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Cyclic derivations, species realizations and potentials

Derivaciones cíclicas, realización por especies y potenciales

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Abstract. In this paper we give an overview of a generalization, introduced by R. Bautista and the author, of the theory of mutation of quivers with potential developed in 2007 by Derksen-Weyman-Zelevinsky. This new construction allows us to consider finite dimensional semisimple F-algebras, where F is any field. We give a brief account of the results concerning this generalization and its main consequences.

Key words and phrases. species realization, mutation, quiver with potential, strongly primitive.

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Resumen. En este artículo daremos un panorama de una generalización, introducida por R. Bautista y el autor, de la teoría de mutación de carcajes con potencial desarrollada en 2007 por Derksen-Weyman-Zelevinsky. Esta nueva construcción nos permite considerar álgebras semisimples de dimensión finita sobre F, donde F es cualquier campo. Daremos un resumen de los resultados de esta generalización y de sus principales consecuencias.

Palabras y frases clave. realización por especies, mutación, carcaj con potencial, fuertemente primitivo.

1. Introduction

Since the development of the theory of quivers with potentials created by Derksen-Weyman-Zelevinsky in [4], the search for a general concept of *mutation of a quiver with potential* has drawn a lot of attention. The theory of quivers with potentials has proven useful in many subjects of mathematics such as cluster algebras, Teichmüller theory, KP solitons, mirror symmetry, Poisson geometry, among many others. There have been different generalizations of the notion of a quiver with potential and mutation where the underlying F-algebra, F being a field, is replaced by more general algebras, see [3, 6, 7]. This paper is organized as follows. In Section 2, we review the preliminaries taken from [1] and [2]. Instead of working with an usual quiver, we consider the completion of the tensor algebra of M over S, where M is an S-bimodule and S is a finite dimensional semisimple F-algebra. We will then see how to construct a cyclic derivation, in the sense of Rota-Sagan-Stein [9], on the completion of the tensor algebra of M. Then we introduce a natural generalization of the concepts of potential, right-equivalence and cyclical equivalence as defined in [4]. In Section 3, we describe a generalization of the so-called *Splitting theorem* ([4, Theorem 4.6]) and see how this theorem allows us to lift the notion of mutation of a quiver with potential to this more general setting. Finally, in Section 4, we recall the notion of species realizations and describe how the generalization given in [2] allows us to give a partial affirmative answer to a question raised by J. Geuenich and D. Labardini-Fragoso in [5].

2. Preliminaries

The following material is taken from [1] and [2].

Definition 2.1. Let F be a field and let $D_1, \ldots, D_n$ be division rings, each containing F in its center and of finite dimension over F. Let $S = \prod_{i=1}^{n} D_i$ and let M be an S-bimodule of finite dimension over F. Define the algebra of formal power series over M as the set

$$\mathcal{F}_S(M) = \left\{ \sum_{i=0}^{\infty} a(i) : a(i) \in M^{\otimes i} \right\}$$

where $M^0 = S$. Note that $\mathcal{F}_S(M)$ is an associative unital F-algebra where the product is the one obtained by extending the product of the tensor algebra $T_S(M) = \bigoplus_{i=0}^{\infty} M^{\otimes i}$.

Let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of S.

Definition 2.2. An element $m \in M$ is legible if $m = e_i m e_j$ for some idempotents e_i, e_j of S.

Definition 2.3. Let $Z = \sum_{i=1}^{n} F e_i \subseteq S$. We say that M is Z-freely generated by a Z-subbimodule M_0 of M if the map $\mu_M : S \otimes_Z M_0 \otimes_Z S \to M$ given by $\mu_M(s_1 \otimes m \otimes s_2) = s_1 m s_2$ is an isomorphism of S-bimodules. In this case we say that M is an S-bimodule which is Z-freely generated.

Throughout this paper we will assume that M is Z-freely generated by M_0.

Definition 2.4. Let A be an associative unital F-algebra. A cyclic derivation, in the sense of Rota-Sagan-Stein [9], is an F-linear function $\mathfrak{h} : A \to \mathrm{End}_F(A)$ such that

$$\mathfrak{h}(f_1 f_2)(f) = \mathfrak{h}(f_1)(f_2 f) + \mathfrak{h}(f_2)(f f_1) \tag{1}$$

for all $f, f_1, f_2 \in A$. Given a cyclic derivation $\mathfrak{h}$, we define the associated cyclic derivative $\delta : A \to A$ as $\delta(f) = \mathfrak{h}(f)(1)$.

We now construct a cyclic derivative on $\mathcal{F}_S(M)$. First, we define a cyclic derivation on the tensor algebra $A = T_S(M)$ as follows. Consider the map

$$\hat{u} : A \times A \to A$$

given by $\hat{u}(f, g) = \displaystyle\sum_{i=1}^{n} e_i g f e_i$ for every $f, g \in A$. This is an F-bilinear map which is Z-balanced. By the universal property of the tensor product, there exists a linear map $u : A \otimes_Z A \to A$ such that $u(a \otimes b) = \hat{u}(a, b)$. Now we define an F-derivation $\Delta : A \to A \otimes_Z A$ as follows. For $s \in S$, we define $\Delta(s) = 1 \otimes s - s \otimes 1$; and for $m \in M_0$, we set $\Delta(m) = 1 \otimes m$. Then we define $\Delta : M \to T_S(M)$ by

$$\Delta(s_1 m s_2) = \Delta(s_1) m s_2 + s_1 \Delta(m) s_2 + s_1 m \Delta(s_2)$$

for $s_1, s_2 \in S$ and $m \in M_0$. Note that the above map is well-defined since $M \cong S \otimes_Z M_0 \otimes_Z S$ via the multiplication map μ_M. Once we have defined Δ on M, we can extend it to an F-derivation on A. Now we define $\mathfrak{h} : A \to \mathrm{End}_F(A)$ as follows

$$\mathfrak{h}(f)(g) = u(\Delta(f) g)$$

We have

$$\begin{aligned}
\mathfrak{h}(f_1 f_2)(f) &= u(\Delta(f_1 f_2) f) \\
&= u(\Delta(f_1) f_2 f) + u(f_1 \Delta(f_2) f) \\
&= u(\Delta(f_1) f_2 f) + u(\Delta(f_2) f f_1) \\
&= \mathfrak{h}(f_1)(f_2 f) + \mathfrak{h}(f_2)(f f_1).
\end{aligned}$$

It follows that $\mathfrak{h}$ is a cyclic derivation on $T_S(M)$. We now extend $\mathfrak{h}$ to $\mathcal{F}_S(M)$ as follows. Let $f, g \in \mathcal{F}_S(M)$, then $\mathfrak{h}(f(i))(g(j)) \in M^{\otimes(i+j)}$; thus we define $\mathfrak{h}(f)(g)(l) = \displaystyle\sum_{i+j=l} \mathfrak{h}(f(i))(g(j))$ for every *non*-negative integer l.

In [1, Proposition 2.6], it is shown that the F-linear map $\mathfrak{h} : \mathcal{F}_S(M) \to \mathrm{End}_F(\mathcal{F}_S(M))$ is a cyclic derivation. Using this fact we obtain a cyclic derivative δ on $\mathcal{F}_S(M)$ given by

$$\delta(f) = \mathfrak{h}(f)(1).$$

Definition 2.5. Let $\mathcal{C}$ be a subset of M. We say that $\mathcal{C}$ is a right S-local basis of M if every element of C is legible and if for each pair of idempotents e_i, e_j of S, we have that $C \cap e_i M e_j$ is a D_j-basis for $e_i M e_j$.

We note that a right S-local basis $\mathcal{C}$ induces a dual basis $\{u, u^*\}_{u\in\mathcal{C}}$, where $u^* : M_S \to S_S$ is the morphism of right S-modules defined by $u^*(v) = 0$ if $v \in \mathcal{C} \setminus \{u\}$; and $u^*(u) = e_j$ if $u = e_i u e_j$.

Let T be a Z-local basis of M_0 and let L be a Z-local basis of S. The former means that for each pair of distinct idempotents e_i,e_j of S, $T \cap e_i M e_j$ is an F-basis of $e_i M_0 e_j$; the latter means that $L(i) = L \cap e_i S$ is an F-basis of the division algebra $e_i S = D_i$. It follows that the non-zero elements of the set $\{sa : s \in L, a \in T\}$ form a right S-local basis of M. Therefore, for every $s \in L$ and $a \in T$, we have the map $(sa)^* \in \mathrm{Hom}_S(M_S, S_S)$ induced by the dual basis.

Definition 2.6. Let $\mathcal{D}$ be a subset of M. We say that $\mathcal{D}$ is a left S-local basis of M if every element of $\mathcal{D}$ is legible and if for each pair of idempotents e_i, e_j of S, we have that $\mathcal{D} \cap e_i M e_j$ is a D_i-basis for $e_i M e_j$.

Let ψ be any element of $\mathrm{Hom}_S(M_S, S_S)$. We will extend ψ to an F-linear endomorphism of $\mathcal{F}_S(M)$, which we will denote by ψ_*.

First, we define $\psi_*(s) = 0$ for $s \in S$; and for $M^{\otimes l}$, where $l \geq 1$, we define $\psi_*(m_1 \otimes \cdots \otimes m_l) = \psi(m_1) m_2 \otimes \cdots \otimes m_l \in M^{\otimes (l-1)}$ for $m_1, \ldots, m_l \in M$. Finally, for $f \in \mathcal{F}_S(M)$ we define $\psi_*(f) \in \mathcal{F}_S(M)$ by setting $\psi_*(f)(l-1) = \psi_*(f(l))$ for each integer $l > 1$. Then we define

$$\psi_*(f) = \sum_{l=0}^{\infty} \psi_*(f(l)).$$

Definition 2.7. Let $\psi \in M^* = \mathrm{Hom}_S(M_S, S_S)$ and $f \in \mathcal{F}_S(M)$. We define $\delta_\psi : \mathcal{F}_S(M) \to \mathcal{F}_S(M)$ as

$$\delta_\psi(f) = \psi_*(\delta(f)) = \sum_{l=0}^{\infty} \psi_*(\delta(f(l))).$$

Definition 2.8. Given an S-bimodule N we define the *cyclic part* of N as $N_{cyc} := \sum_{j=1}^{n} e_j N e_j$.

Definition 2.9. A *potential* P is an element of $\mathcal{F}_S(M)_{cyc}$.

Motivated by the *Jacobian ideal* introduced in [4], we define an analogous two-sided ideal of $\mathcal{F}_S(M)$.

For each legible element a of $e_i M e_j$, we let $\sigma(a) = i$ and $\tau(a) = j$.

Definition 2.10. Let P be a potential in $\mathcal{F}_S(M)$, we define a two-sided ideal $R(P)$ as the closure of the two-sided ideal of $\mathcal{F}_S(M)$ generated by all the elements $X_{a^*}(P) = \sum_{s \in L(\sigma(a))} \delta_{(sa)^*}(P)s$, $a \in T$.

In [2, Theorem 5.3], it is shown that $R(P)$ is invariant under algebra isomorphisms that fix pointwise S. Furthermore, one can show that $R(P)$ is independent of the choice of the Z-subbimodule M_0 and also independent of the choice of Z-local bases for S and M_0.

Definition 2.11. An algebra with potential is a pair $(\mathcal{F}_S(M), P)$ where P is a potential in $\mathcal{F}_S(M)$ and $M_{cyc} = 0$.

We denote by $[\mathcal{F}_S(M), \mathcal{F}_S(M)]$ the closure in $\mathcal{F}_S(M)$ of the F-subspace generated by all the elements of the form $[f, g] = fg - gf$ with $f, g \in \mathcal{F}_S(M)$.

Definition 2.12. Two potentials P and P' are called cyclically equivalent if $P - P' \in [\mathcal{F}_S(M), \mathcal{F}_S(M)]$.

Definition 2.13. We say that two algebras with potential $(\mathcal{F}_S(M), P)$ and $(\mathcal{F}_S(M'), Q)$ are right-equivalent if there exists an algebra isomorphism $\varphi : \mathcal{F}_S(M) \to \mathcal{F}_S(M')$, with $\varphi|_S = id_S$, such that $\varphi(P)$ is cyclically equivalent to Q.

The following construction follows the one given in [4, p.20]. Let k be an integer in $[1, n]$ and $\bar{e}_k = 1 - e_k$. Using the S-bimodule M, we define a new S-bimodule $\mu_k M = \widetilde{M}$ as:

$$\widetilde{M} := \bar{e}_k M \bar{e}_k \oplus M e_k M \oplus (e_k M)^* \oplus^* (M e_k)$$

where $(e_k M)^* = \mathrm{Hom}_S((e_k M)_S, S_S)$, and ${}^*(M e_k) = \mathrm{Hom}_S({}_S(M e_k), {}_S S)$. One can show (see [2, Lemma 8.7]) that $\mu_k M$ is Z-freely generated.

Definition 2.14. Let P be a potential in $\mathcal{F}_S(M)$ such that $e_k P e_k = 0$. Following [4], we define

$$\mu_k P := [P] + \sum_{sa \in {}_k\hat{T}, bt \in \tilde{T}_k} [btsa]((sa)^*)({}^*(bt))$$

where

$$\begin{aligned} {}_k\hat{T} &= \{sa : s \in L(k), a \in T \cap e_k M\} \\ \tilde{T}_k &= \{bt : b \in T \cap M e_k, t \in L(k)\}. \end{aligned}$$

3. Mutations and potentials

Let $P = \sum_{i=1}^{N} a_i b_i + P'$ be a potential in $\mathcal{F}_S(M)$ where $A = \{a_1, b_1, \ldots, a_N, b_N\}$ is contained in a Z-local basis T of M_0 and $P' \in \mathcal{F}_S(M)^{\geq 3}$. Let L_1 denote the complement of A in T, N_1 be the F-vector subspace of M generated by A and N_2 be the F-vector subspace of M generated by L_1; then $M = M_1 \oplus M_2$ as S-bimodules where $M_1 = SN_1S$ and $M_2 = SN_SS$.

One of the main results proved in [4] is the so-called *Splitting theorem* (Theorem 4.6). Inspired by this result, the following theorem is proved in [2].

Theorem 3.1. ([2, Theorem 7.15]) *There exists an algebra automorphism $\varphi : \mathcal{F}_S(M) \to \mathcal{F}_S(M)$ such that $\varphi(P)$ is cyclically equivalent to a potential of the form $\sum_{i=1}^{N} a_i b_i + P''$ where P'' is a reduced potential contained in the closure of the algebra generated by M_2 and $\sum_{i=1}^{N} a_i b_i$ is a trivial potential in $\mathcal{F}_S(M_1)$.*

Definition 3.2. Let $P \in \mathcal{F}(M)$ be a potential and k an integer in $\{1, \ldots, n\}$. Suppose that there are no two-cycles passing through k. Using Theorem 3.1, one can see that $\mu_k P$ is right-equivalent to the direct sum of a trivial potential W and a reduced potential Q. Following [4], we define the mutation of P in the direction k, as $\overline{\mu}_k(P) = Q$.

One of the main results of [4] is that mutation at an arbitrary vertex is a well-defined involution on the set of right-equivalence classes of reduced quivers with potentials. In [2], the following analogous result is proved.

Theorem 3.3. ([2, Theorem 8.21]) *Let P be a reduced potential such that the mutation $\overline{\mu}_k P$ is defined. Then $\overline{\mu}_k \overline{\mu}_k P$ is defined and it is right-equivalent to P.*

Definition 3.4. Let $k_1, \ldots, k_l$ be a finite sequence of elements of $\{1, \ldots, n\}$ such that $k_p \neq k_{p+1}$ for $p = 1, \ldots, l-1$. We say that an algebra with potential $(\mathcal{F}_S(M), P)$ is $(k_l, \ldots, k_1)$-nondegenerate if all the iterated mutations $\bar{\mu}_{k_1} P$, $\bar{\mu}_{k_2} \bar{\mu}_{k_1} P, \ldots, \bar{\mu}_{k_l} \cdots \bar{\mu}_{k_1} P$ are 2-acyclic. We say that $(\mathcal{F}_S(M), P)$ is nondegenerate if it is $(k_l, \ldots, k_1)$-nondegenerate for every sequence of integers as above.

In [2, p.29], we impose the following condition on each of the bases $L(i)$. For each $s, t \in L(i)$:

$$e_i^*(st^{-1}) \neq 0 \text{ implies } s = t \text{ and } e_i^*(s^{-1}t) \neq 0 \text{ implies } s = t \tag{2}$$

where $e_i^* : D_i \to F$ denotes the standard dual map corresponding to the basis element $e_i \in L(i)$.

Throughout the rest of the paper we will assume that each of the bases $L(i)$ satisfy (2).

4. Species realizations

We begin this Section by recalling the definition of *species realization* of a skew-symmetrizable integer matrix, in the sense of [5] (Definition 2.22).

Definition 4.1. Let $B = (b_{ij}) \in \mathbb{Z}^{n\times n}$ be a skew-symmetrizable matrix, and let $I = \{1, \dots, n\}$. A species realization of B is a pair $(\mathbf{S}, \mathbf{M})$ such that:

(1) $\mathbf{S} = (F_i)_{i\in I}$ is a tuple of division rings;

(2) $\mathbf{M}$ is a tuple consisting of an $F_i - F_j$ bimodule M_{ij} for each pair $(i, j) \in I^2$ such that $b_{ij} > 0$;

(3) for every pair $(i, j) \in I^2$ such that $b_{ij} > 0$, there are $F_j - F_i$-bimodule isomorphisms
$$\mathrm{Hom}_{F_i}(M_{ij}, F_i) \cong \mathrm{Hom}_{F_j}(M_{ij}, F_j);$$

(4) for every pair $(i, j) \in I^2$ such that $b_{ij} > 0$ we have $\dim_{F_i}(M_{ij}) = b_{ij}$ and $\dim_{F_j}(M_{ij}) = -b_{ji}$.

In [5, p.14], motivated by the seminal paper [4], J. Geuenich and D. Labardini-Fragoso raise the following question:

Question [5, Question 2.23] Can a mutation theory of species with potential be defined so that every skew-symmetrizable matrix B have a species realization which admit a nondegenerate potential?

In [4], it is shown that if the underlying base field F is uncountable then a nondegenerate quiver with potential exists for every underlying quiver.

Motivated by the above question, the following theorem is proved in [1].

Theorem 4.2. ([1, Theorem 3.5]) *Let $B = (b_{ij}) \in \mathbb{Z}^{n\times n}$ be a skew-symmetrizable matrix with skew-symmetrizer $D = \mathrm{diag}(d_1, \dots, d_n)$. Suppose that d_j divides b_{ij} for every j and every i. Then the matrix B can be realized by a species that admits a nondegenerate potential provided the underlying field F is uncountable.*

We now give an example ([1, p.8]) of a class of skew-symmetrizable 4×4 integer matrices, which are not globally unfoldable nor strongly primitive, and that have a species realization admitting a nondegenerate potential. This gives an example of a class of skew-symmetrizable 4×4 integer matrices which are not covered by [6].

Let

$$B = \begin{bmatrix} 0 & -a & 0 & b \\ 1 & 0 & -1 & 0 \\ 0 & a & 0 & -b \\ -1 & 0 & 1 & 0 \end{bmatrix} \tag{3}$$

where a, b are positive integers such that $a < b$, a does not divide b and $\gcd(a, b) \neq 1$.

Note that there are infinitely many such pairs (a, b). For example, let p and q be primes such that $p < q$. For any $n \geq 2$, define $a = p^n$ and $b = p^{n-1}q$. Then $a < b$, a does not divide b and $\gcd(a, b) = p^{n-1} \neq 1$. Note that B is skew-symmetrizable since it admits $D = \mathrm{diag}(1, a, 1, b)$ as a skew-symmetrizer.

***Remark* 4.3.** By [6, Example 14.4] we know that the class of all matrices given by (3) does *not* admit a global unfolding. Moreover, since we are not assuming that a and b are coprime, then such matrices are not strongly primitive; hence they are not covered by [6].

We have the following

Proposition 4.4. ([1, Proposition 5.2]) *The class of all matrices given by* (3) *are not globally unfoldable nor strongly primitive, yet they can be realized by a species admitting a nondegenerate potential.*

By Theorem 4.2, we know that a nondegenerate potential exists provided the underlying field F is uncountable. If F is infinite (but not necessarily uncountable) one can show that $\mathcal{F}_S(M)$ admits "locally" nondegenerate potentials. More precisely, we have

Proposition 4.5. ([2, Proposition 12.5]) *Let $B = (b_{ij}) \in \mathbb{Z}^{n \times n}$ be a skew-symmetrizable matrix with skew-symmetrizer $D = \mathrm{diag}(d_1, \dots, d_n)$. Suppose that d_j divides b_{ij} for every j and every i. If $k_1, \dots, k_l$ is an arbitrary sequence of elements of $\{1, \dots, n\}$ and F is infinite, then there exists a species realization (M, S) of B, and a potential $P \in \mathcal{F}_S(M)$ on this species, such that the mutation $\overline{\mu}_{k_l} \cdots \overline{\mu}_{k_1} P$ exists.*

We conclude the paper by giving an example of a class of skew-symmetrizable 4×4 integer matrices that have a species realization via field extensions of the rational numbers. Although in this case we cannot guarantee the existence of a nondegenerate potential, we can guarantee (by Proposition 4.5) the existence of "locally" nondegenerate potentials.

First, we require some definitions.

Definition 4.6. Let E/F be a finite field extension. An F-basis of E, as a vector space, is said to be semi-multiplicative if the product of any two elements of the basis is an F-multiple of another basis element.

It can be shown that every extension E/F which has a semi-multiplicative basis satisfies (2).

Definition 4.7. A field extension E/F is called a simple radical extension if $E = F(a)$ for some $a \in E$, with $a^n \in F$ for some integer $n \geq 2$.

Note that if E/F is a simple radical extension then E has a semi-multiplicative F-basis.

Definition 4.8. A field extension E/F is a radical extension if there exists a tower of fields $F = F_0 \subseteq F_1 \ldots \subseteq F_l = E$ such that F_i/F_{i-1} is a simple radical extension for $i = 1, \ldots, l$.

As before, let

$$B = \begin{bmatrix} 0 & -a & 0 & b \\ 1 & 0 & -1 & 0 \\ 0 & a & 0 & -b \\ -1 & 0 & 1 & 0 \end{bmatrix} \tag{4}$$

but without imposing additional conditions on a or b.

Proposition 4.9. *Let $n, m \geq 2$. The matrix B admits a species realization $(\mathbf{S}, \mathbf{M})$ where $\mathbf{M}$ is a Z-freely generated S-bimodule, S satisfies (2), and such species admits a locally nondegenerate potential.*

Proof. To prove this we will require the following result (cf. [8, Theorem 14.3.2]).

Lemma 4.10. *Let $n \geq 2$, $p_1, \ldots, p_m$ be distinct primes and let $\mathbf{Q}$ denote the set of all rational numbers. Let ζ_n be a primitive nth-root of unity. Then*

$$[\mathbf{Q}(\zeta_n)(\sqrt[n]{p_1}, \ldots, \sqrt[n]{p_m}) : \mathbf{Q}(\zeta_n)] = n^m$$

Now we continue with the proof of Proposition 4.9. Let $F = \mathbf{Q}(\zeta_n)$ be the base field and let p_1 be an arbitrary prime. By Lemma 4.10, $F_2 = F(\sqrt[n]{p_1})/F$ has degree n. Now choose $m - 1$ distinct primes $p_2, p_3, \ldots, p_m$ and also distinct from p_1. Define $F_4 = F(\sqrt[n]{p_1}, \sqrt[n]{p_2}, \ldots, \sqrt[n]{p_m})/F$, then by Lemma 4.10, F_4 has degree n^m. Let $S = F \oplus F_2 \oplus F \oplus F_4$ and $Z = F \oplus F \oplus F \oplus F$. Since $F/\mathbf{Q}$ is a simple radical extension then it has a semi-multiplicative basis; thus it satisfies (2). On the other hand, note that $F_2/\mathbf{Q}(\zeta_n)$ and $F_4/\mathbf{Q}(\zeta_n)$ are radical extensions. Using [2, Remark 6, p.29] we get that both F_2 and F_4 satisfy (2); hence, it is always possible to choose a Z-local basis of S satisfying (2). Finally, for each $b_{ij} > 0$, define $e_i M e_j = (F_i \otimes_F F_j)^{\frac{b_{ij}}{d_j}} = F_i \otimes_F F_j$. It follows that $(\mathbf{S}, \mathbf{M})$ is a species realization of B. ☑

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