Renormalisation via locality morphisms

Pierre Clavier[1,a], Li Guo[2,b], Sylvie Paycha[1,c,✉] ,
Bin Zhang[3,]

[1]University of Potsdam, Potsdam, Germany

[2]Rutgers University, Newark, USA

[3]Sichuan University, Chengdu, China

Abstract. This is a survey on renormalisation in algebraic locality setup
highlighting the role that locality morphisms can play for renormalisation
purposes. After describing the general framework to build locality regularisa-
tion maps, we illustrate renormalisation by locality algebra homomorphisms
on three examples, the renormalisation of conical zeta functions at poles, the
definition of branched zeta functions and their evaluation at poles and finally
the values of iterated integrals stemming from Kreimer's toy model.

Key words and phrases. locality, renormalisation, partial algebra, operated al-
gebra, Hopf algebra, Rota-Baxter algebra, symbols.

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Resumen. Éste es un estudio sobre la renormalización en la configuración de
la localidad algebraica, que resalta el papel que los morfismos de la localidad
pueden desempeñar para los propósitos de la renormalización. Después de des-
cribir el marco general para construir mapas de regularización de la localidad,
ilustramos la renormalización mediante homomorfismos de álgebras de la lo-
calidad en tres ejemplos, la renormalización de las funciones zeta cónica en
los polos, la definición de las funciones zeta ramificadas y su evaluación en los
polos y, finalmente, los valores de las integrales iteradas derivadas del modelo
de juguete de Kreimer.

Palabras y frases clave. localidad, renormalización, álgebra parcial, álgebra op-
erada, álgebra de Hopf, álgebra de Rota-Baxter, símbolos.

Introduction

The study of certain models, might they stem from quantum field theory or be of a pure mathematical nature like that of measuring the volume of a convex cone, gives rise to *formal expressions*. Their evaluation can yield infinities instead of the desired numerical invariants, a situation which calls for renormalisation, a procedure which aims at getting rid of the infinities.

A renormalisation procedure typically consists of two steps. It first calls for a regularisation procedure which turns the formal expressions into mathematically meaningful expressions, typically described in terms of an algebra homomorphism

$$\phi^{\mathrm{reg}} : A \longrightarrow M \tag{1}$$

defined on a certain algebra A with values in a space M of meromorphic functions. The algebra structure on A is expected to reflect the structure of the family of formal expressions. Next, these meromorphic functions are processed to only keep the holomorphic parts (minimal subtraction scheme) which can then be evaluated at zero, giving rise to the *renormalised values* of the formal expressions.

Quantum field theory gives rise to singularities which typically arise in the form of distributions. Just as one cannot multiply two arbitrary distributions, the product on A is typically only partially defined, namely A is only a partial algebra. This gives the first motivation for us to introduce the notion of a locality algebra, where the product is defined only for selected pairs, which we call pairs of independent elements. The domain of this product is the graph of a locality relation.

A basic requirement is that the renormalised values preserve the structure of original formal expressions, typically formal multiple integrals such as Feynman integrals or multiple sums that are expected to factorise over independent sets of variables. Consequently, the renormalised integrals or sums are also expected to factorise over independent sets of variables. Such a formal multiplicative property over independent elements is encoded in the product of A, so in this sense A captures the structure of renormalised values. This partial multiplicativity is reminiscent of the *locality principle* in quantum field theory.

Let us briefly describe the locality principle in the setup of (resp. perturbative) algebraic quantum field theory (also called axiomatic quantum field theory) [7, 12]. To every closed subset $\mathcal{O} \subset X$ of the Minkowski spacetime X there is an associated C^*-algebra (resp. a formal power series algebra in Planck's constant $\hbar$) $\mathcal{A}(\mathcal{O})$; for every inclusion $\mathcal{O}_1 \hookrightarrow \mathcal{O}_2$ of such spacetime regions there is a corresponding inclusion $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ and $\mathcal{A}$ defines a functor from the poset of causally closed subsets to C^*-algebras. In this setup the locality principle states that whenever $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O} \subset X$ are spacelike separated, then the

elements of the corresponding algebras of observables (graded-)commute with each other:

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0,$$

which guarantees the causality independence of observables. States on two local algebras cannot be a priori multiplied. Yet states on the algebras of observables measured on two spacelike separated spacetime regions can be multiplied, a property which expresses the statistical independence of the observables.

We now discuss the analytic aspects of our approach. While the regularisation process depends on the specific problem under consideration, the minimal subtraction process is independent of the particular setup. In the case of an univariate regularisation, the algebra of Laurent series $M = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ and a Rota-Baxter operator $\pi_+ : M \longrightarrow M_+$ gives a projection onto the holomorphic part $M_+ = \mathbb{C}[[\varepsilon]]$ by means of the minimal subtraction scheme. Yet the Rota-Baxter operator itself is not multiplicative so that a mere minimal subtraction scheme $\pi_+ \circ \phi^{\mathrm{reg}}$ does not preserve multiplicativity. However, if the space A carries a suitable Hopf algebra structure, an algebraic Birkhoff factorisation à la Connes and Kreimer [4] implemented on the regularised map $\phi^{\mathrm{reg}} : A \longrightarrow M$ such that $\phi^{\mathrm{reg}} = \phi_-^{\mathrm{reg}\star(-1)} \star \phi_+^{\mathrm{reg}}$ guarantees the multiplicativity of the renormalised map $\phi_+^{\mathrm{reg}} : A \longrightarrow M_+$, that is, the conservation of products after renormalisation.

An alternative approach to a univariate regularisation is a multivariate regularisation, a setup which opens the way to new opportunities, and new challenges. At first glance, the multivariate minimal subtraction scheme, when available, seems ill-suited since it does not yield an algebra homomorphism, and does not even give rise to a Rota-Baxter operator, a major obstacle to the implementation of an algebraic Birkhoff factorisation in order to obtain algebra homomorphism even when A carries a Hopf algebra structure. Yet a closer look shows that multivariate regularisation makes it possible to encode the locality algebra structure, and that an appropriately chosen projection indeed preserves products of independent pairs, thus providing an easy book keeping device to preserve products of independent pairs. In accordance with the locality principle in quantum field theory, multiplicativity then holds for pairs of independent arguments. The latter property, together with the multiplicativity of the evaluation map at zero, allows the renormalisation to preserve products for independent pairs of elements. In this sense, the regularisation and renormalisation are locality algebra homomorphisms, so in particular, they are linear maps that are multiplicative on pairs of independent elements.

On the grounds of these considerations, we choose to work in a multivariate regularisation setup, and generally assume that $\phi^{\mathrm{reg}} : A \longrightarrow M$ is a locality algebra homomorphism, which turns out to be a natural assumption in the cases of interest here.

An algebraic formulation of the locality principle in renormalisation is discussed in [3]. There, we express a locality relation as a symmetric binary relation, study locality versions of algebraic structures, and develop a machinery used to preserve the locality algebra structure during the renormalisation procedure. Not only is the locality setup useful for renormalisation purposes, but it also plays a crucial role when exploring deeper structures as can be seen from the example of lattice cones.

The key asset of a locality setup lies in the fact that an appropriate minimal subtraction scheme is a *locality algebra homomorphism*. This is the case if the regularisation map ϕ^{reg} takes values in the algebra $\mathcal{M}$ of multivariate meromorphic germs with linear poles, which carries a locality algebra structure $(\mathcal{M}, \perp^Q, \cdot)$ (where Q is a chosen inner product). In that case the minimal subtraction, that is, the projection π_+^Q to $\mathcal{M}_+$ arising from the decomposition $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$ (which as we stressed previously is not an algebra homomorphism) is a locality algebra homomorphism. Thus, for a multivariate regularisation, provided the source space A can be equipped with a *locality algebra* $(A, \top_A)$ structure, the regularised map ϕ^{reg} takes its values in $\mathcal{M}$ and is a locality algebra homomorphism, then a *multivariate subtraction scheme* can be implemented on the regularised maps $\phi^{\mathrm{reg}} : (A, \top_A) \longrightarrow (\mathcal{M}, \perp^Q)$. Even though it does not preserve products, in this locality setup, the renormalised map $\pi_+^Q \circ \phi^{\mathrm{reg}}$ preserves products of independent pairs, which is what one needs.

To sum up, we work in a setup which encompasses an algebraic principle of locality; locality detects pairs of independent elements and partial multiplicativity amounts to multiplicativity on pairs of independent elements. The locality setup combined with the multivariate regularisation provides a way to preserve locality multiplicativity while renormalising, in accordance with the locality principle in physics. In our approach, the subtraction process is straighforward and the focus is on the regularisation process, so we put much effort to construct adequate regularised maps.

Based on previous work by the authors [2, 3, 5, 6], the purpose of this survey is

(1) to demonstrate how to achieve a multivariate regularisation of a formal expression so as to build a locality algebra homomorphism (so a multivariate version of (1))

$$\phi^{\mathrm{reg}} : (A, \top_A) \longrightarrow \left(\mathcal{M}, \perp^Q\right) \qquad (2)$$

on a locality algebra $(A, \top_A)$ with values in the locality algebra $(\mathcal{M}, \perp^Q)$, and

(2) to renormalise the resulting regularised locality algebra homomorphism, describing the general theory and illustrating it by examples.

Let us describe the contents of the paper in more detail.

In Section 1, we introduce locality algebras (Definition 1.1), a notion we first illustrate by the pivotal example of $\mathbb{R}^\infty$ (Example 1.2) equipped with an inner product Q which induces an orthogonality relation $\perp^Q$, after which we discuss in Paragraph 1.2, the algebra $\mathcal{M}$ of multivariate meromorphic germs with linear poles at zero, equipped with a locality relation induced by $\perp^Q$, which by a slight abuse of notation is denoted by the same symbol (see Proposition 1.3). Other relevant examples are the locality algebra of *lattice cones* in Paragraph 1.3 and the locality algebra of *properly decorated rooted forests* in Paragraph 1.4.

Section 2 is dedicated to the main protagonists of this paper, namely *locality morphisms* (Definition 2.2) of locality algebras, so maps between locality algebras which, as well as preserving the locality relation and locality vector space structure, further preserve the related product of independent pairs. In view of their importance in our approach, we chose to dedicate a section to locality algebra homomorphisms.

Amongst these is the *locality projection* $\pi_+^Q : \mathcal{M} \longrightarrow \mathcal{M}_+$ onto the space $\mathcal{M}_+$ of holomorphic germs at zero built from the inner product Q, arising from the decomposition $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$ (Eq. (7)) induced by Q. Its locality as a morphism of locality algebras is a consequence of the fact that $\mathcal{M}_+$ (resp. $\mathcal{M}_-^Q$) is a locality subalgebra (resp. locality ideal) of $\mathcal{M}$ (Proposition 1.4).

Composed with the evaluation ev_0 at zero this projection yields a useful *renormalisation schemes* discussed in Paragraph 3.1:

$$\mathrm{ev}_0 \circ \pi_+^Q : \mathcal{M} \longrightarrow \mathbb{C}, \tag{3}$$

which can be viewed as a *multivariate minimal subtraction scheme.*

With this multivariate minimal subtraction scheme, a renormalisation process is reduced to two steps:

(1) to construct the regularised map $\phi^{\mathrm{reg}} : (A, \top_A) \longrightarrow (\mathcal{M}, \perp^Q)$;

(2) to implement renormalisation schemes of the type (3) to the regularised map $\phi^{\mathrm{reg}} : (A, \top_A) \longrightarrow (\mathcal{M}, \perp^Q)$ in order to build the *renormalised map*

$$\phi^{\mathrm{ren}} := \mathrm{ev}_0 \circ \pi_+^Q \circ \phi^{\mathrm{reg}} : A \longrightarrow \mathbb{C}.$$

Various locality maps built in Section 2 are interpreted in Section 3 as regularisation maps $\phi^{\mathrm{reg}} : A \longrightarrow \mathcal{M}$ which need to be renormalised, all of which stem from formal sums and integrals as multivariate regularisations.

We first illustrate (in Paragraphs 2.3 and 3.2) this multivariate approach with *conical zeta functions* (resp. *branched zeta functions*), which to a lattice cone (resp. a decorated rooted forest), assign a renormalised value of the *regularised conical zeta function* (resp. *regularised branched zeta function*) at poles.

The (partial) multiplicativity of the maps encoded in their very construction in our multivariate locality setup, ensures their multiplicativity on orthogonal lattice cones (resp. independent decorated rooted forests).

In [3, 6], *conical zeta functions* (Paragraph 3.2), which generalise multiple zeta functions were built using exponential sums on lattice cones. The exponential sum S (resp. integral I) on a lattice cone corresponds to the discrete (resp. continuous) Laplace transformation of the characteristic function of the lattice cone (Proposition 2.7). One easily checks that Laplace transforms of characteristic functions of smooth cones define meromorphic maps with linear poles; the fact that S and I take their values in $\mathcal{M}$ for any convex lattice cone, then follows from their additivity on disjoint unions combined with the fact that any convex lattice cone can be subdivided into smooth lattice cones. Both maps define locality algebra homomorphisms on the locality algebra of lattice cones for a locality relation induced by the orthogonality relation $\perp^Q$ on $\mathbb{R}^\infty$. Their multiplicativity on orthogonal lattice cones follows from the usual homomorphism property of the exponential maps on these cones.

A second example which provides an alternative generalisation of multiple zeta functions, is given by *branched zeta functions* [2] (discussed in Paragraph 3.3) associated with rooted forests (Paragraph 1.4). These are built by means of a *branching procedure* which strongly relies on the universal properties of properly decorated rooted forests (see Proposition 2.11). Such a branching procedure lifts a map ϕ defined on the decoration set to what we call a *branched map* $\widehat{\phi}$ on the algebra of decorated forests (see (16)). Applied to a summation map $\phi = \mathfrak{S}_\lambda$ on the locality algebra $\mathcal{MS}^{\mathrm{adm}}$ of admissible meromorphic germs of symbols (Definition 1.12), this branching procedure gives rise to a branched sum $\widehat{\mathfrak{S}_\lambda}$ acting on the algebra of properly decorated rooted forests by meromorphic family of symbols on $\mathbb{R}_{\geq 0}$. The universal property underlying the construction ensures the multiplicativity on independent forests. Combining this with the locality morphism given by the Hadamard finite part at infinity (11)-a linear form on polyhomogeneous pseudodifferential symbols which coincides on smoothing symbols with the limit at infinity-extended to $\mathcal{MS}^{\mathrm{adm}}$, gives rise to *branched regularised zeta functions* $\zeta^{\mathrm{reg},\lambda}$ defined on the locality algebra of properly $\mathcal{MS}^{\mathrm{adm}}$-decorated rooted forests.

In Paragraph 2.5 we describe similar constructions based on the universal properties of properly decorated rooted forests [1], which yield a third example (Paragraph 3.4), namely $\mathcal{M}$-valued maps stemming from iterated integrals arising in Kreimer's toy model [9].

Since there are different approaches to explore locality, in Section 4, much of which is borrowed from [14], we review and compare various partial structures with the locality structures introduced in [3]. In particular, we view the locality setup as a symmetric version of the more general $\mathcal{R}$-setup which comprises partial semigroups (Definition 4.7) introduced in [13] and we relate $\mathcal{R}$-monoids to

the selective category of Li-Bland and Weinstein [10] with one object (Proposition 4.9). Thereafter, for the sake of simplicity, we choose to keep to the locality setup which turns out to be sufficient for the renormalisation purposes we have in mind.

1. Locality algebras

Throughout the paper we choose to work in the framework of locality structures, in part for the sake of simplicity but mostly due to the fact the the applications we have in view do not require the more general framework of $\mathcal{R}$-structures discussed in Section 4.

1.1. Basic definitions

We borrow the subsequent definitions from [3]. Among them, locality algebras are fundamental objects in multivariate renormalisation.

Definition 1.1. (1) A **locality set** is a couple $(X, \top)$ where X is a set and $\top \subseteq X \times X$ is a symmetric relation on X, also denoted $X \times_\top X$ and referred to as the **locality relation** (or **independence relation**) of the locality set.

(2) Let $(X, \top)$ be a locality set and $U \subseteq X$ a subset of X. We then define

$$U^\top := \{x \in X | (U, x) \subseteq X \times_\top X\}$$

the **polar subset** of U.

(3) A **locality semigroup** is a locality set $(G, \top)$ together with a product law defined on $\top$:
$$m_G : G \times_\top G \longrightarrow G$$

for which the product is compatible with the locality relation on G, namely
$$\text{for all } U \subseteq G, \quad m_G((U^\top \times U^\top) \cap \top) \subset U^\top \tag{4}$$

and such that the following **locality associativity property** holds:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \text{ for all } (x, y, z) \in G \times_\top G \times_\top G, \tag{5}$$

where for a locality set $(X, \top)$ we have set $\underbrace{X \times_\top \cdots \times_\top X}_{k \text{ times}} :=$ $\{(x_1, \cdots, x_k \in X^k, x_i \top x_j \, \forall i \neq j\}$.

(4) A **locality monoid** is a locality semigroup $(G, \top, m_G)$ together with a **unit element** $1_G \in G$ given by the defining property

$$\{1_G\}^\top = G \quad \text{and} \quad m_G(x, 1_G) = m_G(1_G, x) = x \ \text{ for all } x \in G.$$

(5) A **locality vector space** is a vector space V over a field K equipped with a locality relation $\top$ which is compatible with the linear structure on V in the sense that, for any subset X of V, $X^\top$ is a linear subspace of V.

(6) A **(resp. unital) locality algebra** $(A, \top, +, \cdot, m_A)$ (resp. $(A, \top, +, \cdot, m_A, 1_A)$) over K is a locality vector space $(A, +, \cdot, \top)$ over K together with a locality bilinear map

$$m_A : A \times_\top A \to A$$

such that $(A, \top, m_A)$ is a locality semigroup (resp. a locality monoid with **unit** $1_A \in A$). $(A, \top)$ is called **commutative** if $(A, \top, m_A)$ is a commutative locality semigroup.

(7) A **sub-locality algebra** of a locality algebra $(A, \top, m_A)$ is a linear subspace B of A such that with respect to the locality condition $\top_B := (B \times B) \cap \top$ of $\top$ and the partial product $m_B := m_A|_{\top_B}$ on B, $(B, \top_B, m_B)$ is a locality algebra.

(8) A sub-locality algebra I of a locality commutative algebra $(A, \top, m_A)$ is called a **locality ideal** of A if for any $b \in I$ we have $m_A(c, b) \in I$ for all $c \in A$ such that $c \top b$.

Example 1.2. A pivotal example is the locality vector space $\left(\mathbb{R}^\infty, \perp^Q\right)$, where $\mathbb{R}^\infty = \bigcup_{k \geq 1} \mathbb{R}^k$ is the inductive limit for the standard embeddings $i_k : \mathbb{R}^k \to \mathbb{R}^{k+1}$ and $Q = (Q_k(\cdot, \cdot))_{k \geq 1}$ is an inner product on $\mathbb{R}^\infty$ defined by the Euclidean inner product on $\mathbb{R}^k$

$$Q_k(\cdot, \cdot) : \mathbb{R}^k \otimes \mathbb{R}^k \to \mathbb{R}, \quad k \geq 1,$$

such that $Q_{k+1}|_{\mathbb{R}^k \otimes \mathbb{R}^k} = Q_k$. The inner product induces a locality relation on $\mathbb{R}^\infty$

$$u \perp^Q v \Leftrightarrow Q(u, v) = 0,$$

which makes $\left(\mathbb{R}^\infty, \perp^Q\right)$ a locality vector space.

The inner product also induces a locality set structure on the set of subspaces of $\mathbb{R}^\infty$:

$$U \perp^Q V \Leftrightarrow Q(u, v) = 0 \quad \text{for all } u \in U, v \in V.$$

1.2. The locality algebra of meromorphic germs with linear poles

For the filtered Euclidean space $(\mathbb{R}^\infty, Q)$ as in Example 1.2, stemming from the standard embeddings $i_n : \mathbb{R}^n \to \mathbb{R}^{n+1}$, the inner product Q induces an isomorphism

$$Q_n^* : \mathbb{R}^n \to (\mathbb{R}^n)^*.$$

Let $\mathcal{M}(\mathbb{R}^n \otimes \mathbb{C})$ be the algebra of **meromorphic germs with linear poles at zero** and real coefficients, that is, the algebra of meromorphic germs f at zero, for which there exist real linear forms on $\mathbb{C}^n$ whose product with f is holomorphic at zero with real coefficients power series expansions. Then

$$j_{n+1} := (Q_n^*)^{-1} \circ i_n^* \circ Q_{n+1}^* : \mathbb{R}^{n+1} \to \mathbb{R}^n,$$

induce a direct system

$$j_{n+1}^* : \mathcal{M}(\mathbb{R}^n \otimes \mathbb{C}) \to \mathcal{M}(\mathbb{R}^{n+1} \otimes \mathbb{C}),$$

and we set

$$\mathcal{M} := \mathcal{M}(\mathbb{C}^\infty) := \varinjlim_n \mathcal{M}(\mathbb{C}^n) = \varinjlim_n \mathcal{M}(\mathbb{R}^n \otimes \mathbb{C}), \tag{6}$$

which is the algebra of multivariate meromorphic germs with linear poles and real coefficients [5, 6].

The locality structure on $(\mathbb{R}^\infty, \perp^Q)$ induces a locality structure on $\mathcal{M}$. For $f \in \mathcal{M}(\mathbb{C}^n)$, let $\mathrm{Dep}(f)$ denote the **dependence space** of f, defined as the smallest subspace of $(\mathbb{C}^n)^*$ spanned by the linear forms on which f depends in the sense of [3, Definitions 2.9 and 2.13].

Proposition 1.3. [3, Proposition 3.9] *Equipped with the locality relation*

$$f_1 \perp^Q f_2 \iff \mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2),$$

and the ordinary product of functions restricted to the graph of the locality relation, the locality set $(\mathcal{M}, \perp^Q)$ carries a locality algebra structure.

The inner product Q induces a decomposition of $\mathcal{M}$ [5]

$$\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-^Q, \tag{7}$$

where $\mathcal{M}_+$ is the subspace of holomorphic germs at zero and $\mathcal{M}_-^Q$ (which depends on Q) is the vector subspace generated by polar germs at zero, namely meromorphic germs at zero $\frac{f_1}{f_2}$ with linear poles such that $\mathrm{Dep}(f_1) \perp^Q \mathrm{Dep}(f_2)$.

Further details can be found in [5, Definition 2.3].

Proposition 1.4. [3, Proposition 3.19] *The subspace $\mathcal{M}_+$ is a subalgebra and sub-locality algebra of $\mathcal{M}$. The subspace $\mathcal{M}_-^Q$ is not a subalgebra but a locality ideal of $\mathcal{M}$.*

There is another locality structure on $\mathcal{M}$ which is also compatible with the ordinary product of functions.

Let $\{e_n \mid n \in \mathbb{N}\}$ denote a Q-orthonormal basis of $\mathbb{R}^\infty$. We call the **support of** $f \in \mathcal{M}$, denoted $\mathrm{Supp}(f)$, the smallest subset $J \subset \mathbb{N}$ such that $\mathrm{Dep}(f)$ is contained in the subspace spanned by $\{e_j^* \mid j \in J\}$. We thus equip $\mathcal{M}$ with the locality relation

$$f_1 \top_D^Q f_2 \iff \mathrm{Supp}(f_1) \cap \mathrm{Supp}(f_2) = \varnothing,$$

which makes $\mathcal{M}$ a locality vector space.

Remark 1.5. Since the $\mathbb{R}$-linear span of $e_j^*, j \in \mathrm{Supp}(f)$ contains $\mathrm{Dep}(f)$, for $f_1, f_2 \in \mathcal{M}$ we have $f_1 \top_D^Q f_2 \implies f_1 \perp^Q f_2$. Yet $(e_1^* + e_2^*) \perp^Q (e_1^* - e_2^*)$ whereas these two linear forms are not $\top_D$ independent since $\mathrm{Supp}(e_1^* + e_2^*) = \{1,2\} = \mathrm{Supp}(e_1^* - e_2^*)$.

Proposition 1.6. *The locality set* $\left(\mathcal{M}, \top_D^Q\right)$ *equipped with the product of functions is a locality algebra.*

Proof. This follows from Remark 1.5 and Proposition 1.3. ☑

1.3. Locality algebra of lattice cones

In the filtered Euclidean lattice space $(\mathbb{R}^\infty, \mathbb{Z}^\infty, Q)$ defined by the condition that $Q(u,v)$ lies in $\mathbb{Q}$ for $u, v \in \mathbb{Z}^\infty$, a lattice cone is a pair (C, Λ_C) where C is a convex polyhedral cone in some $\mathbb{R}^k$ generated by elements in $\mathbb{Z}^k$ and Λ_C is a lattice generated by elements in $\mathbb{Q}^k$ in the linear subspace spanned by C. Let $\mathcal{C}_k$ be the set of lattice cones in $\mathbb{R}^k$ and

$$\mathcal{C} = \bigcup_{k \geq 1} \mathcal{C}_k$$

be the set of lattice cones in $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$ which is the direct limit under the standard embeddings. Let $\mathbb{Q}\mathcal{C}_k$ and $\mathbb{Q}\mathcal{C}$ be the linear spans of $\mathcal{C}_k$ and $\mathcal{C}$ over $\mathbb{Q}$.

For two lattice cones (C_i, Λ_i), $i = 1, 2$, then

$$(C_1, \Lambda_1) \perp^Q (C_2, \Lambda_2) \iff \mathrm{span}(C_1) \perp^Q \mathrm{span}(C_2) \tag{8}$$

defines a locality relation on $\mathbb{Q}\mathcal{C}$.

For convex cones $C := \langle u_1, \cdots, u_m \rangle$ and $D := \langle v_1, \cdots, v_n \rangle$ spanned by $u_1, \cdots, u_m$ and $v_1, \cdots, v_n$ respectively, their Minkowski sum is the convex cone

$$C \cdot D := \langle u_1, \cdots, u_m, v_1, \cdots, v_n \rangle.$$

This operation can be extended to a product in $\mathbb{Q}\mathcal{C}$, called the **extended Minkowski sum**:

$$(C, \Lambda_C) \cdot (D, \Lambda_D) := (C \cdot D, \Lambda_C + \Lambda_D), \tag{9}$$

where $\Lambda_C + \Lambda_D$ is the abelian group generated by Λ_C and Λ_D in $\mathbb{Q}^\infty$. This product endows a monoid structure on $\mathcal{C}$ with unit $(\{0\}, \{0\})$, which also restricts to a locality monoid structure on $(\mathcal{C}, \perp^Q)$.

Proposition 1.7. [3, Lemma 3.18] *The locality vector space* $\left(\mathbb{Q}\mathcal{C}, \perp^Q\right)$ *equipped with the extended Minkowski sum is a (graded) locality algebra.*

As with the case for meromorphic germs, there is another subset $\top_D^Q$ of $\perp^Q$ which also makes $\mathbb{Q}\mathcal{C}$ into a locality algebra. Let $\{e_n \,|\, n \in \mathbb{N}\}$ be an orthonormal basis of $\mathbb{R}^\infty$. For a lattice cone (C, Λ_C), we denote by $\mathrm{Supp}(C, \Lambda_C)$ the smallest subset J such that $\mathrm{span}(C)$ is contained in the subspace spanned by $\{e_j^* \,|\, j \in J\}$ and equip $\mathbb{Q}\mathcal{C}$ with the locality relation

$$(C_1, \Lambda_{C_1}) \top_D^Q (C_2, \Lambda_{C_2}) \Longleftrightarrow \mathrm{Supp}(C_1, \Lambda_{C_1}) \cap \mathrm{Supp}(C_2, \Lambda_{C_2}) = \varnothing,$$

which makes $\mathbb{Q}\mathcal{C}$ a locality vector space.

Proposition 1.8. *The locality set* $\left(\mathbb{Q}\mathcal{C}, \top_D^Q\right)$ *equipped with the extended Minkowski sum is a locality algebra.*

1.4. Locality algebra of decorated rooted forests.

Let $(\Omega, \top_\Omega)$ be a locality set. A **properly** $(\Omega, \top_\Omega)$-**decorated rooted forest** is a pair (F, d), where F is a (non-planar) rooted forest and $d : V(F) \to \Omega$ is a map from the set $V(F)$ of vertices of F to Ω such that

$$v \neq v' \Rightarrow d(v) \top_\Omega d(v').$$

Let $\mathcal{F}_{\Omega, \top_\Omega}$ denote the set of properly $(\Omega, \top_\Omega)$-decorated rooted forests and by $K\mathcal{F}_{\Omega, \top_\Omega}$ its linear span where K is a generic field of characteristic 0. The set $\mathcal{F}_{\Omega, \top_\Omega}$ carries a natural locality relation $\top_{\mathcal{F}_{\Omega, \top_\Omega}}$ from $(\Omega, \top_\Omega)$:

$$(F_1, d_1) \top_{\mathcal{F}_{\Omega, \top_\Omega}} (F_2, d_2) \Leftrightarrow d_1(v_1) \top_\Omega d_2(v_2) \text{ for all } v_1 \in V(F_1), v_2 \in V(F_2)$$

and this locality relation induces a locality relation $\top_{\mathcal{F}_{\Omega, \top_\Omega}}$ on $K\mathcal{F}_{\Omega, \top_\Omega}$.

Proposition 1.9. [2, Proposition 1.22] *The space* $K\,\mathcal{F}_{\Omega, \top_\Omega}$ *of properly* $(\Omega, \top_\Omega)$-*decorated rooted forests is a locality algebra for the concatenation product.*

1.5. Locality algebra of meromorphic germs of symbols

In analysis and geometry, the algebra of polyhomogeneous symbols plays an important role. We are in particular interested in the set $\mathcal{S}_{\mathrm{ph}}^\alpha(\mathbb{R}_{\geq 0})$ of polyhomogeneous symbols in $\mathbb{R}_{\geq 0}$ of order $\alpha \in \mathbb{C}$, i.e. of smooth functions on $\mathbb{R}_{\geq 0}$ with asymptotic expansion

$$\sigma \sim \sum_{j=0}^{\infty} a_j \, x^{\alpha - j}.$$

Precisely, for any integer N, the remainder term

$$\sigma_{(N)}^{\chi} : x \longmapsto \sigma(x) - \sum_{j=0}^{N-1} a_j \, x^{\alpha-j} \, \chi(x) \tag{10}$$

satisfies

$$\forall k \in \mathbb{Z}_{\geq 0}, \ \exists D_k \in \mathbb{R}_{>0} : \ \forall x \in \mathbb{R}_{\geq 0}, \ |\partial_x^k \sigma_{(N)}(x)| \leq D_k x^{\Re(\alpha)-N-k}$$

for one (and hence any) compactly supported smooth cut-off function χ which is identically one in a neighborhood of zero. For such a polyhomogeneous symbol σ, the Hadamard finite part at infinity is defined by

$$\underset{+\infty}{\mathrm{fp}} \ \sigma := \sum_{j=0}^{\infty} a_j \, \delta_{\alpha-j,0}, \tag{11}$$

(with $\delta_{i,k}$ the Kronecker symbol). We set

$$\mathcal{S}_{\mathrm{ph}}(\mathbb{R}_{\geq 0}) := \sum_{\alpha \in \mathbb{C}} \mathcal{S}_{\mathrm{ph}}^{\alpha}(\mathbb{R}_{\geq 0}),$$

which corresponds to the algebra of polyhomogeneous symbols in $\mathbb{R}_{\geq 0}$.

Example 1.10. Any polynomial function $\sigma(x) = \sum_{j=0}^{n} a_j \, x^j$ of degree n is a polyhomogeneous symbol of order n and $\underset{+\infty}{\mathrm{fp}} \ \sigma = a_0$.

***Remark* 1.11.** The linear map $\underset{+\infty}{\mathrm{fp}}$ is not an algebra homomorphism on $\mathcal{S}(\mathbb{R}_{\geq 0})$ equipped with the ordinary product of functions as can be seen from the fact that for $\sigma_1(x) = x^{-1} \, \chi(x)$ where χ is some cut-off function as above and $\sigma_2(x) = x$ we have

$$\underset{+\infty}{\mathrm{fp}} \ (\sigma_1 \, \sigma_2) = 1 \neq 0 = \underset{+\infty}{\mathrm{fp}} \ (\sigma_1) \ \underset{+\infty}{\mathrm{fp}} \ (\sigma_2).$$

In the filtered Euclidean space $(\mathbb{R}^{\infty}, Q)$, let $\mathcal{L}_k := (\mathbb{R}^k)^*$ and $\mathcal{L} = \underset{\rightarrow}{\lim} \mathcal{L}_k$ be the direct limit of spaces of linear forms. Recall that for a domain U in $\mathbb{C}^n$, a family $(\sigma(z)_{z \in U})$ of classical symbols is **holomorphic of affine order** $\alpha(z)$ if

(1) for any $z \in U$, $\sigma(z) \in \mathcal{S}_{\mathrm{ph}}^{\alpha(z)}(\mathbb{R}_{\geq 0})$;

(2) $\alpha(z) = L(z) + c$ with $c \in \mathbb{R}$ and $L \in \mathcal{L}_k$;

(3) for any $N \in \mathbb{Z}_{\geq 1}$ and any excision function χ, the remainder (see Eq. (10))

$$z \mapsto \sigma_{(N)}^{\chi}(z) := \sigma(z) - \sum_{j=0}^{N-1} \chi(x) \, a_j(z) \, x^{\alpha(z)-j},$$

satisfies the following uniform estimation: for any $k \in \mathbb{Z}_{\geq 0}$, and for any $x \in \mathbb{R}_{\geq 0}$, the derivatives $\partial_x^k \sigma_{(N)}^\chi$ are holomorphic functions on U, and for any compact subset K of U, and any $n \in \mathbb{Z}_{\geq 0}$ there is a positive constant $C_{k,n,N}(K)$ such that

$$\left| \partial_z^n \left(\partial_x^k \sigma_{(N)}^\chi(z) \right)(x) \right| \leq C_{k,n,N}(K) \langle x \rangle^{\Re(\alpha(z)) - N - k + \epsilon} \quad \text{for all } z \in K \subset U, \epsilon > 0.$$

Such a family is called **a simple holomorphic family of symbols (of affine order α)**. A sum of simple holomorphic families of symbols is called **a holomorphic family of symbols**. A simple **symbol-valued holomorphic germ** or **holomorphic germ of symbols** at zero (with affine order $\alpha(z)$) is an equivalence class of simple holomorphic families around zero of symbols of affine order $\alpha(z)$ under the equivalence relation:

$$(\sigma(z)_{z \in U}) \sim (\tau(z)_{z \in V}) \iff \exists W, \ 0 \in W \subset U \cap V, \ \sigma(z) = \tau(z) \ \text{ for all } z \in W.$$

For any positive integer k, any $\alpha(z) = L(z) + c$ with $c \in \mathbb{R}, L \in \mathcal{L}_k$, let $\mathcal{M}_+ \mathcal{S}^\alpha(\mathbb{C}^k)$ denote the linear space generated by simple holomorphic germs of symbols of order α, and $\mathcal{M}_+ \mathcal{S}(\mathbb{C}^k)$ denote the linear space generated by simple holomorphic germs of symbols.

Let U be a domain of $\mathbb{C}^k$ containing the origin. A simple **meromorphic family** on U of polyhomogeneous symbols with linear poles (with real coefficients) and affine order $\alpha(z)$ is a holomorphic family $(\sigma(z)_{z \in U \setminus X})$ with affine order $\alpha(z)$ of symbols on $U \setminus X$, for which

- $X = \cup_{i=1}^k \{L_i = 0\}$ with $L_1, \cdots L_n \in \mathcal{L}_k$,

- there exists a simple holomorphic family $(\tau(z)_{z \in U})$ with affine order $\alpha(z)$ and nonnegative integers $s_1, \cdots, s_n$, such that

$$L_1^{s_1} \cdots L_n^{s_n} \sigma(z) = \tau(z)$$

   on $U \setminus X$.

A simple symbol-valued **meromorphic germ** or meromorphic germ of symbols at zero on $\mathbb{C}^k$ with linear poles and affine order $\alpha(z)$ is an equivalence class of meromorphic families around zero with linear poles of symbols of affine order $\alpha(z)$ under the equivalence relation:

$$(\sigma(z)_{z \in U \setminus X}) \sim (\tau(z)_{z \in V \setminus Y}) \Leftrightarrow \exists W, \ 0 \in W \subset U \cap V, \ \sigma(z) = \tau(z), \forall z \in W \setminus (X \cup Y),$$

where U and V are domains of $\mathbb{C}^k$ containing the origin. Let $\mathcal{M}\mathcal{S}^\alpha(\mathbb{C}^k)$ denote the linear space generated by simple symbol-valued meromorphic germ at zero on $\mathbb{C}^k$ with linear poles and affine order $\alpha(z)$, and $\mathcal{M}\mathcal{S}(\mathbb{C}^k)$ denote the linear space generated by simple symbol-valued meromorphic germ at zero.

Composing with the projection $(\mathbb{C}^{k+1})^* \to (\mathbb{C}^k)^*$ dual to the canonical inclusion $i_k : \mathbb{C}^k \to \mathbb{C}^{k+1}$, and the isomorphism induced by the inner product $Q_k^* : (\mathbb{C}^k)^* \cong \mathbb{C}^k$, yields the embeddings $\mathcal{M}_+\mathcal{S}(\mathbb{C}^k) \hookrightarrow \mathcal{M}_+\mathcal{S}(\mathbb{C}^{k+1})$ (resp. $\mathcal{M}\mathcal{S}(\mathbb{C}^k) \hookrightarrow \mathcal{M}\mathcal{S}(\mathbb{C}^{k+1})$), thus giving rise to the direct limits:

$$\mathcal{M}_+^\alpha \mathcal{S}(\mathbb{C}^\infty) := \varinjlim \mathcal{M}_+^\alpha \mathcal{S}(\mathbb{C}^k) = \bigcup_{k=1}^\infty \mathcal{M}_+^\alpha \mathcal{S}(\mathbb{C}^k), \qquad (12)$$

$$\mathcal{M}_+\mathcal{S}(\mathbb{C}^\infty) := \varinjlim \mathcal{M}_+\mathcal{S}(\mathbb{C}^k) = \bigcup_{k=1}^\infty \mathcal{M}_+\mathcal{S}(\mathbb{C}^k), \qquad (13)$$

$$(\text{resp. } \mathcal{M}\mathcal{S}(\mathbb{C}^\infty) := \varinjlim \mathcal{M}\mathcal{S}(\mathbb{C}^k) = \bigcup_{k=1}^\infty \mathcal{M}\mathcal{S}(\mathbb{C}^k), \qquad (14)$$

where $\alpha(z) = L(z) + c$ with $c \in \mathbb{R}$ and $L \in \mathcal{L}_k$.

Then under pointwise function multiplication, $\mathcal{M}\mathcal{S}(\mathbb{C}^\infty)$ is a complex algebra and we have the following inclusions of subalgebras

$$\mathcal{M}_+\mathcal{S}(\mathbb{C}^\infty) \subset \mathcal{M}\mathcal{S}(\mathbb{C}^\infty); \quad \mathcal{M}(\mathbb{C}^\infty) \cdot \mathcal{P}(\mathbb{R}_{\geq 0}) \subset \mathcal{M}\mathcal{S}(\mathbb{C}^\infty).$$

where $\mathcal{P}(\mathbb{R}_{\geq 0})$ are polynomial functions.

Definition 1.12. We denote by $\mathcal{M}\mathcal{S}^{\mathrm{adm}}$ the complex linear space generated by the set $\cup_{\alpha'(0) \neq 0} \mathcal{M}\mathcal{S}^\alpha(\mathbb{C}^\infty)$ and the linear space $\mathcal{M}(\mathbb{C}^\infty) \cdot \mathcal{P}(\mathbb{R}_{\geq 0})$, which we call the **space of admissible meromorphic germs of symbols**.

As in the case of $\mathcal{M}$, for $\sigma \in \mathcal{M}\mathcal{S}(\mathbb{C}^\infty)$, we can define the **dependence space** $\mathrm{Dep}(\sigma)$ of σ, and then define a locality relation

$$\sigma_1 \perp^Q \sigma_2 \iff \mathrm{Dep}(\sigma_1) \perp^Q \mathrm{Dep}(\sigma_2),$$

on $\mathcal{M}\mathcal{S}(\mathbb{C}^\infty)$, which by linearity induces one on $\mathcal{M}\mathcal{S}^{\mathrm{adm}}$, also denoted by $\perp^Q$.

Proposition 1.13. [2, Proposition 4.15] *The triple $\left(\mathcal{M}\mathcal{S}^{\mathrm{adm}}, \perp^Q, m_{\mathcal{M}\mathcal{S}^{\mathrm{adm}}}\right)$ is a commutative and unital locality algebra, with unit given by the constant function 1 and $m_{\mathcal{M}\mathcal{S}^{\mathrm{adm}}}$ is the restriction of the pointwise function multiplication to the graph $\perp^Q \subset \mathcal{M}\mathcal{S}^{\mathrm{adm}} \times \mathcal{M}\mathcal{S}^{\mathrm{adm}}$.*

2. Locality morphisms

We now consider morphisms between locality sets and algebras.

2.1. Basic notions and examples

Definition 2.1. A **locality map** from a locality set $(X, \top_X)$ to a locality set $(Y, \top_Y)$ is a map $\phi : X \to Y$ such that $(\phi \times \phi)(\top_X) \subseteq \top_Y$. More generally, maps $\phi, \psi : (X, \top_X) \to (Y, \top_Y)$ are called **independent** and denoted $\phi \top \psi$ if $(\phi \times \psi)(\top_X) \subseteq \top_Y$.

So a locality map is a map independent of itself.

Definition 2.2. Let $(U, \top_U)$ and $(V, \top_V)$ be locality vector spaces. A linear map $\phi : (U, \top_U) \to (V, \top_V)$ is called a **locality linear map** if it is a locality map.

Definition 2.3. A locality linear map $f : (A, \top_A, \cdot_A) \to (B, \top_B, \cdot_B)$ between two (not necessarily unital) locality algebras is called a **locality algebra homomorphism** if

$$f(u \cdot_A v) = f(u) \cdot_B f(v) \quad \text{for all } (u, v) \in \top_A. \tag{15}$$

By the definition, the composition of locality morphisms is again a locality morphism, so we have the category **LA** of locality algebras over K.

2.2. Locality morphisms on the algebra of meromorphic germs of symbols

Here are fundamental examples of locality morphisms on $\mathcal{M}$. The first one plays a central role in our multivariate minimal subtraction renormalisation scheme.

As a onsequence of the fact that $\mathcal{M}_-^Q$ is a locality ideal of $\mathcal{M}$, we have

Proposition 2.4. [3, Proposition 3.19] **(The Q-orthogonal projection onto holomorphic germs).** *The projection $\pi_+^Q : (\mathcal{M}, \perp^Q) \longrightarrow (\mathcal{M}_+, \perp^Q)$ is a locality algebra homomorphism.*

Since $\top_D^Q \subset \perp^Q$ and $\mathrm{Supp}(\pi_+^Q f) \subset \mathrm{Supp}(f)$, the projection π_+^Q is also a locality algebra homomorphism on $\left(\mathcal{M}, \top_D^Q \right)$.

Remark **2.5.** We view the fact of going from the locality relation $\perp^Q$ to the locality relation $\top_D^Q$ with a smaller graph $\top_D^Q \subset \perp^Q$, as a reduction of the locality relation, which rigidifies the setup in a manner similar to the fact that the structure group of a principal bundle to a subgroup rigidifies the underlying geometric setup.

On the locality algebra $(\mathcal{MS}^{\mathrm{adm}}, \perp^Q)$ of meromorphic germs of symbols, we can define several important locality maps:

- the Hadamard finite part at infinity map $\underset{+\infty}{\mathrm{fp}} : \mathcal{MS}^{\mathrm{adm}} \to \mathcal{M}$;

- locality maps: $\mathfrak{S}_\lambda : \mathcal{MS}^{\mathrm{adm}} \to \mathcal{MS}^{\mathrm{adm}}$ with $\lambda = 0, \pm 1$.

These maps are constructed as follows. In spite of the fact that the Hadamard finite part at infinity map is not an algebra homomorphism on $\mathcal{S}(\mathbb{R}_{\geq 0})$, its extension to $\mathcal{MS}^{\mathrm{adm}}$ enjoys the following property.

Proposition 2.6. [2, Proposition 4.17] *The Hadamard finite part at infinity map* $\underset{+\infty}{\mathrm{fp}}$ *extends to a locality algebra homomorphism*

$$\underset{+\infty}{\mathrm{fp}} : (\mathcal{MS}^{\mathrm{adm}} \perp^Q) \to (\mathcal{M}, \perp^Q).$$

For $\lambda = 1$ (resp. $\lambda = -1$) we define

$$\mathfrak{S}_1(\sigma)(n) := \sum_{k=1}^{n} \sigma(k) \quad \left(\text{resp. } \mathfrak{S}_{-1}(\sigma)(n) := \sum_{k=1}^{n-1} \sigma(k)\right),$$

both maps can be interpolated by means of the Euler-MacLaurin formula [8, Eqn. (13.1.1)] to take values in $\mathcal{MS}^{\mathrm{adm}}$; and for $\lambda = 0$ we define

$$\mathfrak{S}_0(\sigma)(x) := \mathfrak{I}(\sigma)(x) := \int_1^x \sigma(y)\, dy.$$

2.3. Locality morphisms on lattice cones

On a strongly convex lattice cone (C, Λ_C) with interior C^o, discrete (resp. continuous) Laplace transforms of characteristic functions lead to exponential sums (resp. integrals) and give rise to meromorphic functions

$$\sum_{\vec{n} \in C^o \cap \Lambda_C} e^{\langle \vec{\epsilon}, \vec{n} \rangle} \quad \left(\text{resp. } \int_C e^{\langle \vec{\epsilon}, \vec{x} \rangle}\, d\vec{x}_{\Lambda_C}\right).$$

These can be extended by linearity and subdivisions to any convex lattice cone, to build maps S^o (resp. I) from $\mathbb{Q}\mathcal{C}$ to $\mathcal{M}$.

The idempotency $(C, \Lambda_C) \cdot (C, \Lambda_C) = (C, \Lambda_C)$ for any lattice cone (C, Λ_C) implies that S^o and I are not algebra homomorphisms for the Minkowski sum, since otherwise they can only assume values t with $t^2 = t$, meaning $t = 0$ or 1. But in the locality setting, we have

Proposition 2.7. ([6, Proposition 3.7]) *The maps S^o and I are locality algebra homomorphisms from $\left(\mathbb{Q}\mathcal{C}, \perp^Q\right)$ to $\left(\mathcal{M}, \perp^Q\right)$.*

Similarly, S^o and I are locality morphisms from $\left(\mathbb{Q}\mathcal{C}, \top_D^Q\right)$ to $(\mathcal{M}(\mathbb{C}^\infty), \top_D)$, a useful property of these maps which shows the importance of locality algebra.

2.4. Linear operators lifted to the algebra of rooted forests.

Let us briefly recall some definitions and results borrowed from [1].

Definition 2.8. Let $(\Omega, \top)$ be a locality set. A **locality $(\Omega, \top)$-operated set** or simply a **locality operated set** is a locality set $(X, \top_X)$ together with a

partial action β of Ω on X: there is a subset $\top_{\Omega,X} := \Omega \times_{\top} X \subseteq \Omega \times X$ and a map

$$\beta : \Omega \times_{\top} X \longrightarrow X, \ (\omega, x) \mapsto \beta^{\omega}(x)$$

satisfying the following compatibility conditions

(1) For

$$\Omega \times_{\top} X \times_{\top} X := \{(\omega, u, u') \in \Omega \times X \times X \mid (u, u') \in \top_X, (\omega, u), (\omega, u') \in \Omega \times_{\top} X\},$$

we have

$$\beta \times \mathrm{Id}_X : \Omega \times_{\top} X \times_{\top} X \longrightarrow X \times_{\top} X.$$

In other words, If $(\omega, u, , u')$ is in $\Omega \times_{\top} X \times_{\top} X$, then $(\beta^{\omega}(u), u')$ is in $\top_X$.

(2) For

$$\Omega \times_{\top} \Omega \times_{\top} X := \{(\omega, \omega', u) \in \Omega \times \Omega \times X \mid (\omega, \omega') \in T_{\Omega}, (\omega, u), (\omega', u) \in \Omega \times_{\top} X\},$$

we have

$$\mathrm{Id}_{\Omega} \times \beta : \Omega \times_{\top} \Omega \times_{\top} X \longrightarrow \Omega \times_{\top} X,$$

that is, if $(\omega, \omega') \in \top_{\Omega}, (\omega, u), (\omega', u) \in \Omega \times_{\top} X$, then $(\omega', \beta^{\omega}(u)) \in \Omega \times_{\top} X$.

Definition 2.9. Let $(\Omega, \top)$ be a locality set.

(1) A **locality $(\Omega, \top)$-operated semigroup** is a quadruple $(U, \top_U, \beta, m_U)$, where $(U, \top_U, m_U)$ is a locality semigroup and $(U, \top_U, \beta)$ is a $(\Omega, \top)$-operated locality set such that if (ω, u, u') is in $\Omega \times_{\top} U \times_{\top} U$, then (ω, uu') is in $\Omega \times_{\top} U$.

(2) A **locality $(\Omega, \top)$-operated monoid** is a quintuple $(U, \top_U, \beta, m_U, 1_U)$, where $(U, \top_U, m_U, 1_U)$ is an locality monoid and $(U, \top_U, \beta, m_U)$ is a $(\Omega, \top)$-operated locality semigroup such that $\Omega \times 1_U \subset \Omega \times_{\top} U$.

(3) A $(\Omega, \top)$**-operated locality nonunital algebra** (resp. $(\Omega, \top)$**-operated locality unital algebra**) is a quadruple $(U, \top_U, \beta, m_U)$ (resp. quintuple $(U, \top_U, \beta, m_U, 1_U))$ which is at the same time a locality algebra (resp. unital algebra) and a locality $(\Omega, \top)$-operated semigroup (resp. monoid), satisfying the additional condition that for any $\omega \in \Omega$, the set $\{\omega\}^{\top_{\Omega,U}} := \{u \in U \mid \omega \top_{\Omega,U} u\}$ is a subspace of U on which the action of ω is linear. More precisely, the last condition means

> let $u_1, u_2 \in U$. If $u_1, u_2 \in \{\omega\}^{\top_{\Omega,U}}$ then for all $k_1, k_2 \in K$, we have $k_1 u_1 + k_2 u_2 \in \{\omega\}^{\top_{\Omega,U}}$ and $\beta^{\omega}(k_1 u_1 + k_2 u_2) = k_1 \beta^{\omega}(u_1) + k_2 \beta^{\omega}(u_2)$.

(resp. this condition and $\Omega \times 1_U \subset \Omega \times_\top U$).

Definition 2.10. Given $(\Omega, \top_\Omega)$-operated locality structures (sets, semigroups, monoids, nonunital algebras, algebras) $(U_i, \top_{U_i}, \beta_i)$, $i = 1, 2$, a **morphism of locality operated locality structures** is a locality map $f : U_1 \to U_2$ such that $f \circ \beta_1^\omega = \beta_2^\omega \circ f$ for all $\omega \in \Omega$.

The key property of $K\mathcal{F}_{\Omega, \top_\Omega}$ is the following universal property.

Proposition 2.11. $K\mathcal{F}_{\Omega, \top_\Omega}$ *is a commutative* $(\Omega, \top_\Omega)$-*operated algebra, and it is the initial object in the category of commutative* $(\Omega, \top_\Omega)$-*operated algebra.*

Let $(\Omega, \top_\Omega)$ be a locality algebra. By the universal property of the initial object, a linear map $\phi : \Omega \longrightarrow \Omega$ such that $\phi \top Id_\Omega$ induces a $(\Omega, \top_\Omega)$ locality operation on itself, and ϕ lifts uniquely to a locality morphism of $(\Omega, \top_\Omega)$-operated locality algebra for this action [2, Corollary 1.24]

$$\widehat{\phi} : \mathbb{R}\mathcal{F}_{\Omega, \top_\Omega} \longrightarrow \Omega. \tag{16}$$

This can be applied to the space $(\mathcal{MS}^{\mathrm{adm}}, \perp^Q)$ of multivariate meromorphic germs of symbols on $\mathbb{R}_{\geq 0}$. The interpolated summation maps $\mathfrak{S}_\lambda$ on Ω (with $\lambda = \pm 1$) give rise to what we call branched[1] maps

$$\widehat{\mathfrak{S}_\lambda} : (\mathbb{R}\mathcal{F}_{\Omega, \perp^Q}, \top_{\mathcal{F}_{\Omega, \perp^Q}}) \to (\mathcal{MS}^{\mathrm{adm}}, \perp^Q). \tag{17}$$

Proposition 2.12. *The branched map* $\widehat{\mathfrak{S}_\lambda}$ *is a locality algebra homomorphism.*

2.5. Operations lifted to the algebra of rooted forests

An operation $\beta : \Omega \times U \longrightarrow U$ of a locality set $(\Omega, \top_\Omega)$ on a locality monoid $(U, \top_U)$ induces a locality algebra morphism [1, Proposition 2.6]

$$\Phi_\beta : (\mathbb{R}\mathcal{F}_{\Omega, \top_\Omega}, \top_{\mathcal{F}_{\Omega, \top_\Omega}}) \longrightarrow (U, \top_U).$$

On the filtered Euclidean space $(\mathbb{R}^\infty, Q)$ we have a direct system

$$j_{n+1}^* : (\mathbb{R}^n)^* \to (\mathbb{R}^{n+1})^*.$$

Let

$$\mathcal{L} = \varinjlim_n (\mathbb{R}^n)^*$$

be the direct limit of spaces of linear forms. To an element $L \in \mathcal{L}$, regarded as a linear function on $\mathbb{R}^\infty \otimes \mathbb{C}$, one assigns a homogeneous pseudodifferential

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[1]also called "arborified" in the literature [11].

symbol $x \longmapsto x^L$ on $(0, +\infty)$ of order L, i.e. for any $z \in \mathbb{R}^\infty \otimes \mathbb{C}$, it defines a smooth function on $(0, +\infty)$ which is homogeneous in x of degree $L(z)$. Let

$$\Omega := \mathcal{L}; \quad U := \mathcal{M}[\mathcal{L}],$$

where $\mathcal{M}[\mathcal{L}]$ is the group ring over $\mathcal{M}$ generated by the additive monoid $\mathcal{L}$, equipped with the locality relation $\perp^Q$ induced by that on $\mathcal{M}$ and on $\mathcal{L}$:

$$\left(\sum_i f_i x^{L_i} \perp^Q \sum_j g_j x^{\ell_j} \right) \iff \left(\{f_i, L_i\}_i \perp^Q \{g_j, \ell_j\}_j \right),$$

where the sums are taken over finite sets. The map

$$\mathcal{I} : (L, f) \longmapsto \left(x \longmapsto \int_0^\infty \frac{f(y)\, y^{-L}}{y + x}\, dy \right),$$

defines an operation

$$\mathcal{I} : \mathcal{L} \times \mathcal{M}[\mathcal{L}] \longrightarrow \mathcal{M}[\mathcal{L}], \quad (L, f) \longmapsto \mathcal{I}(L, f) \tag{18}$$

and can therefore be lifted to a map [1, Eqs. (33)-(35)]

$$\mathcal{R} : (\mathbb{R}\mathcal{F}_{\mathcal{L}, \perp^Q}, \top_{\mathcal{F}_{\mathcal{L}, \perp^Q}}) \longrightarrow (\mathcal{M}[\mathcal{L}], \perp^Q).$$

Composing the resulting map $\mathcal{R}$ with the evaluation of the maps at $x = 1$, gives rise to a $\mathcal{M}$-valued locality morphism

$$\mathcal{R}_1 = ev_{x=1} \circ \mathcal{R} : (\mathbb{R}\mathcal{F}_{\mathcal{L}, \perp^Q}, \top_{\mathcal{F}_{\mathcal{L}, \perp^Q}}) \longrightarrow (\mathcal{M}, \perp^Q).$$

3. Renormalisation by locality morphisms

In this section we describe a general renormalisation scheme via multivariate regularisations, and implement it to renormalise formal sums on lattice cones, branched formal sums and branched formal integrals.

3.1. A renormalisation scheme

If a regularised theory is realised by a locality algebra morphism

$$\phi^{\mathrm{reg}} : (\mathcal{A}, \top) \longrightarrow \left(\mathcal{M}, \perp^Q \right), \tag{19}$$

then by Proposition 1.4, the projection

$$\pi_+^Q : \mathcal{M} \to \mathcal{M}_+$$

is a locality homomorphism. Therefore we have a locality algebra homomorphism

$$ev_0 \circ \pi_+^Q \circ \phi^{\mathrm{reg}} : (\mathcal{A}, \top) \quad \longrightarrow \quad \mathbb{C},$$

where ev_0 is the evaluation at 0 and the locality relation on $\mathbb{C}$ is $\mathbb{C} \times \mathbb{C}$. This locality algebra homomorphism $\mathrm{ev}_0 \circ \pi_+^Q \circ \phi^{\mathrm{reg}}$ is taken as a renormalisation of ϕ^{reg}. This gives a renormalisation scheme in this setting. When $(\mathcal{A}, \top)$ is equipped with a suitable locality Hopf algebra structure, this renormalisation agrees with the one which arises from the locality variant of algebraic Birkhoff factorisation [3, Theorem 5.9].

3.2. Renormalised conical zeta values

For a lattice cone (C, Λ) in $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$, were they well-defined, the formal sums

$$\sum_{n \in C^o \cap \Lambda_C} 1 \quad \text{and} \quad \sum_{n \in C \cap \Lambda_C} 1,$$

would yield values characteristic of the lattice cone, but they are unfortunately divergent. In order to extract information from these divergent expressions, a univariate regularisation is shown to be less appropriate (see [6]) than multivariate regularisations, which appear as very natural:

$$S^o(C, \Lambda_C) := \sum_{n \in C^o \cap \Lambda_C} e^{\langle n, z \rangle} \quad \text{and} \quad S^c(C, \Lambda_C) := \sum_{n \in C \cap \Lambda_C} e^{\langle n, z \rangle}.$$

By subdivision techniques, we can extend S^o and S^c to linear maps from $\mathbb{Q}\mathcal{C}$ to $\mathcal{M}$, which are locality algebra homomorphisms as discussed in Section 2.3. These are regularised maps for the formal expressions.

Therefore we have renormalised open conical zeta values for a lattice cone (C, Λ_C)

$$\zeta^o(C, \Lambda_C) := (\mathrm{ev}_0 \circ \pi_+^Q \circ S^o)(C, \Lambda_C)$$

and renormalised closed conical zeta values for a lattice cone (C, Λ_C)

$$\zeta^c(C, \Lambda_C) := (\mathrm{ev}_0 \circ \pi_+^Q \circ S^c)(C, \Lambda_C).$$

In fact, the function $(\pi_+^Q \circ S^o)(C, \Lambda_C)$ or $(\pi_+^Q \circ S^c)(C, \Lambda_C)$ contains important geometry information for lattice cones – they are building blocks of Euler-MacLaurin formula for lattice cones. Because of their geometric nature, these formal expression can easily be renormalised by means of locality morphisms.

3.3. Branched zeta values

In [2], this multivariate renormalisation scheme was applied to renormalise a branched generalisation of multiple zeta values. Renormalised multiple zeta values are related to the renormalisation of the formal sum

$$\sum_{n_1 > \cdots > n_k > 0} 1.$$

This formal sum is an iterated sum corresponding to a totally ordered structure, and can therefore be viewed as a sum over ladder trees. It generalises to branched sums on more general partially ordered structures such as rooted trees.

In order to renormalise such branched formal sums, we construct the regularisation maps from the locality morphisms in Section 2.4 and Section 2.2

$$\mathcal{Z}^\lambda = \operatorname*{fp}_{+\infty} \circ \widehat{\mathfrak{S}_\lambda} : (\mathbb{R}\mathcal{F}_{\mathcal{MS}^{\mathrm{adm}}, \perp^Q}, \top_{\mathcal{F}_{\mathcal{MS}^{\mathrm{adm}}, \perp^Q}}) \to (\mathcal{M}, \perp^Q). \qquad (20)$$

Once the regularisation is chosen, a specific choice of meromorphic germs of symbols $x \longmapsto \sigma(s)(x) := \chi(x)\, x^{-s}$ on $\mathbb{R}_{\geq 0}$ [2, Definition 5.1], where χ is an excision function around zero, leads to a generalisation of multiple zeta functions, namely **regularised branched zeta functions**

$$\zeta^{\mathrm{reg},\lambda} : \mathbb{R}\mathcal{F}_{\mathcal{MS}^{\mathrm{adm}}, \perp^Q} \to \mathcal{M}.$$

Due to the locality of the morphisms involved in its construction, $\zeta^{\mathrm{reg},\lambda}$ is a locality morphism of locality algebras. Composing on the left with the renormalised evaluation at zero $\mathrm{ev}_0 \circ \pi_+^Q$ leads to **renormalised branched zeta values**

$$\zeta^{\mathrm{ren},\lambda} : \mathbb{R}\mathcal{F}_{\mathcal{MS}^{\mathrm{adm}}, \perp^Q} \to \mathbb{R}.$$

3.4. Kreimer's toy model

In [1], this multivariate renormalisation scheme was applied to Kreimer's toy model [9] which recursively assigns formal iterated integrals to rooted forests induced by the formal grafting operator:

$$\beta_+(f)(x) = \int_0^\infty \frac{f(y)}{y + x} dy,$$

which defines a linear map on $\mathcal{M}[\mathcal{L}]$. There are different ways to regularise these divergent integrals. We adapt the regularisation by universal property of rooted forests studied in Section 2.5:

$$\mathcal{R}_1 : \mathbb{R}\mathcal{F}_{\mathcal{L}, \perp^Q} \to \mathcal{M}.$$

Applying the renormalisation scheme to this locality map, we have the renormalised value of a properly decorated forest (F, d)

$$(\mathrm{ev}_0 \circ \pi_+^Q \circ \mathcal{R}_1)(F, d).$$

We refer the reader to [1] for further details.

4. Partial versus locality structures

We review and compare various partial product structures with the locality structures introduced in [3]; although the concept of algebraic locality structures is to our knowledge new in the context of renormalisation, partial products have been used in other contexts, hence the need to relate the two concepts, partial and locality products. This section is based on [14].

4.1. Partial semigroups

We start with a generalisation of the notion of a locality set introduced in [3], by dropping the symmetry property of the relation required in [3]:

Definition 4.1.   (1) An $\mathcal{R}$-**set** is a couple $(X, \top)$ with X a set and $\top \subset X \times X$ a binary relation on X. We also write $X \times_\top X$ for $\top$.

   (2) Let $(X, \top)$ be an $\mathcal{R}$-set and $U \subset X$. We write $^\top U$ (resp. $U^\top$) the **left polar set** (resp. **right polar set**) of U; defined by

$$^\top U := \{x \in X \,|\, (x, u) \in \top \quad \text{for all } u \in U\} \tag{21}$$

   (resp.

$$U^\top := \{x \in X \,|\, (u, x) \in \top \quad \text{for all } u \in U\}). \tag{22}$$

If $\top$ is a symmetric binary relation, we call, as in [3], the couple $(X, \top)$ a **locality set**, in which case $^\top U = U^\top$.

Let **RS** (resp. **LS**) denote the category of $\mathcal{R}$-sets (resp. locality sets) whose morphisms are maps $\phi : (X, \top_X) \longrightarrow (Y, \top_Y)$ such that $(\phi \times \phi)(\top_X) \subset \top_Y$, called $\mathcal{R}$-**maps** (resp. **locality maps**).

We equip an $\mathcal{R}$-**set** with four distinct, however related, partial product structures, the first one is a generalisation (dropping the symmetry condition) taken from [14] of the locality relation introduced in [3]:

Definition 4.2. An $\mathcal{R}$-**semigroup** is an $\mathcal{R}$-set $(X, \top)$ together with a partial product map

$$\mu : \top \longrightarrow X, \quad (x, y) \mapsto x\,y$$

which we denote by $(X, \top, \mu)$, such that:

   (1) For any subset $U \subset X$,

$$\mu((^\top U \times {}^\top U) \cap \top) \subseteq {}^\top U, \tag{23}$$

   (2) For any subset $U \subset X$,

$$\mu((U^\top \times U^\top) \cap \top) \subseteq U^\top. \tag{24}$$

(3) For any a, b, c in X such that any couple lies in $\top$ we have $(a\,b)\,c = a\,(b\,c)$.

If $\top$ is a symmetric binary relation, condition (23) coincides with (24) and $(X, \top, \mu)$ is the locality semigroup defined in Section 1.

Let us denote by **RSg** (resp. **LSg**) the category of $\mathcal{R}$- (resp. locality) semigroups whose morphisms are $\mathcal{R}$-maps (resp. locality maps)

$$\phi : (X, \top_X, \mu_X) \longrightarrow (Y, \top_Y, \mu_Y),$$

which are partially multiplicative

$$(a, b) \in \top_X \Longrightarrow \phi(\mu_X(a, b)) = \mu_Y\left(\phi(a), \phi(b)\right).$$

They are called $\mathcal{R}$-**morphisms** (resp. **locality morphisms**).

Remark 4.3. Note that a map between two locality semigroups is a locality morphism if and only if it is an $\mathcal{R}$-morphism.

Remark 4.4. It is easy to check that

- Eq. (23) is equivalent to

$$(x\top z \text{ and } y\top z \text{ and } x\top y) \Longrightarrow (x\,y)\top z \quad \text{for all } (x, y, z) \in X^3, \qquad (25)$$

- Eq. (24) is equivalent to

$$(z\top x \text{ and } z\top y \text{ and } x\top y) \Longrightarrow z\top(x\,y) \quad \text{for all } (x, y, z) \in X^3. \qquad (26)$$

The following definitions are taken from [14].

Definition 4.5. (see [14, Definition 3.1])

(1) A **strong $\mathcal{R}$-semigroup** is an $\mathcal{R}$-set $(X, \top)$ together with a partial product map

$$\mu : \top \longrightarrow X, \quad (x, y) \mapsto x\,y$$

also denoted by $(X, \top, \mu)$, such that for any $x, y, z \in X$:

$$((x, y), (y, z) \in \top) \Longrightarrow ((x\,y, z), (x, y\,z) \in \top \text{ and } (x\,y)\,z = x\,(y\,z)).$$

Let us denote by **SRSg** the category of strong $\mathcal{R}$-semigroups whose morphisms are $\mathcal{R}$-**morphisms**.

(2) A **refined $\mathcal{R}$-semigroup** is an $\mathcal{R}$-set $(X, \top)$ together with a partial product map

$$\mu : \top \mapsto X, \quad (x, y) \mapsto x\,y$$

such that:

- $(x, y) \in \top \Longrightarrow ((y, z) \in \top \Leftrightarrow (x\,y, z) \in \top \quad$ for all $z \in X)$,
- $(y, z) \in \top \Longrightarrow ((x, y) \in \top \Leftrightarrow (x, y\,z) \in \top \quad$ for all $x \in X)$,
- For any $(x, y) \in \top$ and $(y, z) \in \top$ we have $(x\,y)\,z = x\,(y\,z)$.

Let us denote by **RRSg** the category of refined $\mathcal{R}$-semigroups whose morphisms are $\mathcal{R}$-morphisms.

Remark **4.6.** ([14, Proposition 3.3]) Every strong $\mathcal{R}$-semigroup is clearly an $\mathcal{R}$-semigroup, but the converse does not hold. See e.g. [14, Counterexample 3.4] and the subsequent paragraph.

The following definition is taken from [13]. See also [14, Definition 2.20].

Definition 4.7. A **partial semigroup** is an $\mathcal{R}$-set $(X, \top)$ together with a partial product map
$$\mu : \top \longrightarrow X, \quad (x, y) \mapsto x\,y$$
such that for any $x, y, z \in X$
$$((x, y) \in \top \text{ and } (x\,y, z) \in \top) \Leftrightarrow ((y, z) \in \top \text{ and } (x, y\,z) \in \top) \qquad (27)$$
in which case $(x\,y)\,z = x\,(y\,z)$ also holds. Let us denote by **PSg** the category of partial semigroups whose morphisms are $\mathcal{R}$**-morphisms**.

The notion of partial semigroup relates to a particular instance of the **selective category** of Li-Bland and Weinstein introduced in [10, Definition 2.1], whose definition we now recall.

Definition 4.8. A **selective category** is a category $\mathcal{C}$ whose set of morphisms (resp. objects) we denote by Mor (resp. Ob) together with a distinguished class $\mathcal{S} \subset$ Mor of morphisms, called **suave**, and a class $\top_{\mathcal{S}} \subset \mathcal{S} \times \mathcal{S}$ of pairs of suave morphisms called **congenial pairs**, such that:

(1) Any identity morphism is suave so Id_x is suave for any $x \in$ Ob which we write for short Id $\subset \mathcal{S}$;

(2) If $f : X \longrightarrow Y$ is suave, (Id_Y, f) and (f, Id_X) are congenial;

(3) If f is a suave isomorphism, its inverse f^{-1} is suave as well, and the pairs (f, f^{-1}) and (f^{-1}, f) are both congenial;

(4) If f and g are suave and (f, g) is congenial, then $f \circ g$ is suave, i.e., the composition is a map $\circ : \top_{\mathcal{S}} \longrightarrow \mathcal{S}$;

(5) If $f, g, h \in \mathcal{S}$, then
$$((f, g) \in \top_{\mathcal{S}} \text{ and } (f \circ g, h) \in \top_{\mathcal{S}}) \Longleftrightarrow ((g, h) \in \top_{\mathcal{S}} \text{ and } (f, g \circ h) \in \top_{\mathcal{S}}),$$
in which case (f, g, h) is called a **congenial triple**.

A **selective functor** between selective categories is one which takes congenial pairs to congenial pairs.

Recall that a category $\mathcal{C} = (\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$ is **small** if $\mathrm{Obj}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{C})$ are sets and not proper classes.

Proposition 4.9. *A small selective category with one object reduces to a partial semigroup $(S, \top \subset S \times S, m)$ built from a nonempty subset $S \subset M$ of a monoid (M, μ) with unit 1 such that,*

(1) $1 \in S$, $1 \top S$ and $S \top 1$;

(2) $(S, \top)$ is stable under taking inverse (in M) in the following sense: if $s \in S$ is invertible in M, then its inverse s^{-1} is in S and $(s, s^{-1}), (s^{-1}, s)$ are in $\top$.

A selective morphism between selective categories with one object reduces to $\mathcal{R}$-morphisms of partial semigroups that preserve the identity (and hence inverses).

Proof. With exactly one object, a small category $\mathcal{C} = (\mathrm{Obj}(\mathcal{C}), \mathrm{Mor}(\mathcal{C}))$ boils down to a monoid $M := \mathrm{Mor}(\mathcal{C})$, its distinguished class $\mathcal{S}\mathrm{Mor}$ of suave morphisms boils down to a subset $S \subset M$, and the class of congenial pairs of suave elements boils down to a subset $\top \subset S \times S$. Further conditions $(1) - (3)$ of a selective category boil down to the two conditions in the lemma, while conditions $(4) - (5)$ boil down to the condition that $(S, \top)$ is a partial semigroup.

Finally a selective functor $f : (S_1, \top_1) \to (S_2, \top_2)$ between selective categories $(S_i, \top_i)$ with one object boils down to a $\mathcal{R}$-morphism of partial semigroups that preserve the identity. ☑

Remark 4.10. We need the category $\mathcal{C}$ to be small, as even a category with only one object can be large. For example, take $\mathcal{C}$ the category whose only object **Set** is the category of sets, and whose morphisms are the endofunctors of **Set**. In this example, $\mathrm{Mor}(\mathcal{C})$ has no monoid structure as it is not a set.

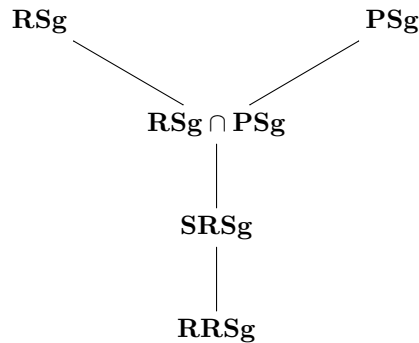## 4.2. Relating various partial structures

We quote from [14] with the reference to the statements. We start with some general comparisons:

- **RRSg** $\subsetneq$ **SRSg** [14, Example 4.3].

- **SRSg** $\subsetneq$ **RSg** [14, Proposition 3.3 and Counterexample 3.4].

- **SRSg** $\subsetneq$ **PSg** [14, Example 3.6].

- **SRSg** $\subsetneq$ **RSg** $\cap$ **PSg** [14, Proposition 3.7].

There are examples of $\mathcal{R}$-semigroups that are not partial semigroups and vice-versa:

- **RSg** $\nsubseteq$ **PSg** [14, Example 3.8].

- **PSg** $\nsubseteq$ **RSg** [14, Example 3.10].

Note that the last two conditions mean that $\mathbf{RSg} \cap \mathbf{PSg} \subsetneq \mathbf{PSg}$ and $\mathbf{RSg} \cap \mathbf{PSg} \subsetneq \mathbf{RSg}$. Thus in summary, we have strict inclusions shown by the following Hasse diagram.

$$
\begin{array}{ccc}
\mathbf{RSg} & & \mathbf{PSg} \\
& \diagdown \quad \diagup & \\
& \mathbf{RSg} \cap \mathbf{PSg} & \\
& | & \\
& \mathbf{SRSg} & \\
& | & \\
& \mathbf{RRSg} &
\end{array}
$$

Here are examples of locality sets with a partial product which fulfills the following equivalence relation:

$$(x \top y \text{ and } (x\, y) \top z) \iff (x \top y \text{ and } y \top z \text{ and } x \top z) \iff (y \top z \text{ and } x \top (y\, z)),$$

namely, conditions (25), (26) (which are equivalent for locality semigroups) are equivalent to (27). So they are both locality and partial semigroups.

Example 4.11. (1) The set $\mathbb{N}$ of natural numbers equipped with the co-prime relation $n \top m \Leftrightarrow n \wedge m = 1$ and the usual product of real numbers is a partial semigroup since

$$a \wedge b = 1 \text{ and } a\, b \wedge c = 1 \iff a \wedge b = 1 \text{ and } a \wedge c = 1 \text{ and } b \wedge c = 1$$
$$\iff c \wedge b = 1 \text{ and } a \wedge b\, c = 1,$$

and a locality semigroup since

$$a \wedge c = 1 \text{ and } b \wedge c = 1 \Longrightarrow a\, b \wedge c = 1.$$

(2) The power set $\mathcal{P}(X)$ of a set X equipped with the disjointness relation $A\top B \Leftrightarrow A\cap B = \varnothing$ and the product law given by the union $\cup$ is a partial semigroup and we have

$$
\begin{aligned}
&A\cap B = \varnothing \text{ and } (A\cup B)\cap C = \varnothing \\
\Longleftrightarrow\ &A\cap B = \varnothing \text{ and } A\cap C = \varnothing \text{ and } B\cap C = \varnothing \\
\Longleftrightarrow\ &B\cap C = \varnothing \text{ and } A\cap (B\cup C) = \varnothing.
\end{aligned}
$$

It is also a locality semigroup since

$$
A\cap C = \varnothing \text{ and } B\cap C = \varnothing \Longrightarrow (A\cup B)\cap C = \varnothing.
$$

4.3. Transitive partial structures

Here is a useful property of partial structures.

Definition 4.12. A locality set $(X, \top)$ is called **transitive** if the relation $\top$ is transitive, namely if for any $a, b, c \in X$

$$
((a, b) \in \top \text{ and } (b, c) \in \top) \Longrightarrow (a, c) \in \top.
$$

A partial structure $(X, \top, \mu)$ such that $(X, \top)$ is transitive is called a **transitive partial structure**. We write **tLSg** (resp. **tSLSg, tRSg, tPSg**) for the category of transitive locality semigroups (resp. transitive strong locality semigroups, transitive refined locality semigroups, transitive partial semigroups).

Remark **4.13.** Transitive partial structures, which are interesting in their own right, are not relevant in the context of locality understood in the sense of quantum field theory, since we do not expect the event A to be independent of the event C under the assumption that the event A is independent of the event B and the event B is independent of the event C. In fact, a transitive locality structure $\top$ is *almost* reflexive, in that for every event a, if there exists b such that $b\top a$, then a is independent of itself.

We saw that locality semigroups and partial semigroups are distinct structures. However, we have the following result:

Proposition 4.14. [14, Proposition 3.9] **tLSg** $\subsetneq$ **tPSg**.

The statement of [14] involves a non-strict inclusion $\subseteq$, yet [14, Example 3.10] gives a transitive partial semigroup which is not a locality semigroup.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF POTSDAM
CAMPUS GOLM, HAUS 9
KARL-LIEBKNECHT-STR. 24-25
D-14476 POTSDAM OT
GOLM, GERMANY
e-mail: `clavier@math.uni-potsdam.de`

DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
NEWARK CAMPUS
101 WARREN STREET
NEWARK, NEW JERSEY 07102
USA
e-mail: `liguo@rutgers.edu`

INSTITUTE OF MATHEMATICS
UNIVERSITY OF POTSDAM, D-14469 POTSDAM, GERMANY
ON LEAVE FROM THE UNIVERSITÉ CLERMONT-AUVERGNE
CLERMONT-FERRAND, FRANCE
e-mail: `paycha@math.uni-potsdam.de`

SCHOOL OF MATHEMATICS
SICHUAN UNIVERSITY
YANGTZE CENTER OF MATHEMATICS
CHENGDU, 610064
CHINA
e-mail: `zhangbin@scu.edu.cn`