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Connes-Landi spheres are homogeneous spaces

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Abstract. In this paper, we review some recent developments of compact quantum groups that arise as θ-deformations of compact Lie groups of rank at least two. A θ-deformation is merely a 2-cocycle deformation using an action of a torus of dimension higher than 2. Using the formula (Lemma 5.3) developed in [11], we derive the noncommutative 7-sphere in the sense of Connes and Landi [3] as the fixed-point subalgebra.

Key words and phrases. Noncommutative geometry, quantum homogeneous space, compact quantum group, Connes-Landi deformation, θ-deformation.

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Resumen. En este artículo nosotros revisamos algunos desarrollos recientes de grupos cuánticos compactos que surgen en θ-deformaciones de grupos compactos de Lie de rango al menos dos. Una θ-deformación es simplemente una deformación por 2-cociclo, usando una acción de un toro de dimensión superior a 2. Usando la fórmula (Lemma 5.3) desarrollada en [11], nosotros derivamos la 7-esfera no conmutativa, en el sentido de Connes y Landi [3], como la subálgebra de puntos fijos.

Palabras y frases clave. Geometría no conmutativa, espacio cuántico homogéneo, grupo cuántico compacto, deformación de Connes-Landi, θ-deformación..

1. Introduction

Lie groups play crucial roles both in mathematics and physics. For example, most field theories of particle physics are based on certain symmetries with gauge group. For instance, quantum chromodynamics is a gauge theory with the symmetry group $SU(3)$ while quantum electromagnetism is a gauge theory with the symmetry group $U(1)$ and the standard model is a gauge theory with

the symmetry group $SU(3) \times SU(2) \times U(1)$. Also, $SU(5)$ had been proposed as a gauge group in grand unification theory.

The main purpose of this article is to survey the construction of compact quantum groups from compact Lie groups of rank at least 2 using both the left and the right action of the torus on the algebra generated by coordinate functions [7]. Once we establish an action by $\mathbb{T}^n$, we can apply the θ-deformation on the coordinate functions using an antisymmetric bicharacter or a 2-cocycle to deform the multiplication of the algebra [3, 7, 6]. These noncommutative manifolds are called θ-deformation because the original work used θ as the parameter.

We consider a noncommutative version of the group $SU(n)$ in the framework of θ-deformation of manifolds introduced in [3]. This version of deformations of compact Lie groups to compact quantum groups was first discovered by Rieffel in [7] using his theory of strict deformation quantization along the action of $\mathbb{R}^n$ [6], which is a 2-cocycle twist of the multiplication of the algebra structure. Later, Connes and Landi in [3] considered isometric actions of $\mathbb{T}^n$ on compact spin manifolds to deform the algebra of continuous functions to obtain a noncommutative compact spin manifold. Since in their case, the spectrum of the Dirac operator does not change, it is often referred to as the isospectral deformation. More generally, since the new product on the algebra can be defined as long as there is an action of the n-torus on the algebra, the deformed algebra is also called a toric noncommutative manifold. Although this type of deformation was done in the C*-algebraic framework, we will deform only the algebraic part to obtain generators and relations. At the algebraic level we will be able to give very explicit formulae in terms of the generators. The resulting compact quantum group will be a compact matrix quantum group in the sense of Woronowicz [12].

In [2], Connes and Dubois-Violette constructed some quantum matrix groups as quotients of the θ-deformation $M_{n\times n}(\mathbb{R})_\theta$ of the algebra generated by the coordinates of $M_{n\times n}(\mathbb{R})$. However, their method does not yield any nontrivial deformation of $SU(n)$ whereas [7] yields nontrivial deformations of $SU(n)$, $n \geq 3$. Since a compact Lie group G of rank higher than 2 admits a nontrivial θ-deformation to a compact quantum group, it is natural to ask whether a Lie group action $G \times X \to X$ on a compact manifold X endowed with a $\mathbb{T}^n$ action, $n \geq 2$, extends to an action $C(X_{\theta'}) \to C(G_\theta) \otimes C(X_{\theta'})$ of the compact quantum group G_θ. Of course, the original action will induce an action at the vector space level, but, in general, it is no longer an algebra homomorphism. For instance, Landi and van Suijlekom in [5] studied for exactly which values θ'_{ij} of $S^7_{\theta'}$ the diagonal action of the group $SU(2)$ on S^7 extends to an action of $SU(2)$ on $S^7_{\theta'}$. On the contrary, the group $SU(2)$ does not admit any nontrivial θ-deformation because the group is of rank 1. We recite a lemma from [11] in order to determine the dependence of the deformation parameters θ and θ', which generalizes the computation in [5]. Although the group $SU(2)$ does not

admit a nontrivial deformation in the framework of [7], a similar construction of Hopf fibration was generalized to the quantum group $SU_q(2)$ in [4], which is, a priori, a different deformation than our framework.

An interesting consideration of G_θ is the construction of quantum homogeneous space. Suppose $\mathbb{T}^n \subset K \subset G$, $n \geq 2$ with K a closed subgroup of the compact Lie group G. Using the isomorphism $C(G/K) \cong C(G)^K$, we can define a quantum homogeneous space to be the fixed-point subalgebra for the action of K. This definition can be generalized to essentially any quantum subgroup acting on the quantum group. In our present work, we present an example of the 'quotient' $SU(4)_\lambda/SU(3)_\theta$. To make this statement precise, we give an action $\rho : SU(4)_\lambda \to SU(4)_\lambda \otimes SU(3)_\theta$ and compute the invariant elements. We show that the subalgebra generated by the invariant elements is isomorphic to a noncommutative 7-sphere $S^7_{\theta'}$ for some θ'.

Quantum homogeneous spaces using the θ-deformation of compact Lie groups had been treated by Varilly in [9]. However, Varilly does not consider an action of quantum groups nor the fixed-point subalgebra for the action. Rather than computing the fixed-point subalgebra, Varilly endows $C(G/K)$ with a new product consistent in a way that it is embedded in $C(G_\theta)$. We take an approach to endow $C(G)$ with the action (by the left multiplication) $C(G) \to C(K) \otimes C(G)$ by its subgroup K then extend the action to the θ-deformation. Only then, we can compute the fixed-point subalgebra for the action. This subalgebra will be our notion of quantum homogeneous spaces.

This paper is organized as follows. In Section 2, we relate the strict deformation quantization of periodic action in [6] and the θ-deformation in [2]. Section 2.2 is devoted to the review of the construction of the θ-deformation of compact quantum groups [7], and we survey some main results. In Section 2.3, we compute relations on the generators of the deformation $SU(3)_\theta$ of $SU(3)$ as a compact quantum matrix group. In Section 3, we restate the necessary and the sufficient condition as to when an action of a group on an algebra of functions extends to the setting of θ-deformations [11, Lemma 5.3] as Lemma 3.2. We recall an example from [11] in Section 3.1 where it endows the noncommutative 5-sphere S^5_λ with an action of $SU(3)_\theta$. In this case, the fixed-point subalgebra is trivial. Finally in Section 3.2, we construct an action $\rho : SU(4)_\lambda \to SU(4)_\lambda \otimes SU(3)_\theta$ of $SU(3)_\theta$ on $SU(4)_\lambda$ whose fixed-point subalgebra $SU(4)_\lambda^{SU(3)_\theta}$ is isomorphic to the noncommutative 7-sphere $S^7_{\theta'}$.

2. The θ-deformation of compact quantum groups

In this section, we review the construction of toric noncommutative manifolds as a special case of Rieffel's deformation quantization along the action of $\mathbb{R}^n$ [6] and use it to obtain a deformation of compact Lie group of rank at least 2 [7] in Section 2.2. Deforming the algebraic structure is a simple matter while

obtaining a compatible coalgebra structure is more subtle; the 2-cocycle has to be chosen carefully.

2.1. θ-deformation as Rieffel's strict deformation quantization

The construction of toric noncommutative manifolds proceeds as follows [3, 6]. Given a compact manifold M endowed with an action of an n-torus $\mathbb{T}^n$ or equivalently with a periodic action of $\mathbb{R}^n$, $n \geq 2$, the algebra $C^\infty(M)$ of smooth functions on M can be decomposed into isotypic components $C^\infty(M)_{\vec{r}} = \left\{ f \in C^\infty(M) : \alpha_t(f) = e^{2\pi i t\cdot \vec{r}} f \right\}$ indexed by $\vec{r} \in \widehat{\mathbb{T}^n} = \mathbb{Z}^n$. A deformed algebra structure can, then, be given by the linear extension of the product of two functions $f_{\vec{r}}$ and $g_{\vec{s}}$ in some isotypic components. A new product $\times_\theta$ on these elements is given by

$$f_r \times g_s = \chi(r,s) f_r g_s, \tag{1}$$

where $\chi : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ is an antisymmetric bicharacter on the Pontryagin dual $\mathbb{Z}^n = \widehat{\mathbb{T}^n}$ of $\mathbb{T}^n$ i.e. $\overline{\chi(s,r)} = \chi(r,s)$. The bicharacter relation

$$\chi(r,s+t) = \chi(r,s)\chi(r,t), \qquad \chi(r+s,t) = \chi(r,t)\chi(s,t), \qquad s,t \in \mathbb{Z}^n$$

ensures the associativity of the new product. For instance,

$$\chi(r,s) := \exp\left(\pi i \ r \cdot \theta s\right)$$

where $\theta = (\theta_{jk})$ is a real antisymmetric $n \times n$ matrix is a typical choice. The involution for the new product is given by the complex conjugation of the functions. We denote the algebra obtained by extending the new product $\times_\theta$ to all continuous functions by $C(M_\theta)$ [6].

The definition of a new product in (1) is a discretized version of the Rieffel's deformation quantization [6] by viewing the action as a periodic action of $\mathbb{R}^n$. First of all, Rieffel expressed the deformed product of an algebra endowed with an action of $\mathbb{R}^n$ in the integral form

$$a \times_J b = \int_{V\times V} \alpha_{Js}(a)\alpha_t(b)\ e^{2\pi i s\cdot t}\ ds\ dt \tag{2}$$

where a and b belong to an algebra A and J is a real antisymmetric real $n \times n$ matrix [6]. This integral may be interpreted as an oscillatory integral and $\alpha : V \hookrightarrow \mathrm{Aut}(A)$ is a strongly continuous action of a finite dimensional vector space $V \cong \mathbb{R}^n$ on A. The oscillatory integral (2) makes sense a priori only for elements a, b of the smooth subalgebra A^∞ (which forms a Fréchet algebra) of A under the action α i.e. $v \mapsto \alpha_v(a)$ is smooth. However, Rieffel showed that if A is a C^*-algebra, then the new algebra A_J can also be given a C*-algebra structure.

First, A^∞ can be given a suitable pre C^*-norm for which the product $\times_J$ is continuous in a way that the completion with respect to this norm obtains

the deformed algebra A_J. Thus, the deformed product can be extended to the entire algebra A_J.

It should be remarked that the smooth subalgebra remains unchanged as vector spaces $(A_J)^\infty = A^\infty$, even though they have different products [7, Theorem 7.1].

If the action α of V is periodic, then $\alpha_v(a) = a$ for all v in some lattice $\Lambda \subset V$ and $a \in A$. α can, then, be viewed as an action of the compact abelian group $H = V/\Lambda$. Then, A^∞ admits a decomposition into a direct sum of isotypic components indexed by $\Lambda = \widehat{H}$. It is shown in [6, Proposition. 2.21] that if $\alpha_s(a_p) = e^{2\pi i p\cdot s} a_p$ and $\alpha_t(b_q) = e^{2\pi i q\cdot t} b_q$ with $p, q \in \Lambda$, then (1) defines an associative product.

The toric noncommutative manifolds [3] based on the deformation quantization (1), as far as the algebra structure is concerned, is a special case of Rieffel's strict deformation quantization theory. The same remarks are also made in [8] and [9]. We call the θ-deformed algebra the toric noncommutative manifold.

2.2. Compact quantum Lie groups associated with n-torus

We now review the θ-deformation of compact Lie groups in [7].

Let G be a compact Lie group of rank at least 2 or equivalently $\mathbb{T}^n \subset G$, $n \geq 2$. We use the natural action $\alpha_{(t,s)}(f)(x) = f\left(t^{-1}xs\right)$ of $\mathbb{T}^n \times \mathbb{T}^n$ to deform the algebra $C(G)$ of continuous functions and write down the relations for the matrix coefficients in the case of $G = SU(3)$ and $SU(4)$. However, not every choice of deformation of this kind respects the original coalgebra structure [6]. If the antisymmetric matrix θ is chosen to be of the form $K \oplus (-K)$ where $K^T = -K$ is an $n \times n$ real antisymmetric matrix, we would obtain a θ-deformation of compact quantum groups.

Definition 2.1. A coalgebra C is an associative unital algebra with linear maps $\triangle : C \to C \otimes C$ called coproduct and $\epsilon : C \to \mathbb{C}$ called counit such that

$$(\mathrm{id} \otimes \triangle) \circ \triangle = (\triangle \otimes \mathrm{id}) \circ \triangle$$
$$(\epsilon \otimes \triangle) \circ \triangle = (\triangle \otimes \epsilon) \circ \triangle = \mathrm{id}\,.$$

If these maps are algebra homomorphisms, then C is called a bialgebra.

If H is a unital bialgebra endowed with an anti-homomorphism $S : H \to H$ such that

$$m \circ (\mathrm{id} \otimes S) \circ \triangle = m \circ (S \otimes \mathrm{id}) \circ \triangle = \eta \circ \epsilon$$

where $m : H \otimes H \to H$ is the multiplication map and $\eta : \mathbb{C} \to H$ is the embedding of the unit element is called a compact quantum group.

For example, if G is a compact Lie group, then the usual (C*-)compact quantum group structure can be given as follows:

$$\begin{aligned}
&\triangle : C(G) \to C(G)\hat{\otimes}C(G) \cong C(G \times G), \qquad \triangle(f)(x,y) = f(xy)\\
&\epsilon(f) = f(e)\\
&S(f(x)) = f(x^{-1})
\end{aligned}$$

where $e \in G$ is the identity element and $\hat{\otimes}$ is the completed tensor product. Although in this paper we do not address the C*-algebraic setting, when looking at the algebra of continuous functions on a compact Lie group, it is the correct setting.

With the notations above, let $K^T = -K$ be a real antisymmetric matrix. It was shown in [7] that the formula (1) with $\theta = K \oplus (-K)$ with the unaltered coalgebra structure endows the algebra of continuous functions on the compact Lie group with a quantum group structure. Other choices of θ will not afford a quantum group with the original coalgebra structure. We denote the resulting compact quantum group by G_θ. We view $\theta_{ij} \in \mathbb{R}$ as parameters of the deformation Moreover, the deformation $(G_\theta)^\infty$ of smooth functions on G remains dense in $C(G_\theta)$ [7]. In fact, if G is a compact matrix group, then so too is G_θ a compact matrix quantum group in the sense of Woronowicz [12]. We will restrict our deformation to the coordinate functions of a compact matrix Lie group in order to be able to compute the commutation relations on them.

It is shown in [7] that the coproduct defined by $\triangle(f)(x,x') = f(xx')$ extends to a continuous homomorphism $\triangle : C(G_\theta) \to C(G_\theta) \otimes C(G_\theta)$. Here, the interpretation of $C(G_\theta) \otimes C(G_\theta)$ is given by the isomorphism $C\left((G \times G)_{\theta\oplus\theta}\right) \cong C(G_\theta) \otimes C(G_\theta)$ where the tensor product is the minimal tensor product [6] (in fact, $C(G_\theta)$ would be nuclear so distinguishing the kind of tensor product is not necessary in this setting but since this deformation applies to any C* quantum group endowed with an action of $\mathbb{T}^n$, we stress that this analysis is valid for the minimal tensor for a future consideration). Moreover, the counit $\epsilon(f)(x) = f(e)$ remain a homomorphism and the coinverse $S(f)(x) = f(x^{-1})$ remain an anti-homomorphism, and they satisfy all the compatibility conditions in the deformation.

Lastly, since the normalized Haar measure determines a linear functional μ on $C(G)$, μ becomes a state on $C(G_\theta)$. In fact, this state is a tracial state $\mu_\theta : C(G_\theta) \to \mathbb{C}$ defined by $\mu_\theta(f) := \mu(f) = \int_G f(x)dx$. Moreover, this tracial state is a Haar state [7, Theorem 4.2] as we restate in the following:

Theorem 2.2. *The Haar measure μ on $C(G)$ determines a Haar state on the quantum group $C(G_\theta)$. That is, a continuous linear functional μ_θ that is*

unimodular in the sense that

$$\begin{aligned}(id \otimes \mu_\theta) \circ \Delta &= \iota \circ \mu_\theta, \\ (\mu_\theta \otimes id) \circ \Delta &= \iota \circ \mu_\theta,\end{aligned} \tag{3}$$

$$(id \otimes \mu_\theta)\,(1 \otimes a)\,(\Delta b) = (id \otimes \mu_\theta)\,((S \otimes id)\,\Delta a)\,(1 \otimes b)\,, \tag{4}$$

$$\mu_\theta \circ S = \mu_\theta. \tag{5}$$

We denote this faithful trace simply by μ when the presence of θ is understood.

In [7], Rieffel determined the unitary dual of G_θ. He showed that irreducible representations of G are irreducible representations of G_θ and vice versa. Here, a representation π of the compact Lie group G means a continuous group homomorphism $\pi : G \to GL(V)$ for some finite dimensional complex vector space such that $\pi(g)\pi(g)^* = I_{\dim V} = \pi(g)^*\pi(g)$. π is called irreducible if there is no nontrivial proper subspace $\{0\} \neq W \subset V$ such that $\pi(g) \cdot W \subset W$. In the quantum group case, we dualize the action $(g, v) \mapsto \pi(g)v$. A (unitary) representation of the quantum group G_θ is a unitary element $U \in M_n(G_\theta)$, $UU^* = U^*U = I_n$ such that $(id \otimes \triangle)(U) = U_{12}U_{13}$.

Unlike in the case of classical representation theory, if π and ρ are representations of G on V and on W, respectively, then $V \otimes W$ and $W \otimes V$ are equivalent, but the equivalence is not given by the flip map $\sigma_{\pi\rho} : V \otimes W \to W \otimes V$, $\sigma_{\pi\rho}(v \otimes w) = w \otimes v$. It is possible to decompose a G-representation V into $\mathbb{T}^n$-representations where $\mathbb{T}^n \subset G$. Then, every $v \in V$ admits the decomposition $v = \sum_{\vec{p} \in \mathbb{Z}^n} v_{\vec{p}}$ where $v_{\vec{p}} = \int_{\mathbb{T}^n} \alpha_t(v) e^{-2\pi i p \cdot t} dt$.

Then, the equivalence $\pi \otimes \rho \sim \rho \otimes \pi$ is induced by the linear extension of

$$R_{\pi\rho}(v_{\vec{p}} \otimes w_{\vec{q}}) = e^{\pi i \theta(\vec{p},\vec{q})} v_{\vec{p}} \otimes w_{\vec{q}}\ . \tag{6}$$

This is not surprising since [12, Proposition 2.4] showed that a compact matrix group with the property that if the map σ for every pair of representations is an interwining operator then the quantum group is necessarily commutative. Moreover, one can view the above map $R_{\pi\rho}$ as the corresponding θ-deformation of the tensor product.

2.3. θ-deformations of $SU(3)$

In this section, we explicitly compute the θ-deformation $SU(3)_\theta$ of the Lie group $SU(3)$ using the approach in Section 2.2.

We use the maximal torus $\mathbb{T}^2 \subset SU(3)$ to deform the algebra H generated by the matrix coordinate functions in $C(SU(3))$. The coordinate functions u_{ij} of $U \in SU(3)$ satisfy the relation

$$\sum_{k=1}^{3} u_{jk}\overline{u_{jl}} = \delta_{kl}, \qquad \sum_{k=1}^{3} \overline{u_{kj}}u_{lj} = \delta_{kl}, \qquad \det U = 1.$$

Let $K = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$ and we use the diagonal representation of the maximal torus

$$\mathbb{T}^2 = \left\{ t = \begin{pmatrix} e^{2\pi i\varphi_1} & 0 & 0 \\ 0 & e^{2\pi i\varphi_2} & 0 \\ 0 & 0 & e^{2\pi i\varphi_3} \end{pmatrix} : \varphi_j \in \mathbb{R},\ \varphi_3 = -(\varphi_1+\varphi_2) \right\}.$$

In fact, monomials in coordinates are the isotypic components of the action

$$\begin{aligned} &\alpha_{(t,s)}(f(U)) = f\left(tUs\right) \\ &= f \begin{pmatrix} e^{2\pi i(\varphi_1+\psi_1)}u_{11} & e^{2\pi i(\varphi_1+\psi_2)}u_{12} & e^{2\pi i(\varphi_1+\psi_3)}u_{13} \\ e^{2\pi i(\varphi_2+\psi_1)}u_{21} & e^{2\pi i(\varphi_2+\psi_2)}u_{22} & e^{2\pi i(\varphi_2+\psi_3)_2}u_{23} \\ e^{2\pi i(\varphi_3+\psi_1)}u_{31} & e^{2\pi i(\varphi_3+\psi_2)}u_{32} & e^{2\pi i(\varphi_3+\psi_3)}u_{33} \end{pmatrix} \end{aligned}$$

of $\mathbb{T}^2 \times \mathbb{T}^2$. In the above, we omitted taking the inverse of t because $\mathbb{T}^n$ is Abelian. Since K is a 2×2 real antisymmetric matrix, there is only one independent parameter present in the deformation, we denote the resulting compact quantum group by $SU(3)_\theta$. It is evident that each coordinate function is in an isotypic component $A_{\vec{n}}$, $\vec{n} = (n_1, n_2, n_3, n_4)$, of this action. Then using the formula (1),

$$u_{ij} \times_\theta u_{kl} = e^{\pi i\theta(-n_1m_2+n_2m_1+n_3m_4-n_4m_3)}u_{ij}u_{kl},$$

$u_{ij} \in A_{\vec{n}}$ and $u_{kl} \in A_{\vec{m}}$. Thus, we obtain the following commutation relations:

$$\begin{array}{lll}
u_{11}u_{12} = e^{-2\pi i\theta}u_{12}u_{11}, & u_{11}u_{13} = e^{2\pi i\theta}u_{13}u_{11}, & u_{11}u_{21} = e^{2\pi i\theta}u_{21}u_{11}, \\
[u_{11}, u_{22}] = 0, & u_{11}u_{23} = e^{4\pi i\theta}u_{23}u_{11}, & u_{11}u_{31} = e^{-2\pi i\theta}u_{31}u_{11}, \\
u_{11}u_{32} = e^{-4\pi i\theta}u_{32}u_{11}, & [u_{11}, u_{33}] = 0, & u_{12}u_{13} = e^{-2\pi i\theta}u_{13}u_{12}, \\
u_{12}u_{21} = e^{4\pi i\theta}u_{21}u_{12}, & u_{12}u_{22} = e^{2\pi i\theta}u_{22}u_{12}, & [u_{12}, u_{23}] = 0, \\
[u_{12}, u_{31}] = 0, & u_{12}u_{32} = e^{-2\pi i\theta}u_{32}u_{12}, & u_{12}u_{33} = e^{-4\pi i\theta}u_{33}u_{12}, \\
[u_{13}, u_{21}] = 0, & u_{13}u_{22} = e^{4\pi i\theta}u_{22}u_{13}, & u_{13}u_{23} = e^{2\pi i\theta}u_{23}u_{13}, \\
u_{13}u_{31} = e^{-4\pi i\theta}u_{31}u_{13}, & [u_{13}, u_{32}] = 0, & u_{13}u_{33} = e^{-2\pi i\theta}u_{33}u_{13}, \\
u_{21}u_{22} = e^{-2\pi i\theta}u_{22}u_{21}, & u_{21}u_{23} = e^{2\pi i\theta}u_{23}u_{21}, & u_{21}u_{31} = e^{2\pi i\theta}u_{31}u_{21}, \\
[u_{21}, u_{32}] = 0, & u_{21}u_{33} = e^{4\pi i\theta}u_{33}u_{21}, & u_{22}u_{23} = e^{-2\pi i\theta}u_{23}u_{22}, \\
u_{22}u_{31} = e^{4\pi i\theta}u_{31}u_{22}, & u_{22}u_{32} = e^{2\pi i\theta}u_{32}u_{22}, & [u_{22}, u_{33}] = 0, \\
[u_{23}, u_{31}] = 0, & u_{23}u_{32} = e^{4\pi i\theta}u_{32}u_{23}, & u_{23}u_{33} = e^{2\pi i\theta}u_{33}u_{23}, \\
u_{31}u_{32} = e^{-2\pi i\theta}u_{32}u_{31}, & u_{31}u_{33} = e^{2\pi i\theta}u_{33}u_{31}, & u_{32}u_{33} = e^{-2\pi i\theta}u_{33}u_{32}.
\end{array}$$

The coalgebra structure restricted these elements is given by

$$\Delta(u_{ij}) = \sum_{k=1}^{3} u_{ik} \otimes u_{kj}, \qquad \epsilon(u_{ij}) = \delta_{ij}, \qquad S(u_{ij}) = u_{ji}^*.$$

Thus, $SU(3)_\theta$ is a compact quantum group.

While we consider only the $*$-algebra generated by elements to be our compact quantum group, Theorem 3.9 of [10] shows that the algebra generated by $\{u_{ij}\}$ is dense in $SU(3)_\theta$, and therefore, that $SU(3)_\theta$ can be completed to a compact matrix quantum group with the C*-algebra structure.

3. actions of $SU(3)_\theta$

In this section we present examples of actions of $SU(3)_\theta$. Classically, $SU(3)$ acts most naturally on S^5 (transitively) and on $SU(4)$ as a subgroup. In the first subsection, we review the action of $SU(3)_\theta$ on S^5_λ as a compact quantum group. This example was also presented in [11], but it remains relevant for the present article because the fixed-point subalgebra for this action is shown to be trivial. Thus, it gives one way of generalizing homogeneous spaces. On the other hand, the action of $SU(3)_\theta$ on $SU(4)_\lambda$ as a quantum group is a new example in Section 3.2. In this example, we derive the noncommutative 7-sphere $S^7_{\theta'}$ as the fixed-point algebra for this action. Therefore, it is a homogeneous space. In the noncommutative geometric framework, these two generalizations of homogeneous spaces seem to be different.

3.1. action of $SU(3)_\theta$ on S^5_λ

We construct the action of $SU(3)_\theta$ on S^5_λ [11]. It is a generalization of the classical action of $SU(3)$ on S^5. The fixed-point subalgebra is trivial, which is analogous to the classical action of $SU(3)$ on S^5. However, the extent to which it remains an action in the deformation depends on the choice of the parameters of the deformations.

An odd dimensional noncommutative (2n-1)-sphere S^{2n-1}_λ is a θ-deformation of the algebra generated by the coordinate functions on the (2n-1)-sphere S^{2n-1}, which can be constructed from the action of $\mathbb{T}^n$. It is the $*$-algebra generated by n normal elements $z_1, z_2, \ldots, z_n$ satisfying the commutation relations

$$z_j z_k = e^{2\pi i \lambda_{jk}} z_k z_j, \qquad z_j z_k^* = e^{-2\pi i \lambda_{jk}} z_k^* z_j, \qquad \sum_{k=1}^{n} z_k z_k^* = 1$$

where $\lambda_{jk} = -\lambda_{kj}$. These spheres are often referred to as Connes-Landi spheres.

Definition 3.1. Let H be a compact quantum group and A an algebra. An algebra homomorphism

$$\rho : A \to H \otimes A$$

is called a (left) action of the compact quantum group if

i) $(\Delta \otimes id) \circ \rho = (id \otimes \rho) \circ \rho$ and

ii) $(\epsilon \otimes id) \circ \rho = id$.

The set denoted by $A^H := \{a \in A : \rho(a) = 1 \otimes a\}$ is called the fixed-point subalgebra for the action by H. The elements of A^H are called invariant elements.

We restate the following criterion, which gives the necessary and sufficient condition of which an action of the undeformed algebras extends to the deformed setting [11].

Lemma 3.2. *Let A be an algebra and H a compact quantum group both equipped with n-torus $\mathbb{T}^n$ actions, α_t and β_s, respectively. Let λ and $\theta = K \oplus (-K)$ be real antisymmetric matrices. Suppose $\rho : A \to H \otimes A$ is an action of the quantum group H on A. Then, it extends to an action $\rho_{\theta,\lambda} : A_\lambda \to H_\theta \otimes A_\lambda$ if and only if for each $a = \sum_{\vec{n}} a_{\vec{n}}$ and $b = \sum_{\vec{m}} b_{\vec{m}}$ the following equation holds.*

$$\begin{aligned} & e^{\pi i \lambda(\vec{n},\vec{m})} \int_{\mathbb{T}^n \times \mathbb{T}^n} \rho(\alpha_t(a_{\vec{n}}))\rho(\alpha_s(b_{\vec{m}}))e^{-2\pi i(t\cdot\vec{n}+s\cdot\vec{m})} \\ &= \sum_{(\vec{n'},\vec{p}),(\vec{m'},\vec{q})} e^{2\pi i(\theta(\vec{n'},\vec{m'})+\lambda(\vec{p},\vec{q}))} \times \\ & \qquad \int_{\mathbb{T}^{2n}\times\mathbb{T}^{2n}} (\beta_{t'} \otimes \alpha_t)(\rho(a_n))(\beta_{s'} \otimes \alpha_s)(\rho(b_n)) \times \\ & \qquad\qquad e^{-2\pi i(t'\cdot\vec{n'}+t\cdot\vec{p}+s'\cdot\vec{m'}+s\cdot\vec{q})} dt'dtds'ds \end{aligned} \tag{7}$$

for each $\vec{n}$ and $\vec{m}$ in $\mathbb{Z}^n \times \mathbb{Z}^n$. Moreover, the fixed-point subalgebra $A_\lambda^{H_\theta} \subset A_\lambda$ is isomorphic to A^H.

Proof. Note that $a_{\vec{n}} = \int_{\mathbb{T}^n} \alpha_t(a) e^{-2\pi i t\cdot\vec{n}}$. For ρ to be a homomorphism with respect to the new products, we need to have

$$\rho(a \times_\lambda b) = \rho(a) \times_{\theta\oplus\lambda} \rho(b).$$

The left hand side gives

$$\begin{aligned} \rho(a \times_\lambda b) &= \rho\left(\sum_{\vec{n},\vec{m}} e^{\pi i\lambda(\vec{n},\vec{m})} a_{\vec{n}} b_{\vec{m}}\right) \\ &= \sum_{\vec{n},\vec{m}} e^{\pi i\lambda(\vec{n},\vec{m})} \rho(a_{\vec{n}})\rho(b_{\vec{m}}) \\ &= \sum_{\vec{n},\vec{m}} e^{\pi i\lambda(\vec{n},\vec{m})} \int_{\mathbb{T}^n\times\mathbb{T}^n} \rho(\alpha_t(a_{\vec{n}}))\rho(\alpha_s(b_{\vec{m}}))e^{-2\pi i(t\cdot\vec{n}+s\cdot\vec{m})} dtds \end{aligned}$$

while the right hand side gives

$$\begin{aligned}\rho(a)\times_{\theta\oplus\lambda}\rho(b) &= \left(\sum_{\vec{n}}\rho(a_{\vec{n}})\right)\times_{\theta\oplus\lambda}\left(\sum_{\vec{m}}\rho(b_{\vec{m}})\right)\\ &= \sum_{\vec{n},\vec{m}}\rho(a_{\vec{n}})\times_{\theta\oplus\lambda}\rho(b_{\vec{m}})\\ &= \sum_{\vec{n},\vec{m}}\sum_{(\vec{n'},\vec{p}),(\vec{m'},\vec{q})} e^{2\pi i(\theta(\vec{n'},\vec{m'})+\lambda(\vec{p},\vec{q}))}\times\\ &\qquad \int_{\mathbb{T}^{2n}\times\mathbb{T}^{2n}}(\beta_{t'}\otimes\alpha_t)(\rho(a_{\vec{n}}))(\beta_{s'}\otimes\alpha_s)(\rho(b_{\vec{m}}))\times\\ &\qquad e^{-2\pi i(t'\cdot\vec{n'}+t\cdot\vec{p}+s'\cdot\vec{m'}+s\cdot\vec{q})}dt'dtds'ds\ ,\end{aligned}$$

which shows that ρ is a homomorphism if and only if this condition (7) is satisfied.

To show that ρ is an action, note that the coproduct Δ and the action itself ρ are unchanged. Therefore, the conditions i) and ii) of left action are automatically satisfied.

Since the action ρ is unchanged, we see that the invariant elements remain unchanged and $A_\lambda^{H_\theta}$ is a subalgebra because ρ is an algebra homomorphism. ☑

Although the proof is simple, the previous lemma an important criterion of when an action extends to the deformed algebras. It suffices to show that the generators satisfy (7).

For instance, using $S^5\subset\mathbb{C}^3$, the action of $SU(3)$ on S^5 can be given by the matrix multiplication $(z_1,z_2,z_3)^T\mapsto U(z_1,z_2,z_3)^T$, $(z_1,z_2,z_3)^T\in S^5\subset\mathbb{C}^3$ and $U\in SU(3)$. The dual version of this action is the action given by $z_j\mapsto\sum_{k=1}^3 u_{jk}\otimes z_k$.

Let $H_\theta=SU(3)_\theta$ and $A_\lambda=S^5_\lambda$. Then, an action of the compact quantum group H_θ on A_λ can be given by the following proposition.

Proposition 3.3. *Let $\lambda=(\lambda_{ij})$ be an real antisymmetric matrix whose entries satisfy $\lambda_{12}=-\lambda_{13}=\lambda_{23}=\theta$. Then for this particular choice of the values of the parameters, the linear map $\delta:A_\lambda\to H_\theta\otimes A_\lambda$ defined by $\delta(z_j)=\sum_{k=1}^3 u_{jk}\otimes z_k$ is a left action of the compact quantum group H_θ on the algebra A_λ. This action is ergodic in the sense that $A_\lambda^{H_\theta}\cong\mathbb{C}$.*

Proof. We show that the equation (7) holds for those choices of values of λ_{jk}. For instance,

$$\begin{aligned}\delta(z_1) &= u_{11}\otimes z_1+u_{12}\otimes z_2+u_{13}\otimes z_3\\ \delta(z_2) &= u_{21}\otimes z_1+u_{22}\otimes z_2+u_{23}\otimes z_3.\end{aligned}$$

Now,

$$\delta(z_1 \times_\lambda z_2) = e^{\pi i \lambda_{12}} \delta(z_1 z_2) = \delta(z_1) \times_{\theta \oplus \lambda} \delta(z_2)$$

while

$$\begin{aligned}
&\delta(z_1) \times_{\theta \oplus \lambda} \delta(z_2) \\
&= e^{\pi i \theta} u_{11} u_{21} \otimes z_1 z_1 + e^{\pi i \lambda_{12}} u_{11} u_{22} \otimes z_1 z_2 + e^{\pi i (2\theta + \lambda_{13})} u_{11} u_{23} \otimes z_1 z_3 \\
&+ e^{\pi i (2\theta - \lambda_{12})} u_{12} u_{21} \otimes z_2 z_1 + e^{\pi i \theta} u_{12} u_{22} \otimes z_2 z_2 + e^{\pi i \lambda_{23}} u_{12} u_{23} \otimes z_2 z_3 \\
&+ e^{-\pi i \lambda_{13}} u_{13} u_{21} \otimes z_3 z_1 + e^{\pi i (2\theta - \lambda_{23})} u_{13} u_{22} \otimes z_3 z_2 + e^{\pi i \theta} u_{13} u_{23} \otimes z_3 z_3.
\end{aligned} \tag{8}$$

From such relation (8), the necessary condition for the values of $\lambda = (\lambda_{jk})$ are already restricted to $\lambda_{12} = -\lambda_{13} = \lambda_{23} = \theta$. The equation (7) shows that it is enough to prove such relations for the isotypic components of the algebra. The commutation relations for other generators can be ccomputed similarly. Thus, δ is an action whose fixed-point subalgebra is $\mathbb{C}$ by Lemma 3.2. ☑

3.2. θ-deformation of 7-sphere as a homogeneous space

In this subsection, we give the noncommutative 7-sphere $S^7_{\theta'}$ an interpretation of a homogeneous space. We achieve this notion by interpreting $S^7_{\theta'}$ as the fixed-point subalgebra for the action by $SU(3)_\theta$ on $SU(4)_\lambda$. More generally, we can consider homogeneous spaces $M = G/K$ where G is a compact Lie group and K is a closed subgroup. Using the isomorphism $C(G)^K \to C(M)$ of algebras between the algebra $C(M)$ of functions on M the K-invariant functions $C(G)$, the algebra of functions on the homogeneous space is defined to be $C(G)^K$. This notion can be generalized to the noncommutative setting.

A 3-parameter deformation $SU(4)_\lambda$ of $SU(4)$ using the action of its maximal torus $\mathbb{T}^3$ can be computed using the same method for $SU(3)_\theta$ in Section 2.3. Now we can determine exactly when $SU(3)_\theta$ acts on $SU(4)_\lambda$ using Lemma 3.2 and explicitly determine the invariant elements. The proof is essentially identical to Proposition 3.3 with a minor modification. As a subgroup, $SU(3) \subset SU(4)$ acts on $SU(4)$ by the left matrix multiplication on the left upper 3×3 block. The following theorem is the quantum counterpart of this action.

Proposition 3.4. *Let $u_{ij} \in SU(3)_\theta$ and $v_{kl} \in SU(4)_\lambda$. The map defined by*

$$\begin{aligned}
\rho(v_{kl}) &= \sum_{\alpha=1}^{3} u_{k\alpha} \otimes v_{\alpha l}, \qquad k,l = 1,2,3 \\
\rho(v_{4l}) &= 1 \otimes v_{4l}, \qquad k,l = 1,2,3,4
\end{aligned}$$

is an action $\rho : SU(4)_\lambda \to SU(3)_\theta \otimes SU(4)_\lambda$ if and only if $\theta = \lambda_{12} = -\lambda_{13} = \lambda_{23}$. In this case, the algebra $B = (SU(4)_\lambda)^{SU(3)_\theta}$ of invariant elements is generated by

$$\{1 \otimes v_{4l} : l = 1,2,3,4\}.$$

Set $x_l = 1 \otimes v_{4l}$. Then the algebra B generated by x_l is isomorphic to the noncommutative 7-sphere $S^7_{\theta'}$ as in [1, 3, 5] with

$$\theta' = \begin{pmatrix} 0 & -\theta & \theta & 0 \\ \theta & 0 & -\theta & 0 \\ -\theta & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exact relations can be computed using the commutation relations of $SU(4)_\lambda$ and $\theta = \lambda_{12} = -\lambda_{13} = \lambda_{23}$ as in the above theorem:

$$\begin{aligned} &x_1x_2 = e^{-2\pi i\theta}x_2x_1, && x_1x_3 = e^{2\pi i\theta}x_3x_1, && x_1x_4 = x_4x_1, \\ &x_2x_3 = e^{-2\pi i\theta}x_3x_2, && x_2x_4 = x_4x_2, && x_3x_4 = x_4x_3, \\ &x_1x_1^* + x_2x_2^* + x_3x_3^* + x_4x_4^* = 1. \end{aligned}$$

Although the θ-deformations of compact manifolds had been considered as homogeneous spaces, our result yields a general construction. It shows that it is enough to compute the dependence of the deformation parameters according to (7). For instance, Varilly in [9] concluded that the some of the generators of the odd dimensional noncommutative spheres commute with everything else, which is consistent with our result since x_4 in the centre. On the other hand, the noncommutative 3-sphere $S^3_{\theta'}$ is not a homogeneous space unless $\theta = 0$ while S^5_λ is only a homogeneous space in the sense that the fixed-point algebra for the $SU(3)_\theta$ action is trivial in the present work. Indeed, the natural left action by $SU(2)$ on $SU(3)$ given by

$$\rho\left(u_{ij}\right) = \begin{cases} \alpha u_{1j} + \beta u_{2j} & i = 1 \\ -\bar{\beta} u_{1j} + \alpha u_{2j} & i = 2 \\ u_{3j} & i = 3 \end{cases} \tag{9}$$

does not extend on $SU(3)_\theta$. Simply, $\rho(u_{11}u_{12}) \neq \rho(e^{-2\pi i\theta}u_{12}u_{11})$. Varilly's method differs from ours. Rather than computing the invariant elements, Varilly endows $C(G/K)$ with a new product consistent in a way that it is embedded in $C(G)_\theta$. We took a more direct approach to endow $C(G)$ with the natural action by $C(K)$ and studied the extent to which the original action is an action in the θ-deformation context. In that sense, it does not make sense to consider S^5_λ as an embeddable homogeneous space in $SU(3)_\theta$ for $SU(2)$.

The ubiquity of the θ-deformation in noncommutative geometry is already familiar. While the present work shows merely an example of symmetry in θ-deformation, it explains the pervasiveness of symmetry in the θ-deformation. There are already numerous research in mathematical physics using a toric noncommutative manifold as models, whether they are realized as θ-deformations

or not. We believe that our work has potential to reveal symmetries of toric noncommutative manifolds, and therefore, it has potential to be used in numerous areas of quantum physics.

Mathematically, it would be an interesting problem to classify up to Morita equivalence or isomorphism these objects. In fact, very little is known about the Morita equivalence of the θ-deformations other than the case of the noncommutative 2-torus.

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