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A note on deformations of Gorenstein-projective modules over finite dimensional algebras

Una nota acerca de deformaciones de módulos Gorenstein-proyectivos sobre álgebras de dimensión finita

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Abstract. In this note, we present a survey of results concerning universal deformation rings of finitely generated Gorenstein-projective modules over finite dimensional algebras.

Key words and phrases. Deformations of modules, universal deformation rings, Gorenstein algebras, Gorenstein-projective modules.

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Resumen. En esta nota, nosotros damos una revisión de resultados concernientes a anillos universales de deformación de módulos Gorenstein-proyectivos finitamente generados sobre álgebras de dimensión finita.

Palabras y frases clave. Deformaciones de módulos, anillos universales de deformación, álgebras de Gorenstein, módulos Gorenstein-proyectivos.

1. Introduction

Let $\Bbbk$ be a field of arbitrary characteristic. Let Λ be a finite dimensional $\Bbbk$-algebra and let V be a left Λ-module of finite dimension over $\Bbbk$. F. M. Bleher and the author proved in [8, Prop. 2.1] that V has a well-defined versal deformation ring $R(\Lambda, V)$, which is a complete local commutative Noetherian $\Bbbk$-algebra with residue field $\Bbbk$. Moreover, $R(\Lambda, V)$ is universal provided that $\mathrm{End}_\Lambda(V) = \Bbbk$. The aim of this note is to serve as an introductory article to the deformation theory of finitely generated Gorenstein-projective modules over finite dimensional algebras (in the sense of [8]) as well as a survey of some of the concerning results available in the literature. The main motivation of this work is that the representation theory of finite dimensional algebras provides many sophisticated tools such as stable equivalences and combinatorial description of modules that can be used in order to arrive at a deeper understanding of these universal deformation rings.

This note is organized as follows. In §2, we review the definition of lifts, deformations, and (uni)versal deformation rings in the sense of [8]. We also review the definition of Gorenstein-projective modules in the sense of [17], and review with more detail the known results concerning universal deformation rings of finitely generated Gorenstein-projective modules over finite dimensional algebras. In §3, we review some results concerning universal deformation rings of modules over algebras of dihedral type (as introduced by K. Erdmann in [18]) and over monomial algebras in which there is no an overlap (as introduced by X. W. Chen et al. in [14]). We also introduced a class of finitely generated modules that we call *semi-rigid stable bricks*, and prove that a finitely generated Gorenstein-projective module that is also a semi-rigid stable brick has a universal deformation ring which is isomorphic either to $\Bbbk$ or to $\Bbbk[\![t]\!]/(t^2)$.

We refer the reader to look at [4] and [31] in order to review basic notions from the representation theory of algebras and from the homological algebra used in this note. We also refer the reader to [14] and its references for basic properties of finitely generated Gorenstein-projective modules.

This article is an alternative version of the author's 20-minutes talk at the XXII Coloquio Latinoamericano de Álgebra, which was held in Quito, Ecuador, during August 2017, where the result [5, Thm. 5.2] was presented. The author would like to express his gratitude to the organizers of this event as well as to the coordinators of the session in Representation Theory of Algebras, Professors Raymundo Bautista and Claudia Chaio, for giving the author the opportunity of participating in this event and for inviting him to write this note.

2. Preliminaries

In this section, we assume that $\Bbbk$ is a fixed field of arbitrary characteristic. We denote by $\hat{\mathcal{C}}$ the category of all complete local commutative Noetherian $\Bbbk$-algebras with residue field $\Bbbk$. In particular, the morphisms in $\hat{\mathcal{C}}$ are continuous $\Bbbk$-algebra homomorphisms that induce the identity map on $\Bbbk$. Let Λ be a fixed (but arbitrary) finite dimensional $\Bbbk$-algebra, and let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$. We denote by $R\Lambda$ the tensor product of $\Bbbk$-algebras $R \otimes_{\Bbbk} \Lambda$. We assume that all our modules are finitely generated. We denote by Λ-mod the abelian category of left modules over Λ, and by Λ-$\underline{\mathrm{mod}}$ its stable category, i.e. the objects of Λ-$\underline{\mathrm{mod}}$ are the same as those of Λ-mod, and for all objects V and W in Λ-$\underline{\mathrm{mod}}$, $\mathrm{Hom}_{\Lambda\text{-}\underline{\mathrm{mod}}}(V, W) = \underline{\mathrm{Hom}}_{\Lambda}(V, W)$ is the $\Bbbk$-vector space which is the quotient of $\mathrm{Hom}_{\Lambda}(V, W)$ by $\mathrm{PHom}_{\Lambda}(V, W)$, which is the $\Bbbk$-vector space of Λ-module homomorphisms from V to W that factor through a projective Λ-module. In particular, when $V = W$, we set $\mathrm{End}_{\Lambda}(V) = \mathrm{Hom}_{\Lambda}(V, V)$ (resp. $\underline{\mathrm{End}}_{\Lambda}(V) = \underline{\mathrm{Hom}}_{\Lambda}(V, V)$) and call it the endomorphism (resp. stable endomorphism) ring of V. We also denote by ΩV the first syzygy of V, i.e. ΩV is the kernel of a projective cover $P(V) \to V$ of V over Λ, which is unique up to isomorphism.

Let V be a fixed left Λ-module.

2.1. Liftings and deformations of modules

A *lift* (M, ϕ) of V over R is a finitely generated left $R\Lambda$-module M that is free over R together with an isomorphism of Λ-modules $\phi : \Bbbk \otimes_R M \to V$. Two lifts (M, ϕ) and (M', ϕ') over R are *isomorphic* if there exists an $R\Lambda$-module isomorphism $f : M \to M'$ such that $\phi' \circ (\Bbbk \otimes_R f) = \phi$. If (M, ϕ) is a lift of V over R, we denote by $[M, \phi]$ its isomorphism class and say that $[M, \phi]$ is a *deformation* of V over R. We denote by $\mathrm{Def}_\Lambda(V, R)$ the set of all deformations of V over R. The *deformation functor* corresponding to V is the covariant functor $\hat{\mathrm{F}}_V : \hat{\mathcal{C}} \to \mathrm{Sets}$ defined as follows: for all objects $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, define $\hat{\mathrm{F}}_V(R) = \mathrm{Def}_\Lambda(V, R)$, and for all morphisms $\alpha : R \to R'$ in $\hat{\mathcal{C}}$, let $\hat{\mathrm{F}}_V(\alpha) : \mathrm{Def}_\Lambda(V, R) \to \mathrm{Def}_\Lambda(V, R')$ be defined as $\hat{\mathrm{F}}_V(\alpha)([M, \phi]) = [R' \otimes_{R,\alpha} M, \phi_\alpha]$, where $\phi_\alpha : \Bbbk \otimes_{R'} (R' \otimes_{R,\alpha} M) \to V$ is the composition of Λ-module isomorphisms $\Bbbk \otimes_{R'} (R' \otimes_{R,\alpha} M) \cong \Bbbk \otimes_R M \xrightarrow{\phi} V$. Suppose there exists an object $R(\Lambda, V) \in \mathrm{Ob}(\hat{\mathcal{C}})$ and a deformation $[U(\Lambda, V), \phi_{U(\Lambda,V)}]$ of V over $R(\Lambda, V)$ with the following property. For each $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and for all lifts M of V over R, there exists a morphism $\psi_{R(\Lambda,V)} : R(\Lambda, V) \to R$ in $\hat{\mathcal{C}}$ such that

$$\hat{\mathrm{F}}_V(\psi_{R(\Lambda,V)})[U(\Lambda, V), \phi_{U(\Lambda,V)}] = [M, \phi],$$

and moreover, $\psi_{R(\Lambda,V)}$ is unique if R is the ring of dual numbers $\Bbbk[\epsilon]$ with $\epsilon^2 = 0$. Then $R(\Lambda, V)$ and $[U(\Lambda, V), \phi_{U(\Lambda,V)}]$ are respectively called the *versal deformation ring* and *versal deformation* of V. If the morphism $\psi_{R(\Lambda,V)}$ is unique for all $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and lifts (M, ϕ) of V over R, then $R(\Lambda, V)$ and $[U(\Lambda, V), \phi_{U(\Lambda,V)}]$ are respectively called the *universal deformation ring* and the *universal deformation* of V. In other words, the universal deformation ring $R(\Lambda, V)$ represents the deformation functor $\hat{\mathrm{F}}_V$ in the sense that $\hat{\mathrm{F}}_V$ is naturally isomorphic to the Hom functor $\mathrm{Hom}_{\hat{\mathcal{C}}}(R(\Lambda, V), -)$. Using Schlessinger's criteria [25, Thm. 2.11] and using methods similar to those in [22], it is straightforward to prove that the deformation functor $\hat{\mathrm{F}}_V$ is continuous (see [22, §14] for the definition), that every finitely generated Λ-module V has a versal deformation ring, and that this versal deformation is universal provided that the endomorphism ring of V is isomorphic to $\Bbbk$ (see [8, Prop. 2.1]). Moreover, it follows from [8, Prop. 2.5] that Morita equivalence preserve isomorphism classes of versal deformation rings.

2.2. Gorenstein algebras, Gorenstein-projective modules

Following [17], we say that V is *Gorenstein-projective* provided that there exists an acyclic sequence of projective Λ-modules

$$P^\bullet : \cdots \to P^{-2} \xrightarrow{f^{-2}} P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} P^2 \to \cdots$$

such that $\mathrm{Hom}_\Lambda(P^\bullet, \Lambda)$ is also acyclic and $V = \mathrm{coker}\, f^0$. Note that every projective Λ-module is also Gorenstein-projective, and if V is a Gorenstein-projective Λ-module, then $\mathrm{Ext}^i_\Lambda(V, \Lambda) = 0$ for all $i > 0$. It is important to mention that Gorenstein-projective modules are also known under many names in the literature (see e.g. at the introduction of [14] for more details). Following [2], we say that Λ is a *Gorenstein* $\Bbbk$-algebra provided that Λ has finite injective dimension as a left and right Λ-module. By [32], these dimensions coincide and their common value is called the *virtual Gorenstein dimension* of Λ. Note that if Λ is either self-injective (i.e. the regular Λ-module ${}_\Lambda\Lambda$ is injective) or of finite global dimension, then Λ is Gorenstein. In particular, self-injective $\Bbbk$-algebras have virtual Gorenstein dimension zero. We denote by Λ-Gproj the full subcategory of Λ-mod whose objects are Gorenstein-projective Λ-modules, and by Λ-$\underline{\text{Gproj}}$ the corresponding full subcategory of Λ-$\underline{\text{mod}}$. Note that Λ is self-injective if and only if the categories Λ-mod and Λ-Gproj coincide. Following [13], we say that Λ is *CM-finite* if there are at most a finite number of isomorphism classes of indecomposable Gorenstein projective Λ-modules, and that Λ is *CM-free* if every Gorenstein-projective Λ-module is projective. Note in particular that every algebra of finite global dimension is CM-free. We invite the reader to look at [16, Prop. 3.14] to obtain examples of finite dimensional algebras that are CM-free but not Gorenstein.

2.3. Singular equivalences of Morita type

The following definition was introduced by X. W. Chen and L. G. Sun in [15], which was further studied by G. Zhou and A. Zimmermann in [33], as a way of generalizing the concept of stable equivalence of Morita type introduced by M. Broué in [11].

Definition 2.1. Let Λ and Γ be finite dimensional $\Bbbk$-algebras, and let X be a Γ-Λ-bimodule and Y a Λ-Γ-bimodule. We say that X and Y induce a *singular equivalence of Morita type* between Λ and Γ (and that Λ and Γ are *singularly equivalent of Morita type*) if the following conditions are satisfied:

(i) X is finitely generated and projective as a left Γ-module and as a right Λ-module.

(ii) Y is finitely generated and projective as a left Λ-module and as a right Γ-module.

(iii) There is a finitely generated Γ-Γ-bimodule Q with finite projective dimension such that $X \otimes_\Lambda Y \cong \Gamma \oplus Q$ as Γ-Γ-bimodules.

(iv) There is a finitely generated Λ-Λ-bimodule P with finite projective dimension such that $Y \otimes_\Gamma X \cong \Lambda \oplus P$ as Λ-Λ-bimodules.

The concept of singular equivalence of Morita type was further generalized by Z. Wang in [30], where the concept of *singular equivalence of Morita type with level* is introduced. It was proved by Ø. Skartsæterhagen in [26, Prop. 2.6] that if ${}_\Gamma X_\Lambda$ and ${}_\Lambda Y_\Gamma$ are bimodules that induce a singular equivalence of Morita type, then they induce a singular equivalence of Morita type with level. Therefore, it follows from [26, Lemma. 3.6] that if ${}_\Gamma X_\Lambda$ and ${}_\Lambda Y_\Gamma$ are bimodules which induce a singular equivalence of Morita type between two finite dimensional Gorenstein $\Bbbk$-algebras Λ and Γ as in Definition 2.1, then the functors

$$X \otimes_\Lambda - : \Lambda\text{-mod} \to \Gamma\text{-mod} \qquad \text{and} \qquad Y \otimes_\Gamma - : \Gamma\text{-mod} \to \Lambda\text{-mod}$$

send finitely generated Gorenstein-projective left modules to finitely generated Gorenstein-projective left modules. By [33, Prop. 2.3] and [26, Prop. 3.7] it follows that

$$X \otimes_\Lambda - : \Lambda\text{-}\underline{\mathrm{Gproj}} \to \Gamma\text{-}\underline{\mathrm{Gproj}} \qquad \text{and} \qquad Y \otimes_\Gamma - : \Gamma\text{-}\underline{\mathrm{Gproj}} \to \Lambda\text{-}\underline{\mathrm{Gproj}}$$

are equivalences of triangulated categories that are quasi-inverses of each other.

2.4. Results concerning (uni)versal deformation rings of Gorenstein-projective modules

Recall that Λ is a Frobenius $\Bbbk$-algebra provided that the left Λ-modules ${}_\Lambda\Lambda$ and $\mathrm{Hom}_\Bbbk(\Lambda_\Lambda, \Bbbk)$ are isomorphic. In particular, Frobenius $\Bbbk$-algebras are also self-injective (see e.g. [6, Prop. 1.6.2 (i)]), and basic self-injective algebras are also Frobenius (see e.g. [27, Cor. 4.3]). There are examples of self-injective algebras that are not Frobenius (see e.g. [28, §IV.7]).

The following result summarizes some of the known results concerning versal deformation rings of Gorenstein-projective modules over finite dimensional $\Bbbk$-algebras.

Theorem 2.2. *Let Λ be a finite dimensional $\Bbbk$-algebra, and let V be a Gorenstein-projective Λ-module.*

(i) If V is projective, then $R(\Lambda, V)$ is universal and isomorphic to $\Bbbk$.

(ii) For all finitely generated projective left Λ-modules P, the versal deformation rings $R(\Lambda, V)$ and $R(\Lambda, V \oplus P)$ are isomorphic in $\hat{\mathcal{C}}$.

(iii) *If* $\underline{\mathrm{End}}_\Lambda(V) = \Bbbk$*, then* $R(\Lambda, V)$ *is universal. In particular, the versal deformation ring* $R(\Lambda, \Omega V)$ *of* ΩV *is also universal.*

(iv) *If* Λ *is further Frobenius and* V *is non-projective, then the versal deformation rings* $R(\Lambda, V)$ *and* $R(\Lambda, \Omega V)$ *are isomorphic in* $\hat{\mathcal{C}}$*.*

(v) *If* Λ *is Gorenstein and* Γ *is another finite dimensional Gorenstein* $\Bbbk$*-algebra such that there exist bimodules* ${}_\Gamma X_\Lambda$*,* ${}_\Lambda Y_\Gamma$ *inducing a singular equivalence of Morita type (as in Definition 2.1) between* Λ *and* Γ*, then* $X \otimes_\Lambda V$ *is a finitely generated Gorenstein-projective left* Γ*-module, and the versal deformation rings* $R(\Lambda, V)$ *and* $R(\Gamma, X \otimes_\Lambda V)$ *are isomorphic in* $\hat{\mathcal{C}}$*.*

The proofs of Theorem 2.2 (i), (ii), (iii) and (v) can be found in [5, Thm. 1.2], and the proof of Theorem 2.2 (iv) can be found in [10, Prop. 2.4]. It is important to mention that Theorem 2.2 (i), (ii), (iii) and (v) were first proved in the context of self-injective algebras and stable equivalences of Morita type in [8, Thm. 1.1] and [9, Lemma 3.2.2 & Prop. 3.2.6].

***Remark* 2.3.** (i) Assume that Λ is CM-free. Then it follows from Theorem 2.2 (i) that if V is a Gorenstein-projective Λ-module, then $R(\Lambda, V)$ is isomorphic to $\Bbbk$.

(ii) Let Λ be an arbitrary finite dimensional Gorenstein $\Bbbk$-algebra and let V is a Gorenstein-projective left Λ-module whose stable endomorphism ring is $\Bbbk$. Note that by [3, Prop. 3.1 (c)] we also have that ΩV is also Gorenstein-projective with stable endomorphism ring also isomorphic to $\Bbbk$. In particular, the versal deformation ring $R(\Lambda, \Omega V)$ is also universal. In [5, Remark 5.5], the following question was given: Are $R(\Lambda, V)$ and $R(\Lambda, \Omega V)$ isomorphic in $\hat{\mathcal{C}}$? Currently, this is still open for when A is a non-self-injective algebra.

3. Examples

In this section, we assume that $\Bbbk$ is an algebraically closed field. Recall that a *quiver* Q is a directed graph with a set of vertices Q_0, a set of arrows Q_1 and two functions $\mathbf{s}, \mathbf{t} : Q_1 \to Q_0$, where for all $\alpha \in Q_1$, $\mathbf{s}\alpha$ (resp. $\mathbf{t}\alpha$) denotes the vertex where α starts (resp. ends). A *path* in Q is either an ordered sequence of arrows $p = \alpha_n \cdots \alpha_1$ with $\mathbf{t}\alpha_j = \mathbf{s}\alpha_{j+1}$ for $1 \leq j < n$ (in this situation we say that p has length n), or for each $i \in Q_0$, the symbol e_i such that $\mathbf{s}e_i = i = \mathbf{t}e_i$. We call the symbols e_i the *trivial paths*, which have length zero. For a nontrivial path $p = \alpha_n \cdots \alpha_1$ we define $\mathbf{s}p = \mathbf{s}\alpha_1$ and $\mathbf{t}p = \mathbf{t}\alpha_n$. A non-trivial path p in Q is said to be an oriented cycle provided that $\mathbf{s}p = \mathbf{t}p$. The path algebra $\Bbbk Q$ of a quiver Q is the $\Bbbk$-vector space whose basis consists in all the paths in Q, and for two paths p and q, their multiplication is given by the concatenation pq provided that $\mathbf{s}p = \mathbf{t}q$, or zero otherwise. Let J be the two-sided ideal of $\Bbbk Q$

generated by all the arrows in Q. We say that an ideal I of $\Bbbk Q$ is *admissible* if there exists $d \geq 2$ such that $J^d \subseteq I \subseteq J^2$. In this situation, the quotient $\Bbbk Q/I$ is a finite dimensional $\Bbbk$-algebra. If p is a path in Q, we denote also by p is equivalence class a call it a path in $\Bbbk Q/I$. In particular, a path p in $\Bbbk Q/I$ is zero if and only if p belongs to I. It is well-known that every finite dimensional $\Bbbk$-algebra is Morita equivalent to an algebra of the form $\Bbbk Q/I$, where Q is a finite quiver and I is an admissible ideal of $\Bbbk Q$ (see e.g. [4, §III.1]). Since versal deformation rings of finitely generated modules over finite dimensional algebras are invariants under Morita equivalence (see [8, Prop. 2.5]), it is enough to assume that all our finite dimensional $\Bbbk$-algebras are all basic and of the form $\Lambda = \Bbbk Q/I$, where Q and I are as above.

3.1. Algebras of dihedral type

Consider the $\Bbbk$-algebras Λ of dihedral type (as introduced in [18]) in Figure 2. It follows in particular that these algebras are all symmetric, i.e., the functors $\mathrm{Hom}_\Lambda(-,\Lambda)$ and $\mathrm{Hom}_\Bbbk(-,\Bbbk)$ from Λ-mod to Λ^{op}-mod are natural equivalent, where Λ^{op} is the opposite algebra of Λ. This in particular implies that all these algebras are also self-injective (see e.g. [4, Prop. IV.3.8]). Moreover, it follows from [20, Thm. 3.4] that these algebras are also derived equivalent. Since derived equivalences between self-injective $\Bbbk$-algebras induces stable equivalences of Morita type (see [23, Cor. 5.5]), and thus singular equivalences of Morita type as in Definition 2.1, it follows that the algebras in Figure 2 are all singularly equivalent of Morita type.

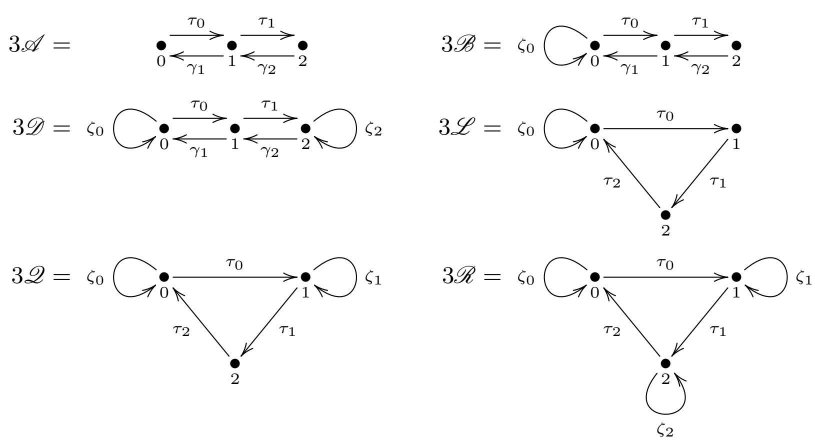

FIGURE 1. Quiver of algebras of dihedral type.

Let $\Lambda_0 = D(3\mathscr{R})^{1,2,2,2}$. In [8, §3], all left Λ_0-modules V with $\underline{\mathrm{End}}_{\Lambda_0}(V) = \Bbbk$ were completely classified. Since self-injective algebras are also Gorenstein, it

$$
\begin{aligned}
D(3\mathscr{A})_2^{2,2} &= \Bbbk[3\mathscr{A}]/\langle \tau_1\tau_0, \gamma_1\gamma_2, (\tau_0\gamma_1)^2 - (\gamma_2\tau_1)^2\rangle\\
D(3\mathscr{B})_2^{1,2,2} &= \Bbbk[3\mathscr{B}]/\langle \zeta_0\gamma_1, \tau_0\zeta_0, \tau_1\tau_0, \gamma_1\gamma_2, \tau_0\gamma_1 - (\gamma_2\tau_1)^2, \zeta_0^2 - \gamma_1\tau_0\rangle\\
D(3\mathscr{B})_2^{2,2,1} &= \Bbbk[3\mathscr{B}]/\langle \zeta_0\gamma_1, \tau_0\zeta_0, \tau_1\tau_0, \gamma_1\gamma_2, (\tau_0\gamma_1)^2 - \gamma_2\tau_1, \zeta_0^2 - (\gamma_1\tau_0)^2\rangle\\
D(3\mathscr{D})_2^{1,1,2,2} &= \Bbbk[3\mathscr{D}]/\langle \zeta_0\gamma_1, \tau_0\zeta_0, \tau_1\tau_0, \gamma_1\gamma_2, \zeta_2\tau_1, \gamma_2\zeta_2, \zeta_0^2 - \gamma_1\tau_0, \zeta_2^2 - \tau_1\gamma_2, \tau_0\gamma_1 - \gamma_2\tau_1\rangle\\
D(3\mathscr{L})^{2,2} &= \Bbbk[3\mathscr{L}]/\langle \tau_0\zeta_0, \zeta_0\tau_2, \zeta_0^2 - (\tau_2\tau_1\tau_0)^2, \tau_1(\tau_0\tau_2\tau_1)^2\rangle\\
D(3\mathscr{Q})^{1,2,2} &= \Bbbk[3\mathscr{Q}]/\langle \tau_0\zeta_0, \zeta_0\tau_2, \zeta_1\tau_0, \tau_1\zeta_1, \zeta_0^2 - \tau_2\tau_1\tau_0, \zeta_1^2 - \tau_0\tau_2\tau_1\rangle\\
D(3\mathscr{R})^{1,2,2,2} &= \Bbbk[3\mathscr{R}]/\langle \tau_0\zeta_0, \zeta_1\tau_0, \tau_1\zeta_1, \zeta_2\tau_1, \tau_2\zeta_2, \zeta_0\tau_2, \zeta_1^2 - \tau_0\tau_2\tau_1, \zeta_2^2 - \tau_1\tau_0\tau_2\rangle
\end{aligned}
$$

FIGURE 2. Algebras of dihedral type.

follows that the next result obtained in [9, §4] can be obtained alternatively from [8, Thm. 1.2] and Theorem 2.2 (v).

Theorem 3.1. ([9, §3]) *Let Λ be a finite dimensional $\Bbbk$-algebra of dihedral type belonging to Figure 2, and let V be a left Λ-module with $\underline{\mathrm{End}}_\Lambda(V) = \Bbbk$. Then the versal deformation ring $R(\Lambda, V)$ is universal and isomorphic either to $\Bbbk$, or to $\Bbbk[\![t]\!]/(t^2)$, or to $\Bbbk[\![t]\!]$.*

***Remark* 3.2.** (i) The results in Theorem 3.1 coincide with the ones obtained by F. M. Bleher and S. N. Talbott in [7] concerning the algebras of dihedral type in Figure 2.

(ii) In [29], the author studied the universal deformation rings of modules over more general cases of algebras of dihedral type of the class $D(3\mathscr{R})$.

(iii) More recently, F. M. Bleher and D. Wackwitz have determined in [10, Thm. 1.3] the universal deformation rings of finitely generated modules over finite dimensional algebras that are stably Morita equivalent to a self-injective split basic Nakayama $\Bbbk$-algebra.

3.2. Monomial algebras in which there is no an overlap

Although the study of finitely generated Gorenstein-projective modules goes back to [1], explicit descriptions of indecomposable Gorenstein-projective modules have been found for only a few classes of not necessarily self-injective $\Bbbk$-algebras (see e.g. [12, 14, 21, 24]). In the following, we recall such description given in [14] for monomial algebras. Recall that an admissible ideal I of $\Bbbk Q$ is said to be monomial if it is generated by paths of length at least two. In this situation we say that the quotient $\Bbbk Q/I$ is a *monomial algebra*. Let $\Lambda = \Bbbk Q/I$ be a monomial algebra. Following [14], we say that a pair (p, q) of non-zero paths in Λ is a *perfect pair* provided that the following conditions are satisfied:

(P1) both p and q are non-trivial with $\mathbf{s}p = \mathbf{t}q$ and $pq \in I$;

(P2) if $pq' \in I$ for a non-zero path q' with $\mathbf{t}q' = \mathbf{s}p$, then $q' = qq''$ for some path q'' in Λ;

(P3) if $p'q \in I$ for a non-zero path p' with $\mathbf{t}q = \mathbf{s}p'$, then $p' = p''p$ for some path p'' in Λ.

A non-zero path p in Λ is *perfect*, provided that there exists a sequence $p = p_1, p_2, \ldots, p_n, p_{n+1} = p$ such that for all $1 \leq i \leq n$, the pair (p_i, p_{i+1}) is a perfect pair. It follows from [14, Thm. 4.1] that a finitely generated indecomposable non-projective left Λ-module V is Gorenstein-projective if and only if $V = \Lambda p$, where p is a perfect path in Λ. This results unifies those descriptions for indecomposable Gorenstein-projective modules over Nakayama algebras given in [24] and over gentle algebras given in [21]. An *overlap* in Λ is given by two perfect paths p and q in Λ that satisfy one of the following conditions:

(O1) $p = q$, and $p = p'x$ and $q = xq'$ for some non-trivial paths x, p' and q' with the path $p'xq'$ non-zero.

(O2) $p \neq q$, and $p = p'x$ and $q = xq'$ for some non-trivial path x with the path $p'xq'$ non-zero.

We refer the reader to look at [14, Examples 4.4 & 5.8] for examples of monomial algebras and their corresponding perfect paths, and [14, Example 5.10] for an example of a monomial algebra in which there is no an overlap.

The following result was proved in [5, Thm. 5.5].

Theorem 3.3. *Let $\Lambda = \Bbbk Q/I$ be a monomial algebra in which there is no an overlap, and let V be a finitely generated indecomposable Gorenstein-projective left Λ-module. Then the versal deformation ring $R(\Lambda, V)$ of V is universal and isomorphic either to $\Bbbk$ or to $\Bbbk[\![t]\!]/(t^2)$.*

***Remark* 3.4.** We say that a monomial algebra $\Lambda = \Bbbk Q/I$ is *quadratic* if I is generated by paths of length two. In particular, gentle algebras are quadratic monomial. Note that there is no an overlap in a quadratic monomial algebra, for in this situation all the perfect paths are arrows. Thus, Theorem 3.3 applies also to gentle algebras.

3.3. Universal deformation rings of a certain type of Gorenstein-projective modules

Definition 3.5. We say that a left Λ-module V is a *semi-rigid stable brick* if $\underline{\mathrm{End}}_\Lambda(V) = \Bbbk$ and $\underline{\mathrm{Hom}}_\Lambda(\Omega V, V) = \delta_{\Omega V, V} \cdot \Bbbk$, where $\delta_{\Omega V, V}$ is the Kronecker delta.

***Remark* 3.6.** It follows from the proofs of [14, Prop. 5.9] and [12, Lemma 3.6 (iii)] that if Λ is either an algebra in which there is not overlap or skewed-gentle (as introduced by Ch. Geiß and J. A. de la Peña in [19]), then every indecomposable non-projective Gorenstein-projective left Λ-module V is a semi-rigid stable brick as in Definition 3.5.

The following result unifies the classification of universal deformation rings of indecomposable non-projective Gorenstein-projective modules over algebras in which there is no an overlap and over skewed-gentle algebras; its proof can be obtained verbatim from that of Theorem 3.3.

Theorem 3.7. *Let Λ be a finite dimensional $\Bbbk$-algebra, and let V be a Gorenstein-projective left Λ-module which is also a semi-rigid stable brick as in Definition 3.5. Then the versal deformation ring $R(\Lambda, V)$ is universal and isomorphic either to $\Bbbk$ or to $\Bbbk[\![t]\!]/(t^2)$.*

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