. As a consequence, (13) becomes

$$P_{n+2}(x) = \frac{R'_{n+3}(x)}{n+3} + b_n \frac{R'_{n+1}(x)}{n+1}, n \geq 0$$

and, according to Favard's theorem, the recurrence relations (33) and (34) mean that $\{Q_n(\lambda)\}_{n\geq 0}$ and $\left\{\widetilde{Q}_n(\lambda)\right\}_{n\geq 0}$ are orthogonal in the standard sense.

***Remark* 3.10.** The recurrence relation satisfied by the sequences $\{Q_n(\lambda)\}_{n\geq 0}$ and $\left\{\widetilde{Q}_n(\lambda)\right\}_{n\geq 0}$ are studied for the first time in [14] and, in the literature, they are known as R_{II} type recurrence relations.

4. Algorithm for Sobolev-Fourier coefficients

In this section we will describe an algorithm to compute the Fourier coefficients in expansions of Sobolev polynomials, orthogonal with respect to

$$\langle p, q\rangle_S = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\mu_1(x), \quad \lambda > 0.$$

For $f \in W_2^1[\mathbb{R}, \mu_0, \mu_1] = \left\{f | f \in L^2(\mu_0), f' \in L^2(\mu_1)\right\}$ we can expand f in terms of monic Sobolev orthogonal polynomials $\left\{S_n^\lambda\right\}_{n\geq 0}$, namely

$$f(x) \sim \sum_{n=0}^{\infty} \frac{\left\langle f, S_n^\lambda\right\rangle_S}{\|S_n^\lambda\|_S^2} S_n^\lambda(x).$$

We denote $s_n^\lambda := \left\|S_n^\lambda\right\|_S^2$, $f_n^\lambda := \langle f, S_n\rangle_S$ and $F_n^\lambda := f_n^\lambda / s_n$. F_n^λ is said to be the $n - th$ Sobolev-Fourier coefficient. To have the basic tools for implementation of the algorithms, we deduce the following result.

Lemma 4.1.

$$f_{n+2}^\lambda + \eta_{n-1}(\lambda) f_n^\lambda = w_n(f), \quad n \geq 0, \tag{37}$$

holds, where

$$w_n(f) = \left\langle f, P_{n+2}(x) + a_{n-1}\frac{n+2}{n} P_n(x)\right\rangle_{\mu_0} + \lambda \left\langle f', P'_{n+2}(x) + a_{n-1}\frac{n+2}{n} P'_n(x)\right\rangle_{\mu_1}, \tag{38}$$

with the initial conditions $\eta_{-1}(\lambda) = 0$, $f_0^\lambda = \langle f, 1\rangle_S = \langle f, 1\rangle_{\mu_0}$, $f_1^\lambda = \langle f, x\rangle_{\mu_0} + \lambda \langle f', 1\rangle_{\mu_1}$ and

$$w_0(f) := \langle f, P_2(x)\rangle_{\mu_0} + \lambda \langle f', P'_2(x)\rangle_{\mu_1}.$$

***Proof*.** From (20) and (13) we get

$$\frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1} = R_{n+2}(x) + b_n R_n(x)$$

$$S^{\lambda}_{n+3}(x)+\eta_n(\lambda)S^{\lambda}_{n+1}(x)=P_{n+3}(x)+\widetilde{a}_nP_{n+1}(x)$$

$$\begin{aligned}\left\langle f,S^{\lambda}_{n+2}\right\rangle_S &= -\eta_{n-1}\left\langle f,S^{\lambda}_{n}\right\rangle_S+\left\langle f,P_{n+2}(x)+a_{n-1}\frac{n+2}{n}P_n(x)\right\rangle_S\\ &= -\eta_{n-1}\left\langle f,S^{\lambda}_{n}\right\rangle_S+\left\langle f,P_{n+2}(x)+a_{n-1}\frac{n+2}{n}P_n(x)\right\rangle_{\mu_0}\\ &\quad+\lambda\left\langle f',P'_{n+2}(x)+a_{n-1}\frac{n+2}{n}P'_n(x)\right\rangle_{\mu_1}\\ &= -\eta_{n-1}\left\langle f,S^{\lambda}_{n}\right\rangle_S+\left\langle f,P_{n+2}(x)+a_{n-1}\frac{n+2}{n}P_n(x)\right\rangle_{\mu_0}\\ &\quad+\lambda\,(n+2)\left\langle f',R_{n+1}(x)+b_{n-1}R_{n-1}(x)\right\rangle_{\mu_1}\end{aligned}$$

and the result follows. ☑

Now we will summarize the results that, together with (37), yield the structure of the algorithm.

- For $n\geq 1$
$$A_n=\widetilde{b}_n(n+1)^2r_n,\qquad B_n=\widetilde{a}_np_{n+1},\qquad D_n=p_{n+1},\tag{39}$$
and for $n\geq 2$
$$C_n=(n+1)^2r_n+\widetilde{a}^2_{n-2}p_{n-1},\tag{40}$$
with $a_{-n}=b_{-n}=0$ for $n\in\mathbb{N}$, $C_1=4r_1$, furthermore, for $n\geq 0$, $\widetilde{a}_n=\dfrac{n+3}{n+1}a_n$ and $\widetilde{b}_n=\dfrac{n+3}{n+1}b_n$.
- With the initial conditions $\widetilde{Q}_{-1}(\lambda)=0$, $Q_0(\lambda)=\widetilde{Q}_0(\lambda)=1$, $Q_1(\lambda)=\lambda+p_1$, $\eta_0(\lambda)=\dfrac{\widetilde{b}_0\lambda+\widetilde{a}_0p_1}{\lambda+p_1}$, and $\eta_1(\lambda)=\dfrac{4\widetilde{b}_1r_1\lambda+\widetilde{a}_1p_2}{4r_1\lambda+p_2}$, we get
$$\eta_{2n}(\lambda)=(A_{2n}\lambda+B_{2n})\,\frac{Q_n(\lambda)}{Q_{n+1}(\lambda)},\ n\geq 1\tag{41}$$
and
$$\eta_{2n+1}(\lambda)=(A_{2n+1}\lambda+B_{2n+1})\,\frac{\widetilde{Q}_n(\lambda)}{\widetilde{Q}_{n+1}(\lambda)},\quad n\geq 1,\tag{42}$$
with
$$Q_{n+1}(\lambda)=(C_{2n}\lambda+D_{2n})\,Q_n(\lambda)-(A_{2n-2}\lambda+B_{2n-2})^2\,Q_{n-1}(\lambda)\tag{43}$$
and
$$\widetilde{Q}_{n+1}(\lambda)=(C_{2n+1}\lambda+D_{2n+1})\,\widetilde{Q}_n(\lambda)-(A_{2n-1}\lambda+B_{2n-1})^2\,\widetilde{Q}_{n-1}(\lambda).\tag{44}$$

- With initial conditions $s_1^\lambda = \lambda + p_1$ and $\eta_1(\lambda)$ in (30), for $n \geq 1$

$$s_{n+1}^\lambda = (n+1)^2 \lambda r_n + p_{n+1} + \widetilde{a}_{n-2}\left(\widetilde{a}_{n-2} - \widetilde{b}_{n-2}\right) p_{n-1} - \eta_{n-2}(\lambda)\left(\eta_{n-2}(\lambda) - \widetilde{b}_{n-2}\right) s_{n-1}^\lambda. \quad (45)$$

In order to describe the algorithms, we assume the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are known.

Algorithm 1. (Even order Fourier-Sobolev coefficients). For n even, the Fourier-Sobolev coefficients $F_n^\lambda = \dfrac{f_n^\lambda}{s_n^\lambda} = \dfrac{\langle f, S_n\rangle_S}{\|S_n^\lambda\|_S^2}$ can be computed using the following algorithm

Starting data. Initial conditions $\lambda, f_0^\lambda, \eta_{-1}, s_0^\lambda, \widetilde{Q}_{-1}, \widetilde{Q}_0, w_0(f), A_{-1}, B_{-1}, C_1$ and D_1.

Step 1. Using the starting data to compute f_2^λ with the relation (37) and $n = 0$, s_2^λ through (45) with $n = 1$ and finally F_2^λ.

Step 2. Using the starting data and the information in step 1 compute: $\widetilde{Q}_1$ taking $n = 0$ in (44) , A_1, B_1 with (39) and $n = 0$, $\eta_1(\lambda)$ taking $n = 0$ in (42), w_2 with $n = 2$ in (38), f_4^λ through (37) with $n = 2$, and finally s_4^λ taking $n = 3$ in (45). Then compute F_4^λ.

Step k. For $k \geq 3$, using the starting data and the information in steps 1 to $k-1$ we can compute A_{2k-3}, B_{2k-3}, C_{2k-1} and D_{2k-1} with $n = k-1$ in (39) and (40), $\widetilde{Q}_k$ taking $n = k-1$ in (44), A_{2k-1}, B_{2k-1} with (39) and $n = k-1$, $\eta_{2k-1}(\lambda)$ taking $n = k-1$ in (42), w_{2k} with $n = 2k$ in (38), f_{2k+2}^λ through (37) with $n = 2k$, and finally s_{2k+2}^λ taking $n = 2k+1$ in (45). Then, compute F_{2k+2}^λ.

Algorithm 2. (Odd order) For n even, the Fourier–Sobolev coefficients $F_n^\lambda = \dfrac{f_n^\lambda}{s_n^\lambda} = \dfrac{\langle f, S_n\rangle_S}{\|S_n^\lambda\|_S^2}$ can be computed using the following algorithm.

Starting data. Initial conditions λ, f_1^λ, η_0, s_1^λ, Q_0, Q_1, $w_1(f)$, A_0, B_0, C_2 and D_2.

Step 1. Using the starting data compute f_3^λ through (37) with $n = 1$, s_3^λ through (45) with $n = 2$ and then compute F_3^λ.

Step 2. Using the starting data and the information in step 1, compute Q_2 taking $n = 1$ in (43), A_2, B_2 with (39) and $n = 1$, $\eta_2(\lambda)$ taking $n = 1$ in (41), w_3 with $n = 3$ in (38), f_5^λ through (37) with $n = 3$, and finally s_5^λ taking $n = 4$ in (45). Then compute F_5^λ.

Step k. For $k \geq 3$, using the starting data and the information in steps 1 to $k-1$ we can compute A_{2k-2}, B_{2k-2}, C_{2k} and D_{2k} with $n = k$ in (39) and (40), Q_{k+1} taking $n = k$ in (44) , A_{2k}, B_{2k} with (39) and $n = k$, $\eta_{2k}(\lambda)$ taking $n = k$

in (42), w_{2k+1} with $n = 2k+1$ in (38), f^{λ}_{2k+3} through (37) with $n = 2k+1$, and finally s^{λ}_{2k+3} taking $n = 2k+2$ in (45). Then, compute F^{λ}_{2k+3}.

4.1. Numerical examples

Next, with the help of MATHEMATICA, we carry out some numerical experiments where the algorithms described above are implemented.

Example 4.2. (Gegenbauer Polynomials). In [8] the Symmetric $(1,1)-$coherent pairs, when u is the classical Gegenbauer functional, are exhibited. In particular, the pair

$$d\mu_0 = (1-x^2)^{\eta-1/2}dx, \quad d\mu_1 = \frac{x^2+a}{x^2+b}(1-x^2)^{\eta-1/2}dx,$$

$a, b \in \mathbb{R}^+$, $a \neq b$, $\eta > -1/2$, $x \in [-1,1]$, is obtained. Let $\left\{C^{(\eta)}_n\right\}_{n\geq 0}$ be the sequence of monic Gegenbauer polynomials, orthogonal with respect to the inner product

$$\langle p, q\rangle_\eta = \int_{-1}^{1} p(x)q(x)(1-x^2)^{\eta-1/2}dx.$$

Also, the Gegenbauer polynomials satisfy the TTRR

$$C^{(\eta)}_{n+1}(x) = xC^{(\eta)}_n(x) - \frac{n(n+2\eta-1)}{4(n+\eta-1)(n+\eta)}C^{(\eta)}_{n-1}(x), \quad n \geq 1,$$

and $C^{(\eta)}_0(x) = 1$. The corresponding norm is

$$\left\|C^{(\eta)}_n\right\|^2_\eta = \frac{4^{n+\eta}\left(\Gamma(n+\eta+1/2)\right)^2\Gamma\left(n+2\eta\right)}{2\left(n+\eta\right)\left(\Gamma(2n+2\eta)\right)^2}n!.$$

According to the results of the previous sections, if $d\mu_0 = (1-x^2)^{\eta-1/2}dx$, then we have the symmetric $(1,1)-$coherent relation

$$C^{(\eta+1)}_n(x) + b_{n-2}C^{(\eta+1)}_{n-2}(x) = Q_n(x) + a_{n-2}Q_{n-2}(x), \qquad n \geq 2,$$

where $\{Q_n\}_{n\geq 0}$ is the SMOP with respect to the measure $d\mu_1$. Moreover, from (20) we get

$$S^{\lambda}_{n+3}(x) + \eta_n(\lambda)S^{\lambda}_{n+1}(x) = C^{(\eta)}_{n+3}(x) + \frac{n+3}{n+1}b_n C^{(\eta)}_{n+1}(x), \qquad n \geq 0.$$

Explicit relations between recurrence coefficients and the sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are given in [8]. To be more precise, if $\{\widetilde{\gamma}_n\}_{n\geq 0}$ are the recurrence coefficients of the polynomials $\{Q_n\}_{n\geq 0}$, we get

$$b_0 - \frac{1}{2(\eta+1)} = a_0 - \widetilde{\gamma}_1,$$

$$\frac{n(n+2\eta-1)}{4(n+\eta-1)(n+\eta)}+b_{n-2}-b_{n-1}=\widetilde{\gamma}_n+a_{n-2}-a_{n-1},\quad n\geq 2,$$

$$b_{n-2}\frac{(n-2)(n+2\eta-3)}{4(n+\eta-3)(n+\eta-2)}=b_{n-3}\left(\frac{n(n+2\eta-1)}{4(n+\eta-1)(n+\eta)}+b_{n-2}-b_{n-1}\right),\ n\geq 5,$$

and

$$a_{n-2}\widetilde{\gamma}_{n-2}=a_{n-3}\left(\widetilde{\gamma}_n+a_{n-2}-a_{n-1}\right),\quad n\geq 5.$$

In addition, the sequence $\{b_n\}_{n\geq 0}$ satisfies the nonlinear quadratic difference equation

$$b_{n+1} \tag{46}$$

$$=\frac{1}{4(n+\eta+1)}\left(\frac{(n+1)(n+2\eta)}{4(n+\eta)}+\frac{(n+2)(n+2\eta+1)}{(n+\eta+2)}\right) \tag{47}$$

$$+\left(\frac{b_3-2}{1-\dfrac{3}{2b_2}}\right)-\frac{n(n+1)(n+2\eta)(n+2\eta-1)}{16(n+\eta)^2\left((n+\eta)^2-1\right)b_{n-1}}, \tag{48}$$

for $n\geq 3$.

For a fixed initial value b_0 we can compute the sequences of parameters $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ as long as the recurrence coefficients are known. A priori, we do not know the recurrence coefficients $\{\widetilde{\gamma}_n\}_{n\geq 0}$. However, it is possible to compute them with the desired precision through an efficient algorithm. For instance, the algorithms 1 and 4 in [12] meet this specific case, where there is a rational perturbation.

We will use the function $f(x)=e^{-100(x-0.2)^2}$. It can be seen that $f\in W_2^1[\mathbb{R},\mu_0,\mu_1]$. On one hand, in order to show the graphics of some partial Fourier-Sobolev sums, we choose $\eta=5$, $\lambda=0.001$, $a=1$, $b=2$. In Table 1 we get the first 16 Fourier-Gegenbauer-Sobolev coefficients.

n	a_n	b_n	$\eta_n(\lambda)$	s_n^λ	f_n^λ	F_n^λ
0	1.003	1	3	0.71	0.1391	0.1963
1	1.5062	1.5	3	0.051	0.0263	0.5195
2	2.0084	2	3.3	0.0059	-0.004	-0.7336
3	2.21	2.2	3.073	0.00086	-0.0036	-4.221
4	2.2084	2.1985	2.94	0.00014	-0.0001	-0.691
5	2.2199	2.21	2.03	0.000026	0.0005	20.22
6	2.228	2.2177	1.57	5.09×10^{-6}	0.0001	22.253
7	2.234	2.224	0.051	1.44×10^{-6}	-0.00007	-48.261
8	2.2387	2.2288	0.038	3.77×10^{-7}	-0.00003	-81.427
9	2.2427	2.2328	0.018	2.457×10^{-7}	-0.00005	-20.938
10	2.246	2.2361	0.019	7.243×10^{-7}	-0.00004	-48.874
11	2.2487	2.2388	0.02	3.395×10^{-7}	0.00002	62.233
12	2.251	2.2411	0.021	7.26×10^{-8}	0.00002	268.769
13	2.253	2.2431	0.0211	1.59×10^{-8}	-5.21×10^{-7}	-32.765
14	2.2547	2.2447	0.022	3.521×10^{-9}	-3.71×10^{-6}	-1053.25
15	2.256	2.2462	0.0221	7.92×10^{-10}	-5.47×10^{-7}	-690.56

TABLE 1. Fourier-Gegenbauer-Sobolev coefficients with $\eta = 5$, $\lambda = 0.001$, $a = 1$, $b = 2$.

Furthermore, in Figure 1 we show the partial sums for $n = 4, 7, 11, 15$ and 17.

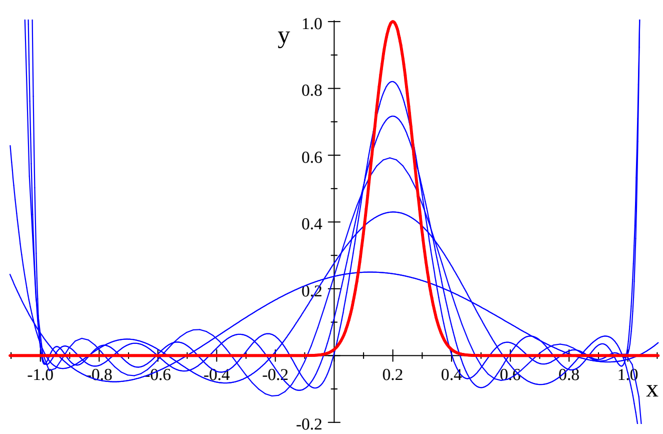

FIGURE 1. Partial sums for $n = 4$, 7, 11, 15 and 17, moreover $\eta = 1$, $\lambda = 0.5$, $a = 1$, $b = 2$. f in red.

On the other hand, in order to analyze the variation of the partial sums with respect to the parameter η. In Figure 2 we set $\lambda = 0.7$, $a = 2$, $b = 1$ and $n = 16$. In particular, we show the partial sums for $\eta = 0.5, 1, 2$ and 2.5.

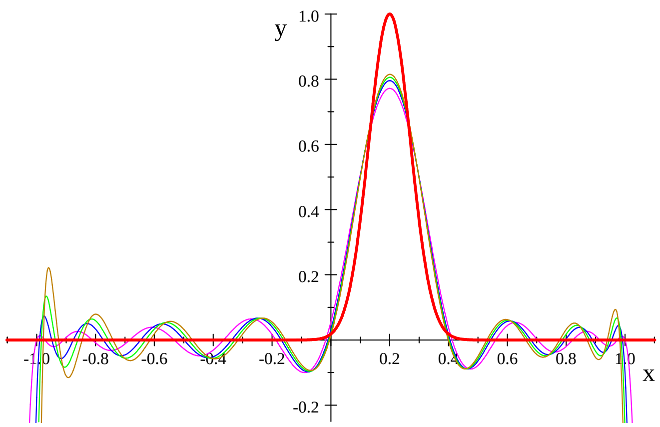

FIGURE 2. $16 - th$ partial sums for $\eta = 0.5$ (magenta), 1.5 (blue), 2 (green) and 2.5 (siena), when $\lambda = 0.7$, $a = 2$, $b = 1$, f in red.

Finally, setting $\eta = 1$, $n = 16$, $a = 1$, $b = 3$, in Figure 4 we exhibit the partial sums for $\lambda = 0.1$, 0.8, 1.8 and 10.

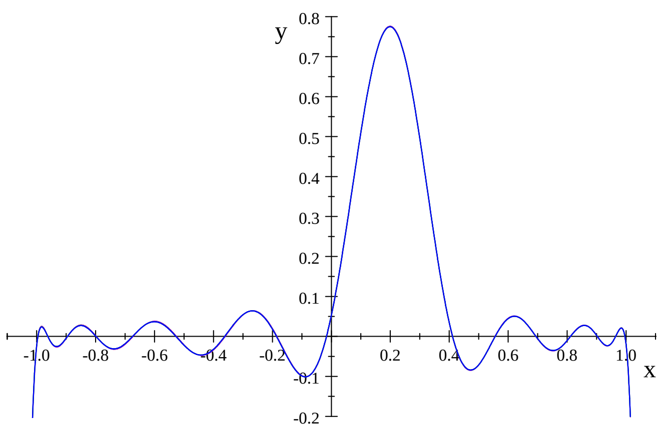

FIGURE 3. $16 - th$ partial sums for $\lambda = 0.1$ (purple), 0.8 (cyan), 1.8 (green) and 10 (blue), when $\eta = 1$, $a = 1$, $b = 3$

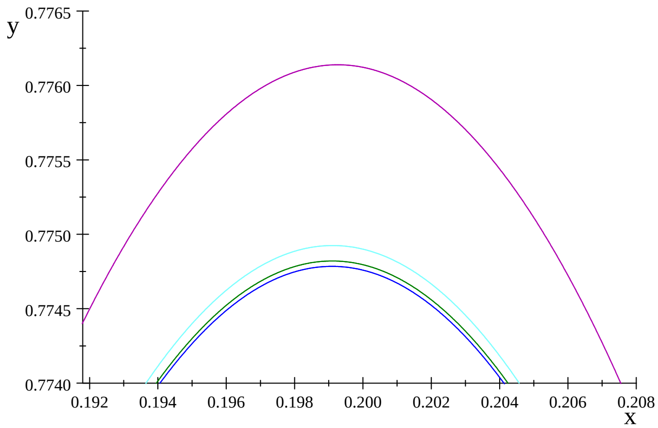

FIGURE 4. $16-th$ partial sums, (Zoom), for $\lambda = 0.1$ (purple), 0.8 (cyan), 1.8 (green) and 10 (blue), when $\eta = 1$, $a = 1$, $b = 3$

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