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Graded modules over simple Lie algebras

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Abstract. The paper is devoted to the study of graded-simple modules and gradings on simple modules over finite-dimensional simple Lie algebras. In general, a connection between these two objects is given by the so-called loop construction. We review the main features of this construction as well as necessary and sufficient conditions under which finite-dimensional simple modules can be graded. Over the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, we consider specific gradings on simple modules of arbitrary dimension.

Key words and phrases. graded Lie algebras, graded modules, simple modules, universal enveloping algebra.

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Resumen. El artículo está dedicado al estudio de módulos graduados simples y graduaciones de módulos simples sobre álgebras de Lie simples de dimensión finita. En general, una conexión entre estos dos objetos viene dada por la llamada construcción de lazos.

Revisaremos las características principales de esta construcción, así como las condiciones necesarias y suficientes bajo las cuales se pueden graduar los módulos simples de dimensión finita. Para el álgebra de Lie $\mathfrak{sl}_2(\mathbb{C})$, consideramos graduaciones específicas en módulos simples de dimensión arbitraria.

Palabras y frases clave. Álgebras de Lie graduadas, módulos graduados, módulos simples, álgebra envolvente universal.

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1. Introduction

Let G be a non-empty set. A *G-grading* on a vector space V over a field $\mathbb{F}$ is a direct sum decomposition of the form

$$V = \bigoplus_{g\in G} V_g. \tag{1}$$

We will sometimes use Greek letters to refer to gradings, for example, we may write $\Gamma : V = \bigoplus_{g\in G} V_g$. If such a grading is fixed, V is called *G-graded*.

Note that the V_g are allowed to be zero subspaces. The subset $S \subset G$ consisting of those $g \in G$ for which $V_g \neq \{0\}$ is called the *support* of the grading Γ and denoted by $\operatorname{Supp}\Gamma$ or $\operatorname{Supp} V$. The subspaces V_g are called the *homogeneous components* of Γ, and the nonzero elements in V_g are called *homogeneous of degree g* (with respect to Γ). A *graded subspace* $U \subset V$ is an $\mathbb{F}$-subspace satisfying $U = \bigoplus_{g\in G} U \cap V_g$ (so U itself becomes G-graded).

Now let Γ and $\Gamma' : V = \bigoplus_{g'\in G'} V'_{g'}$ be two gradings on V with supports S and S', respectively. We say that Γ is a *refinement* of Γ' (or Γ' is a *coarsening* of Γ), if for any $s \in S$ there exists $s' \in S'$ such that $V_s \subset V'_{s'}$. The refinement is *proper* if this inclusion is strict for at least one $s \in S$.

An $\mathbb{F}$-algebra A (not necessarily associative) is said to be *graded by a set* G, or *G-graded* if A is a G-graded vector space and for any $g, h \in G$ such that $A_gA_h \neq \{0\}$ there is $k \in G$ (automatically unique) such that

$$A_gA_h \subset A_k. \tag{2}$$

In this paper, we will always assume that G is an *abelian group* and k in Equation (2) is determined by the operation of G. Thus, if G is written additively (as is commonly done in papers on Lie theory), then Equation (2) becomes $A_gA_h \subset A_{g+h}$. If G is written multiplicatively, then it becomes $A_gA_h \subset A_{gh}$. More generally, one can consider gradings by nonabelian groups (or semigroups). A grading on A is called *fine* if it does not have a proper refinement. Note that this concept depends on the class of gradings under consideration: by sets, groups, abelian groups, etc. It is well known that the latter two classes coincide for simple Lie algebras.

Given a grading $\Gamma : A = \bigoplus_{g\in G} A_g$ with support S, the *universal group* of Γ, denoted by G^u, is the group given in terms of generators and defining relations as follows: $G^u = \langle S \mid R\rangle$, where R consists of all relations of the form $gh = k$ with $\{0\} \neq A_gA_h \subset A_k$. If Γ is a group grading, then S is embedded in G^u and the identity map id_S extends to a homomorphism $G^u \to G$ so that Γ can be viewed as a G^u-grading Γ^u. In fact, any group grading $\Gamma' : A = \bigoplus_{g'\in G'} A'_{g'}$ that is a coarsening of Γ can be induced from Γ^u by a (unique) homomorphism $\nu : G^u \to G'$ in the sense that $A_{g'} = \bigoplus_{g\in\nu^{-1}(g')} A_g$ for all $g' \in G'$. In this situation, one may say that Γ' is a *quotient* of Γ^u. In the above considerations, we can replace "group" by "abelian group" and, in general, this leads to a different G^u. However, there is no difference for gradings on simple Lie algebras.

For example, choose the elements

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

as a basis of $L = \mathfrak{sl}_2(\mathbb{C})$ and consider the following grading by $G = \mathbb{Z}_3$:

$$\Gamma :\ L_1 = \langle x \rangle,\ L_0 = \langle h \rangle,\ L_2 = \langle y \rangle.$$

The support of Γ is G itself, the universal group is $\mathbb{Z}$, and

$$\Gamma^u :\ L_{-1} = \langle x \rangle,\ L_0 = \langle h \rangle,\ L_1 = \langle y \rangle.$$

The following grading by $G' = \mathbb{Z}_2$ is a coarsening of Γ:

$$\Gamma' :\ L_1 = \langle x, y \rangle,\ L_0 = \langle h \rangle.$$

Both Γ and Γ' are quotients of Γ^u, while Γ' is a coarsening but not a quotient of Γ.

A left module M over a G-graded associative algebra A is called *G-graded* if M is a G-graded vector space and

$$A_g M_h \subset M_{gh} \text{ for all } g, h \in G.$$

A G-graded left A-module M is called *graded-simple* if M has no graded submodules different from $\{0\}$ and M. Graded modules and graded-simple modules over a graded Lie algebra L are defined in the same way.

If a Lie algebra L is graded by an abelian group G, then its universal enveloping algebra $U(L)$ is also G-graded. Every graded L-module is a graded left $U(L)$-module and *vice versa*. The same is true for graded-simple modules.

A very general problem is the following: given a module V over a G-graded Lie algebra L, determine if V can be given a G-grading that is compatible with the G-grading on L, i.e., one that makes V a graded L-module. In this paper, we restrict ourselves to the case where L is a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic zero and focus on simple L-modules.

For finite-dimensional V, the answer is given in [12], where the authors classified finite-dimensional graded-simple modules up to isomorphism and, as a corollary, determined which finite-dimensional simple modules can be made graded and which finite-dimensional modules can be made graded-simple. The classification depends on the so-called graded Brauer invariants (see Subsections 4.3 and 4.4 for definitions), which were computed in [12] for all classical simple Lie algebras except D_4 and for the remaining types in [13, 8]. We note that it is difficult to obtain an explicit grading on V using this approach.

If we do not restrict ourselves to finite-dimensional modules, the first question that arises is that, in general, there is no classification of simple modules of arbitrary dimension for any simple Lie algebra, with the exception of $L = \mathfrak{sl}_2(\mathbb{C})$, for which a classification was suggested by R. Block [7]. Despite this, in a number of more recent papers, the authors still try to give a more transparent description of simple $\mathfrak{sl}_2(\mathbb{C})$-modules. We refer the reader to the monograph [17]; some other works in this area are [2, 5, 12, 16, 18, 19].

We start this paper by reviewing the criteria of [12, 13, 8] for the existence of a compatible grading on a finite-dimensional simple module V. Then we focus on the case $L = \mathfrak{sl}_2(\mathbb{C})$, where we give explicit gradings for those V that admit them.

After this we switch to infinite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-modules. We review their construction and determine, for some of these modules, whether they can be made graded or not.

Finally, we turn our attention to reviewing the main results of [14]. Therein, it is described how the so-called loop construction could be used for the classification of graded-simple modules of arbitrary dimension. It should be noted that, even in the case $L = \mathfrak{sl}_2(\mathbb{C})$, this classification remains an interesting open problem.

2. Finite-dimensional simple modules over finite-dimensional simple Lie algebras

Let L be a finite-dimensional simple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0 and suppose L is graded by an abelian group G. In this section, we will give necessary and sufficient conditions for the finite-dimensional simple L-module $V(\lambda)$ of highest weight λ to admit a structure of G-graded L-module.

All G-gradings on L are known (see e.g. the monograph [11, Ch. 3–6]): they have been classified up to isomorphism for all types except E_6, E_7 and E_8, and for these latter, the fine gradings have been classified ([9, 22, 10]), which gives a description of all G-gradings as follows. Every G-grading Γ on L is a coarsening of at least one fine grading Δ, so Γ is induced by a homomorphism $\nu : G^u \to G$, where G^u is the universal group of Δ. In other words, Γ is obtained by assigning the degree $\nu(s) \in G$ to all nonzero elements of L that are homogeneous of degree $s \in G^u$ with respect to Δ. The isomorphism problem for G-gradings on L of types E_6, E_7 and E_8 remains open.

Let $\widehat{G}$ be the group of characters of G, i.e., group homomorphisms $\chi : G \to \mathbb{F}^\times$. If W is a G-graded vector space then $\widehat{G}$ acts on W as follows:

$$\chi \cdot w = \chi(g)w \quad \forall \chi \in \widehat{G},\ g \in G,\ w \in W_g \tag{3}$$

(extended by linearity). For the given G-grading on the Lie algebra L, such action defines a homomorphism $\widehat{G} \to \mathrm{Aut}(L)$ sending $\chi \mapsto \alpha_\chi$ where $\alpha_\chi(x) := \chi \cdot x$ for all $x \in L$. The grading is called *inner* if all α_χ belong to the group of inner automorphisms $\mathrm{Int}(L)$, otherwise it is called *outer*. Let τ_χ be the image of α_χ in the outer automorphism group $\mathrm{Out}(L) := \mathrm{Aut}(L)/\mathrm{Int}(L)$.

Fixing a Cartan subalgebra and a system of simple roots $\alpha_1, \dots, \alpha_r$ for L, we may identify $\mathrm{Out}(L)$ with the group of automorphisms of the Dynkin diagram of L, which permutes $\alpha_1, \dots, \alpha_r$ and hence acts on the lattice of integral weights. Let

$$K_\lambda = \{\chi \in \widehat{G} : \tau_\chi(\lambda) = \lambda\} \text{ and } H_\lambda = K_\lambda^\perp := \{h \in G : \chi(h) = 1 \ \forall \chi \in K_\lambda\}.$$

Observe that $|H_\lambda| = [\widehat{G} : K_\lambda]$ is the size of the $\widehat{G}$-orbit of λ. The nontriviality of H_λ is the first obstruction for $V(\lambda)$ becoming a G-graded L-module (see [12, §3.1]).

We denote the fundamental weights of L by $\pi_1, \dots, \pi_r$ and write $\lambda = \sum_{i=1}^r m_i \pi_i$, $m_i \in \mathbb{Z}_{\geq 0}$. Our numbering of the simple roots is shown for each type of L on the diagrams below. In all cases, $V(\pi_1)$ has the lowest possible dimension among the nontrivial L-modules (which is the reason why we prefer C_2 over B_2). Let $H = H_{\pi_1}$. We have $|H| \leq 2$ for types A_r $(r \geq 2)$ and E_6, $|H| \leq 3$ for D_4, and $|H| = 1$ for all other types.

Consider the homomorphism $\varrho_\lambda : U(L) \to E := \mathrm{End}_{\mathbb{F}}(V(\lambda))$ associated to the L-action on $V(\lambda)$. It turns out that there is a unique G/H_λ-grading on the simple associative algebra E such that ϱ_λ becomes a homomorphism of graded algebras (see [12, §3.2]). For this grading on E, there exist a graded-division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module $\mathcal{V}$ such that E is isomorphic to $\mathrm{End}_{\mathcal{D}}(\mathcal{V})$ as a G-graded algebra (see e.g. [11, Theorem 2.6]), where $\mathcal{D}$ is unique up to graded isomorphism and $\mathcal{V}$ up to graded isomorphism and shift of grading (see e.g. [11, Theorem 2.10]). Here, a *graded-division algebra* is a graded unital associative algebra in which every nonzero homogeneous element is invertible, and the *shift of grading* by an element $g \in G$ replaces a G-graded vector space W with $W^{[g]}$, which equals W as a vector space, but the elements that had degree g' will now have degree $g'g$, for any $g' \in G$. The graded-division algebra $\mathcal{D}$ represents the *graded Brauer invariant* of $V(\lambda)$, and its nontriviality is the second obstruction for $V(\lambda)$ becoming a G-graded L-module (see [12, §3.2]). A generalization of this analysis is outlined in Subsections 4.3 and 4.4 below, following [14].

Group gradings on classical simple Lie algebras were classified by studying $\mathcal{D}$ and $\mathcal{V}$ associated to the 'natural module' $V(\pi_1)$. Since $\mathcal{D}$ is a graded-division algebra, we can find a $\mathcal{D}$-basis $\{v_1, \dots, v_k\}$ of $\mathcal{V}$ that consists of homogeneous elements. Let $g_1, \dots, g_k$ be the degrees of the basis elements. If H is nontrivial, we will write $\bar{g}_1, \dots, \bar{g}_k$ to remind ourselves that these degrees belong to G/H. Let T be the support of $\mathcal{D}$, which is a finite subgroup of G/H. Pick any nonzero elements X_t of $\mathcal{D}_t$, $t \in T$. Note that all homogeneous components of $\mathcal{D}$ are one-dimensional, because $\mathcal{D}_e = \mathbb{F}1$ (being a finite-dimensional division algebra over the algebraically closed field $\mathbb{F}$) and hence $\mathcal{D}_t = \mathcal{D}_e X_t = \mathbb{F}X_t$. Hence,

$$X_s X_t = \beta(s,t) X_t X_s \quad \forall s,t \in T, \tag{4}$$

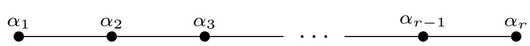
where $\beta : T \times T \to \mathbb{F}^\times$ is an alternating bicharacter, i.e.,

$$\beta(s_1 s_2, t) = \beta(s_1,t)\beta(s_2,t),\ \beta(t, s_1 s_2) = \beta(t,s_1)\beta(t,s_2),\ \text{and } \beta(t,t) = 1$$

for all $s_1, s_2, t \in T$. Bicharacters are analogous to bilinear forms, so we are using the same terminology. In particular, the *radical* of β is the subgroup $\{s \in T : \beta(s,t) = 1\ \forall t \in T\}$. Since the algebra $\mathrm{End}_{\mathbb{F}}(V(\pi_1))$ is central simple, so is $\mathcal{D}$, and hence the radical of β must be trivial. Alternating bicharacters with trivial radical are said to be *nondegenerate*. They admit a 'symplectic basis' (see e.g. [11, Ch. 2, §2]), which implies that there exist subgroups P and Q of T such that $T = P \times Q$, the restrictions of β to these subgroups are trivial, and the mapping $P \to \widehat{Q}$ sending $p \mapsto \beta(p,\cdot)$ is an isomorphism. Therefore, $|T| = \ell^2$ where $\ell = |P| = |Q|$. Note that in our case ℓ is the degree of the matrix algebra $\mathcal{D}$, hence $k\ell = n := \dim V(\pi_1)$.

The bicharacter β is clearly independent of the choice of the elements X_t. Even though the k-tuple $(g_1, \ldots, g_k)$ depends on the choice of the basis $\{v_1, \ldots, v_k\}$, the multiset $\Xi := \{g_1 T, \ldots, g_k T\}$ in G/T is uniquely determined by $\mathcal{V}$. T, β and Ξ are among the parameters that define the grading on L up to isomorphism. Some other parameters will be introduced later as needed.

$A_r \quad (r \geq 1)$



For this type $n = r+1$. Note that if $r \geq 2$ then there are two possibilities for π_1, which lead to L-modules that are dual to one another.

We have $|H| = 1$ if the grading on L is inner and $|H| = 2$ if it is outer. In the latter case, the grading determines a nondegenerate homogeneous φ_0-sesquilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$, where φ_0 is an orthogonal involution on the G/H-graded matrix algebra $\mathcal{D}$ (see [11, Ch. 2, §4 and Ch. 3, §1]). The existence of φ_0 implies that T is an elementary 2-group, so ℓ is a power of 2. The degree $\overline{g}_0 \in G/H$ of B is another parameter of the grading on L. If n is even, set

$$g_{\Xi,\bar{g}_0} := \begin{cases} \overline{g}_0^{\,n/2} (\overline{g}_1 \cdots \overline{g}_k)^\ell & \text{if } \ell \neq 2, \\ (\overline{c}\overline{g}_0)^{n/2} (\overline{g}_1 \cdots \overline{g}_k)^\ell & \text{if } \ell = 2, \end{cases} \tag{5}$$

where, for $\ell = 2$, $\bar{c}$ is the unique element of T such that $\varphi_0(X_{\bar{c}}) = -X_{\bar{c}}$.

Theorem 2.1. [12, Corollaries 16 and 24] *Suppose a simple Lie algebra L of type A_r is given a G-grading with parameters T, β, Ξ and, if the grading is outer, also $\overline{g}_0 \in G/H$ as described above. Consider the finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^r m_i \pi_i$.*

I If the grading on L is inner, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if the number $\sum_{i=1}^{r} im_i$ is divisible by the exponent of the group T.

II If the grading on L is outer (hence $r \geq 2$), then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if the following two conditions are satisfied:

1) $m_i = m_{r+1-i}$ for all i;

2) either r is even or r is odd and at least one of the following holds:

(i) $m_{(r+1)/2}$ is even, or

(ii) $r \equiv 3 \pmod 4$ and $g_{\Xi,\bar{g}_0}$ is the trivial element of G/H, or

(iii) $r \equiv 1 \pmod 4$, $|T| = 1$, and $g_{\Xi,\bar{g}_0}$ is the trivial element of G/H,

where $g_{\Xi,\bar{g}_0}$ is defined by Equation (5). ☑

$B_r \quad (r \geq 3)$

For this type $n = 2r + 1$ is odd and $|H| = 1$. The existence of an involution on $\mathcal{D}$ implies that T is an elementary 2-group, so ℓ is a power of 2 dividing n, hence $\ell = 1$, $k = n$ and $\mathcal{D} = \mathbb{F}$. The grading on L determines a nondegenerate homogeneous symmetric bilinear form $B : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$, which may be assumed to have degree e (at the expense of shifting the grading on $\mathcal{V}$, see [11, Ch. 3, §4]). This implies that the multiset $\Xi = \{g_1, \dots, g_n\}$ is 'balanced' in the sense that, for any $g \in G$, the multiplicities of g and g^{-1} in Ξ are equal to one another. We order the n-tuple $(g_1, \dots, g_n)$ so that $g_i^2 = e$ for $1 \leq i \leq q$ and $g_i^2 \neq e$ for $i > q$, where $1 \leq q \leq n$ and q is odd. For $i = 1, \dots, q$, set

$$\tilde{g}_i := g_1 \cdots g_{i-1} g_{i+1} \cdots g_q. \tag{6}$$

Then $\tilde{g}_i^2 = e$ and $\tilde{g}_1 \cdots \tilde{g}_q = e$. Consider the group homomorphism $f_\Xi : \widehat{G} \to \mathbb{Z}_2^q$ given by

$$f_\Xi(\chi) := (x_1, \dots, x_q) \text{ where } \chi(\tilde{g}_i) = (-1)^{x_i}. \tag{7}$$

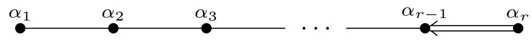
It determines the graded Brauer invariant of the *spin module* $V(\omega_r)$ ([12, §5]), but here we only state the following:

Theorem 2.2. [12, Corollary 29] *Suppose a simple Lie algebra L of type B_r is given a G-grading with parameter Ξ as described above. The finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^{r} m_i \pi_i$ admits a G-grading making it a graded L-module if and only if at least one of the following holds:*

(i) m_r is even, or

(ii) the elements $\tilde{g}_1, \dots, \tilde{g}_q$ of G defined by Equation (6) and the homomorphism $f_\Xi : \widehat{G} \to \mathbb{Z}_2^q$ defined by Equation (7) have the following property: for any $x \in f_\Xi(\widehat{G})$, $\tilde{g}_1^{x_1} \cdots \tilde{g}_q^{x_q} = e$. ☑

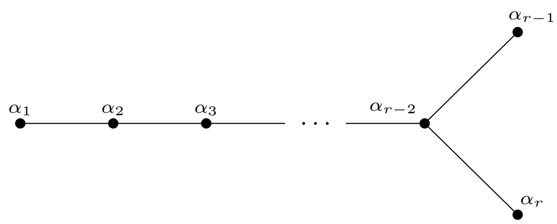
$C_r \quad (r \geq 2)$



For this type $n = 2r$, $|H| = 1$, and again the existence of an involution on $\mathcal{D}$ implies that T is an elementary 2-group.

Theorem 2.3. [12, Corollary 32] *Suppose a simple Lie algebra L of type C_r is given a G-grading with parameter T as described above. The finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^{r} m_i\pi_i$ admits a G-grading making it a graded L-module if and only if either $|T| = 1$ or $\sum_{i=1}^{\lfloor (r+1)/2 \rfloor} m_{2i-1}$ is even.*

$D_r \quad (r \geq 4)$



For this type $n = 2r$ and, unless $r = 4$, $|H| = 1$. For type D_4, we have $|H| \leq 3$ and we can avoid the case $|H| = 2$: if $\widehat{G}$ interchanges two of the outer vertices of the Dynkin diagram, we label by 1 the fixed outer vertex.

Assume that the grading on L is inner. Then $|H| = 1$ and the grading determines a nondegenerate homogeneous φ_0-hermitian form $B : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$, where φ_0 is an orthogonal involution on the G-graded matrix algebra $\mathcal{D}$. Let $g_0 \in G$ be the degree of B. The existence of φ_0 again implies that T is an elementary 2-group, so ℓ is a power of 2 dividing n.

We need to take a closer look at φ_0. Since it preserves degree and all components of $\mathcal{D}$ are one-dimensional, we have

$$\varphi(X_t) = \beta(t)X_t \quad \forall t \in T$$

where $\beta : T \to \{\pm 1\}$, and Equation (4) shows that $\beta(st) = \beta(s)\beta(t)\beta(s,t)$ for all $s, t \in T$, i.e., $\beta(\cdot)$ is a quadratic form with polar form $\beta(\cdot,\cdot)$ if we regard T as a vector space over the field $\mathbb{Z}_2$. Moreover, this quadratic form has Arf invariant 0 because φ_0 is orthogonal.

The multiset $\Xi = \{g_1T, \ldots, g_kT\}$ is 'g_0-balanced' in the following sense: if g' and g'' in G satisfy $g_0g'g'' \in T$ then $g'T$ and $g''T$ have the same mutliplicity in Ξ. We order the k-tuple $(g_1, \ldots, g_k)$ so that $g_0g_i^2 \in T$ for $1 \leq i \leq q$ and $g_0g_i^2 \notin T$ for $i > q$, where $0 \leq q \leq k$ and q has the same parity as k. The cases $\ell = 1$, $\ell = 2$, $\ell = 4$, and $\ell > 4$ require different computations to find the graded Brauer invariants of the *half-spin modules* $V(\pi_{r-1})$ and $V(\pi_r)$ (see [12, §7.3]), so we consider these cases separately. If $q = 0$, the invariants are trivial, so we assume $q \geq 1$.

$\boxed{\ell = 1}$ This case is similar to type B_r: $k = n$, $\mathcal{D} = \mathbb{F}$, and we may assume $g_0 = e$ at the expense of shifting the grading on $\mathcal{V}$ (see [12, Remark 42]). For $i = 1, \dots, q$, we have $g_i^2 = e$, and it can be shown that $g_1 \cdots g_q = e$. Consider the group homomorphism $f_\Xi : \widehat{G} \to \mathbb{Z}_2^q$ given by

$$f_\Xi(\chi) := (x_1, \dots, x_q) \text{ where } \chi(g_i) = (-1)^{x_i}. \tag{8}$$

It determines the graded Brauer invariants of the half-spin modules, which in this case are equal to one another.

In all remaining cases, these invariants are distinct (although related), and the grading on L can be used to define a specific nonscalar central element of the spin group (see [12, §7.3]), whose action determines the designation of one of the half-spin modules as S^+ and the other as S^-. For $i = 1, \dots, q$, set

$$t_i := g_0 g_i^2.$$

These elements of T determine the canonical form of $B : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$ and satisfy $\beta(t_i) = 1$ for all i.

$\boxed{\ell = 2}$ Write $T = \{e, a, b, c\} \simeq \mathbb{Z}_2^2$ where $\beta(a) = \beta(b) = 1$ and $\beta(c) = -1$, so $t_i \in \{e, a, b\}$. For any $t \in T$, define

$$I_t = \{1 \le i \le q : t_i = t\}.$$

Then $I_c = \varnothing$ and the sets I_e, I_a and I_b form a partition of $\{1, \dots, q\}$. It can be seen that $|I_e|$, $|I_a|$ and $|I_b|$ have the same parity as r. Set

$$g_a = g_0^{(|I_e|+|I_a|)/2} \prod_{i \in I_e \cup I_a} g_i \quad \text{and} \quad g_b = g_0^{(|I_e|+|I_b|)/2} \prod_{i \in I_e \cup I_b} g_i. \tag{9}$$

These elements determine the graded Brauer invariant of S^+ and hence of S^-.

$\boxed{\ell = 4}$ Recall that T has a 'symplectic basis': $T = \langle a_1, a_2, b_1, b_2 \rangle \simeq \mathbb{Z}_2^4$ where $\beta(a_i, b_j) = (-1)^{\delta_{ij}}$ and the values of $\beta(\cdot, \cdot)$ on the remaining pairs of basis elements are equal to 1. We choose the basis in such a way that $\beta(a_j) = \beta(b_j) = 1$ for $j = 1, 2$ (in other words, with respect to the quadratic form $\beta(\cdot)$, the subgroups $\langle a_1, a_2 \rangle$ and $\langle b_1, b_2 \rangle$ are totally isotropic). Then the following 4×4 matrix with entries in $\mathbb{Z}_2$ determines the graded Brauer invariant of S^+:

$$M^+_{\Xi, g_0} = \sum_{i=1}^q M^+(t_i), \tag{10}$$

where, for any $t = a_1^{x_1} a_2^{x_2} b_1^{y_1} b_2^{y_2}$, the symmetric matrix $M^+(t)$ is given by

$$M^+(t) = \begin{bmatrix} 0 & (x_1+1)(x_2+1) & 0 & (x_1+1)(y_2+1) \\ & 0 & (x_2+1)(y_1+1) & 1 \\ & & 0 & (y_1+1)(y_2+1) \\ \text{sym} & & & 0 \end{bmatrix}.$$

$\boxed{\ell > 4}$ In this case, the graded Brauer invariant of S^+ is trivial.

Theorem 2.4. [12, Corollaries 47 and 49] and [13, Corollary 24] *Suppose a simple Lie algebra L of type D_r is given a G-grading and consider the finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^{r} m_i\pi_i$.*

I If the grading on L is inner, with parameters T, β, Ξ and g_0 as described above, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if one of the following conditions is satisfied:

1) $|T| = 1$ and at least one of the following holds:

(i) $m_{r-1} \equiv m_r \pmod 2$, or

(ii) the elements $g_1, \dots, g_q$ and the homomorphism $f_\Xi : \widehat{G} \to \mathbb{Z}_2^q$ defined by Equation (8) have the following property: for any $x \in f_\Xi(\widehat{G})$, $g_1^{x_1} \cdots g_q^{x_q} = e$;

2) $|T| > 1$, $m_{r-1} \equiv m_r \pmod 2$ and one of the following holds:

(i) r is even and $\sum_{i=1}^{r/2} m_{2i-1}$ is even, or

(ii) r is odd and $\sum_{i=1}^{(r-1)/2} m_{2i-1} - (m_{r-1} - m_r)/2$ is even;

3) $m_{r-1} \not\equiv m_r \pmod 2$, r is even, $\sum_{i=1}^{r/2} m_{2i-1}$ is even, and one of the following holds:

(i) $|T| = 4$ and the elements g_a and g_b defined by Equation (9) belong to T, or

(ii) $|T| = 16$ and the matrix M^+_{Ξ,g_0} defined by Equation (10) is 0, or

(iii) $|T| > 16$,

where in 3) we assume that the numbering of the simple roots is chosen so that $V(\pi_r) = S^+$.

II If the grading on L is outer and, in the case $r = 4$, the $\widehat{G}$-action is not transitive on the outer vertices of the Dynkin diagram, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if the following two conditions are satisfied:

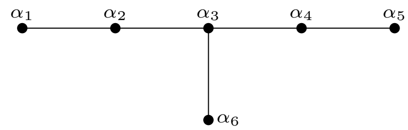
1) $m_{r-1} = m_r$;

2) $|T| = 1$ or $\sum_{i=1}^{\lfloor r/2\rfloor} m_{2i-1}$ is even;

where in the case $r = 4$ we assume that the numbering of the simple roots is chosen so that π_1 is fixed by $\widehat{G}$.

III If $r = 4$ and the $\widehat{G}$-action is transitive on the outer vertices of the Dynkin diagram, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if $m_1 = m_3 = m_4$. ☑

E_6



For this type the dimension of $V(\pi_1)$ is 27 (there are two possibilities for π_1, which lead to dual modules), and we have $|H| = 1$ if the grading on L is inner and $|H| = 2$ if it is outer.

Out of the 14 fine gradings on L (up to equivalence), 5 are inner, with universal groups $\mathbb{Z}^6$, $\mathbb{Z}^2 \times \mathbb{Z}_3^2$, $\mathbb{Z}^2 \times \mathbb{Z}_2^3$, $\mathbb{Z}_3^4$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$, and 9 are outer, with universal groups $\mathbb{Z}^4 \times \mathbb{Z}_2$, $\mathbb{Z}^2 \times \mathbb{Z}_2^3$, $\mathbb{Z} \times \mathbb{Z}_2^5$, $\mathbb{Z} \times \mathbb{Z}_2^4$, $\mathbb{Z}_2 \times \mathbb{Z}_3^3$, $\mathbb{Z}_2^7$, $\mathbb{Z}_2^6$, $\mathbb{Z}_4^3$ and $\mathbb{Z}_4 \times \mathbb{Z}_2^4$.

For each of the inner fine gradings on L with $G^u = \mathbb{Z}^2 \times \mathbb{Z}_3^2$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$ and $\mathbb{Z}_3^4$, there is a distinguished subgroup $T \simeq \mathbb{Z}_3^2$ of G^u, which is associated to the graded Brauer invariant of $V(\pi_1)$. For all other fine gradings, this invariant is trivial (see [8, §4]).

Theorem 2.5. [8, Corollaries 4.2 and 4.5] *Suppose a simple Lie algebra L of type E_6 is given a G-grading induced by a homomorphism $\nu : G^u \to G$ from one of the fine gradings. Consider the finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^6 m_i\pi_i$.*

I If the grading on L is inner, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if one of the following conditions is satisfied:

1) G^u is not one of the groups $\mathbb{Z}^2 \times \mathbb{Z}_3^2$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$ and $\mathbb{Z}_3^4$;

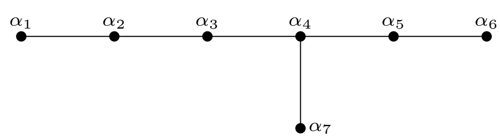
2) G^u is $\mathbb{Z}^2 \times \mathbb{Z}_3^2$, $\mathbb{Z}_2^3 \times \mathbb{Z}_3^2$ or $\mathbb{Z}_3^4$ and at least one of the following holds:

(i) $m_1 - m_2 + m_4 - m_5 \equiv 0 \pmod 3$, or

(ii) ν is not injective on the distinguished subgroup $T \subset G^u$.

II If the grading on L is outer, then $V(\lambda)$ admits a G-grading making it a graded L-module if and only if $m_1 = m_5$ and $m_2 = m_4$. ☑

E_7



For this type the dimension of $V(\pi_1)$ is 56 and we have $|H| = 1$. There are 14 fine gradings on L (up to equivalence), with universal groups $\mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$, $\mathbb{Z} \times \mathbb{Z}_3^3$, $\mathbb{Z}_2^2 \times \mathbb{Z}_3^3$, $\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_4^2$, $\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$, $\mathbb{Z}_2 \times \mathbb{Z}_4^3$, $\mathbb{Z}^4 \times \mathbb{Z}_2^2$, $\mathbb{Z}^2 \times \mathbb{Z}_2^4$, $\mathbb{Z} \times \mathbb{Z}_2^5$, $\mathbb{Z} \times \mathbb{Z}_2^6$, $\mathbb{Z}_2^7$, $\mathbb{Z}_2^5 \times \mathbb{Z}_4$ and $\mathbb{Z}_2^8$.

For the fine gradings on L with $G^u = \mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ and $\mathbb{Z} \times \mathbb{Z}_3^3$, the graded Brauer invariant of $V(\pi_1)$ is trivial. For each of the remaining fine gradings, this invariant gives a distinguished subgroup $T \simeq \mathbb{Z}_2^2$ of G^u (see [8, §5]).

Theorem 2.6. [8, Corollary 5.7] *Suppose a simple Lie algebra L of type E_7 is given a G-grading induced by a homomorphism $\nu : G^u \to G$ from one of the fine gradings. The finite-dimensional simple L-module $V(\lambda)$ of highest weight $\lambda = \sum_{i=1}^{7} m_i \pi_i$ admits a G-grading making it a graded L-module if and only if one of the following conditions is satisfied:*

1) *G^u is $\mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ or $\mathbb{Z} \times \mathbb{Z}_3^3$;*
2) *G^u is not one of the groups $\mathbb{Z}^7$, $\mathbb{Z}^3 \times \mathbb{Z}_2^3$ and $\mathbb{Z} \times \mathbb{Z}_3^3$ and at least one of the following holds:*
   - *(i)* $m_1 + m_3 + m_7 \equiv 0 \pmod 2$, *or*
   - *(ii) ν is not injective on the distinguished subgroup $T \subset G^u$.* ☑

For the remaining types, the algebraic group $\mathrm{Aut}(L)$ is connected and simply connected, which implies that every dominant integral weight λ is fixed by $\widehat{G}$ and the graded Brauer invariant of $V(\lambda)$ is trivial (see [13, Appendix A]).

Theorem 2.7. [13, Corollary 22] *Suppose a simple Lie algebra L of type E_8, F_4 or G_2 is given a G-grading. Then any finite-dimensional L-module admits a G-grading making it a graded L-module.* ☑

3. Group gradings of $\mathfrak{sl}_2(\mathbb{C})$-modules

In this section we restrict our attention to modules over the Lie algebra of type A_1, which can be realized as $\mathfrak{sl}_2(\mathbb{C})$.

3.1. Group gradings of $\mathfrak{sl}_2(\mathbb{C})$

All group gradings on $\mathfrak{sl}_2(\mathbb{C})$ are well-known, see e.g [11]. We will use the following bases:

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{11}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{12}$$

Up to equivalence, there are precisely two fine gradings on $\mathfrak{sl}_2(\mathbb{C})$ (see [11, Theorem 3.55]):

(1) *the Cartan grading* with the universal group $\mathbb{Z}$,

$$\Gamma^1_{\mathfrak{sl}_2} : \mathfrak{sl}_2(\mathbb{C}) = L_{-1} \oplus L_0 \oplus L_1 \text{ where } L_0 = \langle h \rangle,\ L_1 = \langle x \rangle,\ L_{-1} = \langle y \rangle;$$

(2) *the Pauli grading* with the universal group $\mathbb{Z}_2^2$,

$$\Gamma^2_{\mathfrak{sl}_2} : \mathfrak{sl}_2(\mathbb{C}) = L_{(1,0)} \oplus L_{(0,1)} \oplus L_{(1,1)}$$

where

$$L_{(1,0)} = \langle A \rangle \,, \, L_{(0,1)} = \langle B \rangle \,, \, L_{(1,1)} = \langle C \rangle \,.$$

Hence, up to isomorphism, any G-grading on $\mathfrak{sl}_2(\mathbb{C})$ is a coarsening of one of the two gradings: Cartan or Pauli.

Note that any grading Γ of a Lie algebra L uniquely extends to a grading $U(\Gamma)$ of its universal enveloping algebra $U(L)$. The grading $U(\Gamma)$ is a grading in the sense of associative algebras but also as L-modules where $U(L)$ is either a (left) regular L-module or an adjoint L-module. In our study of gradings on $\mathfrak{sl}_2(\mathbb{C})$-modules we will often consider a $\mathbb{Z}_2$-coarsening of $U(\Gamma^2_{\mathfrak{sl}_2})$, in which the component of the coarsening labeled by 0 is the sum of components of the original grading labeled by $(0,0)$ and $(1,0)$ while the component labeled by 1 is the sum of components labeled by $(0,1)$ and $(1,1)$.

3.2. Algebras $U(I_\lambda)$

Let $c \in U(\mathfrak{sl}_2(\mathbb{C}))$ be the Casimir element for $\mathfrak{sl}_2(\mathbb{C})$. With respect to the basis $\{h, x, y\}$ of $\mathfrak{sl}_2(\mathbb{C})$, this element can be written as

$$c = (h+1)^2 + 4yx = h^2 + 1 + 2xy + 2yx. \tag{13}$$

It is well-known that the center of $U(\mathfrak{sl}_2(\mathbb{C}))$ is the polynomial ring $\mathbb{C}[c]$. Note that c is a homogeneous element of degree zero, with respect to the Cartan grading of $U(\mathfrak{sl}_2(\mathbb{C}))$. One can write the Casimir element with respect to the basis $\{h, B, C\}$ of $\mathfrak{sl}_2(\mathbb{C})$.

Namely,

$$\begin{aligned} c = 2xy + 2yx + h^2 + 1 &= 2\left(\frac{B+C}{2}\right)\left(\frac{B-C}{2}\right) \\ &+ 2\left(\frac{B-C}{2}\right)\left(\frac{B+C}{2}\right) + h^2 + 1 \\ &= \frac{1}{2}(B^2 + CB - BC - C^2) + \frac{1}{2}(B^2 + BC - CB - C^2) + h^2 + 1, \end{aligned}$$

and so

$$c = B^2 - C^2 + h^2 + I = A^2 + B^2 - C^2 + 1.$$

It follows that c is also homogeneous, of degree (0,0), with respect to the Pauli grading of $U(\mathfrak{sl}_2(\mathbb{C}))$.

Let R be an associative algebra (or just an associative ring), and V be a left R-module. The annihilator of V, denoted by $\mathrm{Ann}_R(V)$, is the set of all elements r in R such that, for all v in V, $r.v = 0$:

$$\mathrm{Ann}_R(V) = \{r \in R \mid r.v = 0 \text{ for all } v \in V\}.$$

Given $\lambda \in \mathbb{C}$, let I_λ be the two-side ideal of $U(\mathfrak{sl}_2(\mathbb{C}))$, generated by the central element $c - (\lambda + 1)^2$.

Theorem 3.1. [17, Theorem 4.7] *For any simple $U(\mathfrak{sl}_2(\mathbb{C}))$-module M, there exists $\lambda \in \mathbb{C}$ such that $I_\lambda \subset \mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(M)$.*

Proposition 3.2. *Let R be a graded algebra and M be a graded R-module, then $\mathrm{Ann}_R(M)$ is graded.*

Proof. Let $I = \mathrm{Ann}_R(M) = \{x \in R \mid x.M = 0\}$, and $0 \neq x \in I \subseteq R$, then $x = x_1 + x_2 + \cdots x_k$, where x_i are homogeneous elements in R (belonging to different homogeneous components). Let $v \in M$ be an arbitrary homogeneous element, then $0 = x.v = x_1.v + x_2.v + \cdots x_k.v$. Since the components $x_i.v$ belong to different homogeneous subspaces, it follows that $x_i.v = 0$ for all i. and since v is an arbitrary homogeneous element, then $x_i \in I$ for all i. ☑

Proposition 3.3. *The ideal I_λ is both $\mathbb{Z}$ - and $\mathbb{Z}_2^2$-graded ideal.*

Proof. Since $c - (\lambda+1)^2$ is homogeneous of degree 0 (resp., (0,0)) with respect to the $\mathbb{Z}$-grading (resp., $\mathbb{Z}_2^2$- grading), then I_λ is graded. ☑

Now for any $\lambda \in \mathbb{C}$, we write $U(I_\lambda) := U(\mathfrak{sl}_2(\mathbb{C}))/I_\lambda$. Using Proposition 3.3, $U(I_\lambda)$ is a $\mathbb{Z}$-graded algebra and $\mathbb{Z}_2^2$-graded algebra. It is well-known (see e.g. [17]) that the algebra $U(I_\lambda)$ is a free $\mathbb{C}[h]$-module with basis

$$\mathcal{B}_0 = \left\{1, x, y, x^2, y^2, \ldots\right\},$$

and so it is free over $\mathbb{C}$ with basis $\mathcal{B} = \left\{1, h, h^2, \ldots\right\}.\mathcal{B}_0$. Note that the basis $\mathcal{B}$ is a basis of $U(I_\lambda)$ consisting of homogeneous elements with respect to the Cartan grading by $\mathbb{Z}$. A basis of $U(I_\lambda)$ over $\mathbb{C}$ consisting of homogeneous elements with respect to the Pauli grading by $\mathbb{Z}_2^2$ can be computed as follows. Set

$$\widehat{B}_0 = \{1, B, C, BC, B^2, B^2C, B^3, B^3C, \ldots\}.$$

Then easy calculations, using induction by the natural filtration in B and the relation $C^2 = h^2 + B^2 - \lambda^2 - 2\lambda$ show that the set $\widehat{B} = \{1, h, h^2, \ldots\} \cdot \widehat{B}_0$ is a $\mathbb{Z}_2^2$-homogeneous basis of $U(I_\lambda)$.

Let $p(t) = \frac{1}{4}((\lambda^2 + 2\lambda) - 2t - t^2) \in \mathbb{C}[t]$. Then, inside $U(I_\lambda)$, for any $q(t) \in \mathbb{C}[t]$, we have the following relations:

$$\begin{aligned} x^k q(h) &= q(h - 2k)x^k \\ y^j q(h) &= q(h + 2j)y^j. \end{aligned}$$

If $k \geq j$ then

$$x^k y^j = p(h-2k)\cdots p(h-2(k-j+1))x^{k-j}$$
$$y^j x^k = p(h+2(j-1))\cdots p(h)x^{k-j}.$$

If $j \geq k$ then

$$x^k y^j = p(h-2k)\cdots p(h-2)y^{j-k}$$
$$y^j x^k = p(h+2(j-1))\cdots p(h+2(j-k))y^{j-k}.$$

Moreover, $U(I_\lambda)$ is a generalized Weyl algebra (see e.g [4]) and has the following properties.

Theorem 3.4. [17, Theorem 4.15]

(1) $U(I_\lambda)$ is both left and right Noetherian.

(2) $U(I_\lambda)$ is a domain.

(3) The algebra $U(I_\lambda)$ is simple for all $\lambda \in \mathbb{C} \setminus \mathbb{Z}$.

(4) For every $n \in \mathbb{N}_0$, the algebra $U(I_n)$ has a unique proper ideal.

One more property that is important for us is the following.

Theorem 3.5. [17, Theorem 4.26] *For any non-zero left ideal $I \subset U(I_\lambda)$, the $U(I_\lambda)$-module $U(I_\lambda)/I$ has finite length.*

3.3. Weight modules over $\mathfrak{sl}_2(\mathbb{C})$

Let V be an $\mathfrak{sl}_2(\mathbb{C})$-module, $\mathfrak{h} = \langle h \rangle$ be the Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. Since $\dim(\mathfrak{h}) = 1$, we can think of $\mathfrak{h}^*$ as $\mathbb{C}$. We call

$$V_\mu = \{v \in V \mid h.v = \mu v\}, \text{ for } \mu \in \mathbb{C},$$

the *weight spaces* for V, and if V_μ is nontrivial we call $\mu \in \mathbb{C}$ the *weight* of V. If V is the direct sum of these weight spaces, we say that V is *a weight module.* The set of all weights is called the *support* of V, denoted $\mathrm{Supp}(V)$. In the case of a weight module, if $\lambda \in \mathrm{Supp}(V)$ and $\lambda+2 \notin \mathrm{Supp}(V)$, λ is called the *highest weight* of V, and the elements of the space V_λ are called *highest weight vectors.* Similarly, if $\lambda \in \mathrm{Supp}(V)$ and $\lambda - 2 \notin \mathrm{Supp}(V)$, then λ is called the *lowest weight* and the elements of the space V_λ are called *lowest weight vectors.* If the weight module is generated by v_λ, where v_λ is a highest (resp., lowest) weight vector, then V is called *highest (resp., lowest) weight module* of weight λ.

Lemma 3.6. *Any h-invariant subspace of a weight $\mathfrak{sl}_2(\mathbb{C})$-module is spanned by weight vectors.*

Proof. Let V be a weight $\mathfrak{sl}_2(\mathbb{C})$-module and W an h-invariant subspace of V. Let $w \in W \subset V$, so $w = v_1 + v_2 + \cdots + v_k$, where v_i is a nonzero weight vector of weight $\mu_i \in \mathbb{C}$, for all $i = 1, 2, \ldots, k$, where we may assume that $\mu_1, \mu_2, \ldots, \mu_k$ are distinct. Define the elements $h_i \in U(\mathfrak{h})$, $i = 1, \ldots, k$, by

$$h_i = \prod_{l \neq i} (h - \mu_l).$$

Then

$$h_i.v_j = \begin{cases} 0 & \text{if } i \neq j; \\ \prod_{l \neq i} (\mu_i - \mu_l) v_i & \text{if } i = j. \end{cases}$$

Hence,

$$W \ni h_i.w = \sum_{j=1}^{k} h_i.v_j = h_i.v_i = \prod_{l \neq i} (\mu_i - \mu_l) v_i,$$

which means that $v_i \in W$. ☑

3.3.1. Simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules

Let $V = V(n)$ be a finite - dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-module of dimension $n+1$, with a highest weight vector $v_0 \in V_n$ and highest weight n. Define $v_i = \frac{1}{i!} y^i.v_0$ for $i = 0, 1, \ldots, n$. This is a basis of V. It is convenient to set $v_{-1} = 0$. The module action is given by

$$\begin{aligned} h.v_i &= (n - 2i) v_i, \\ x.v_i &= (n - (i-1)) v_{i-1}, \\ y.v_i &= (i+1) v_{i+1}, \end{aligned} \tag{14}$$

hence

$$V(n) = V_n \oplus V_{n-2} \oplus \ldots.. \oplus V_{-(n-2)} \oplus V_{-n}.$$

Note that any finite-dimensional simple $\mathfrak{sl}_2(\mathbb{C})$-module is a highest weight module of weight $n = \dim(V) - 1$, see e.g [15, 17].

3.3.2. Verma modules of $\mathfrak{sl}_2(\mathbb{C})$

The general construction for the Verma modules over a semisimple Lie algebra L is given by the following: consider $B(\Delta) = \mathfrak{h} \oplus N$ be the standard Borel subalgebra of the semisimple Lie algebra L, where $\mathfrak{h}$ is the Cartan subalgebra of L, Δ is the basis of the root system of L with respect to $\mathfrak{h}$, and N the sum of the positive root spaces. For any $\lambda \in \mathfrak{h}^*$, start with a 1-dimensional $B(\Delta)$-module, say D_λ, with trivial N-action and $\mathfrak{h}$ acting through λ, and set $Z(\lambda) = U(L) \otimes_{U(B(\Delta))} D_\lambda$. Then $Z(\lambda)$ is a $U(L)$-module called *the Verma module of weight* λ. In the case of $L = \mathfrak{sl}_2(\mathbb{C})$, we have $B(\Delta) = \langle h, x \rangle$ and $N = \langle x \rangle$. In view of the general Definition of the Verma module, Verma $\mathfrak{sl}_2(\mathbb{C})$-module of highest weight $\lambda \in \mathbb{C}$, is

$$Z(\lambda) = U(\mathfrak{sl}_2(\mathbb{C})) \otimes_{U(B(\Delta))} D_\lambda.$$

In [17], Mazorchuk introduces the Verma $\mathfrak{sl}_2(\mathbb{C})$-module explicitly, he just uses the mathematical induction to generalize from the case of simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules to the Verma $\mathfrak{sl}_2(\mathbb{C})$-modules, and takes $v_i = \frac{1}{i!} y^i.v_0$, for $i \in \mathbb{N}_0$. Then

$$Z(\lambda) = \langle v_0, v_1, v_2, \ldots \rangle$$

and the action is given by the formulas (14). Thus,

$$Z(\lambda) = \bigoplus_{i \in \mathbb{N}_0} V_{\lambda - 2i},$$

where $V_{\lambda-2i} = \mathbb{C}v_i$.

The module $Z(\lambda)$ is a simple $\mathfrak{sl}_2(\mathbb{C})$-module if and only if $\lambda \notin \mathbb{N}_0$. If n is a non-negative integer, then $Z(n)$ is indecomposable and has a unique nontrivial submodule $Z(-n-2)$, with $V(n) \cong Z(n)/Z(-n-2)$. It is well-known see e.g. [17] that I_λ is the annihilator of the Verma module $Z(\lambda)$.

3.3.3. Anti-Verma modules of $\mathfrak{sl}_2(\mathbb{C})$

Let V be the formal vector space with the basis $\{v_i \mid i \in \mathbb{N}_0\}$. Now set $v_{-1} = 0$ and define the action on V for $\lambda \in \mathbb{C}$ as:

$$\begin{aligned} h.v_i &= (\lambda + 2i)v_i, \\ x.v_i &= v_{i+1}, \\ y.v_i &= -i(\lambda + i - 1)v_{i-1}, \end{aligned} \tag{15}$$

then V is a lowest weight $\mathfrak{sl}_2(\mathbb{C})$-module with lowest weight λ, denoted by $\overline{Z}(\lambda)$ and called *anti-Verma module*.

The support of the anti-Verma module is

$$\mathrm{Supp}(\overline{Z}(\lambda)) = \{\lambda + 2i \mid i \in \mathbb{N}_0\}$$

and the Casimir element acts on it as the scalar $(\lambda - 1)^2$. The module $\overline{Z}(\lambda)$ is a simple $\mathfrak{sl}_2(\mathbb{C})$-module if and only if $-\lambda \notin \mathbb{N}_0$. If n is a negative integer, then $\overline{Z}(n)$ has a unique maximal submodule $\overline{Z}(-n+2)$, with $V(n) \cong \overline{Z}(n)/Z(-n+2)$.

3.3.4. Dense modules of $\mathfrak{sl}_2(\mathbb{C})$

A weight $\mathfrak{sl}_2(\mathbb{C})$-module is called a *dense module* if it has no highest nor lowest weights. In other words, the weight module V is dense if $\mathrm{Supp}(V) = \lambda + 2\mathbb{Z}$ for some $\lambda \in \mathbb{C}$. Now we will study a big class of the dense modules.

For $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$, consider V to be the formal vector space with the basis $\{v_\mu \mid \mu \in \xi\}$. Define the action on V as:

$$\begin{aligned} h.v_\mu &= \mu v_\mu, \\ x.v_\mu &= \tfrac{1}{4}(\tau - (\mu+1)^2)v_{\mu+2}, \\ y.v_\mu &= v_{\mu-2}, \end{aligned} \tag{16}$$

then V is a dense weight $\mathfrak{sl}_2(\mathbb{C})$-module, denoted by $V(\xi,\tau)$. In this case the module $V(\xi,\tau)$ is simple if and only if $\tau \neq (\lambda+1)^2$ for all $\lambda \in \xi$, but if the module $V(\xi,\tau)$ is not simple, then it contains a unique maximal submodule isomorphic to a Verma module for some highest weight.

Theorem 3.7. [17, Theorem 3.32] *Up to isomorphism, any simple weight* $\mathfrak{sl}_2(\mathbb{C})$*-module is one of the following modules*

(1) $V(n)$ *for some* $n \in \mathbb{N}$.

(2) $Z(\lambda)$ *for some* $\lambda \in \mathbb{C}\backslash\mathbb{N}_0$.

(3) $\bar{Z}(-\lambda)$ *for some* $\lambda \in \mathbb{C}\backslash\mathbb{N}_0$

(4) $V(\xi,\tau)$ *for some* $\xi \in \mathbb{C}/2\mathbb{Z}$ *and* $\tau \in \mathbb{C}$, *with* $\tau \neq (\lambda+1)^2$ *for all* $\lambda \in \xi$. □

Proposition 3.8. [17] *Let* $J_n := \mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(V(n))$, *where* $V(n)$ *is a finite-dimensional simple* $\mathfrak{sl}_2(\mathbb{C})$*-module. Then*

(1) $I_n \subset J_n$.

(2) $\mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(\overline{Z}(\lambda)) = I_{\lambda-2}$.

(3) Let $\xi \in \mathbb{C}/2\mathbb{Z}$ *and* $\tau = (\lambda+1)^2 \in \mathbb{C}$, *then* $\mathrm{Ann}_{U(\mathfrak{sl}_2(\mathbb{C}))}(V(\xi,\tau)) = I_\lambda$. □

3.4. Torsion-free modules over $\mathfrak{sl}_2(\mathbb{C})$

Definition 3.9. Let M be an $\mathfrak{sl}_2(\mathbb{C})$-module, then the module M is called *torsion* if for any $m \in M$ there exists non-zero $p(t) \in \mathbb{C}[t]$ such that $p(h).m = 0$. The module M is *torsion-free* if $M \neq 0$ and $p(h).m \neq 0$ for all $0 \neq m \in M$ and all non-zero $p(t) \in \mathbb{C}[t]$. If M a torsion-free $\mathbb{C}[h]$-module of rank n, we say that M is of *rank n*.

Theorem 3.10. [17, Theorem 6.3] *A simple* $\mathfrak{sl}_2(\mathbb{C})$*-module is either a weight or a torsion-free module.* □

Theorem 3.10 means that if h has at least one eigenvector on M, then M is a weight module.

As a consequence of Theorem 3.1, it is sufficient to describe simple torsion-free $U(I_\lambda)$-modules instead of simple $U(\mathfrak{sl}_2(\mathbb{C}))$-modules (see e.g [17]).

A further reduction can be achieved as follows. We consider the field of rational functions in h, $\mathbb{K} = \mathbb{C}(h)$, and set $\mathbb{A}$ to be the *algebra of skew Laurent polynomials* over $\mathbb{K}$, that is

$$\mathbb{A} = \mathbb{K}[X, X^{-1}, \sigma] = \left\{ \sum_{i \in \mathbb{Z}} q_i(h) X^i \mid q_i(h) \in \mathbb{K}, \text{ almost all } q_i(h) = 0 \right\},$$

with the usual addition and scalar multiplication, and the product

$$(\sum_{i \in \mathbb{Z}} p_i(h) X^i)(\sum_{j \in \mathbb{Z}} q_j(h) X^j) = \sum_{i,j \in \mathbb{Z}} p_i(h) \sigma^i(q_j(h)) X^{i+j},$$

where $\sigma(h) = h - 2$. Note that $\mathbb{A}$ is an Euclidean domain and it is isomorphic to $S^{-1}U(I_\lambda)$, the localization of the generalized Weyl algebra $U(I_\lambda)$, where $S = \mathbb{C}[h] \setminus \{0\}$. An embedding of $\Phi_\lambda : U(I_\lambda) \to \mathbb{A}$ is the unique extension of the following map:

$$\Phi_\lambda(h) = h, \; \Phi_\lambda(x) = X, \; \Phi_\lambda(y) = \frac{(\lambda+1)^2 - (h+1)^2}{4} X^{-1}.$$

Thanks to this embedding, $\mathbb{A}$ becomes a $\mathbb{A} - U(I_\lambda)$-bimodule and given an $U(I_\lambda)$-module M one can define an $\mathbb{A}$-module $\mathcal{F}(M)$ by

$$\mathcal{F}(M) = \mathbb{A} \underset{U(I_\lambda)}{\otimes} M.$$

Theorem 3.11. [17, Theorem 6.24] *The following are true.*

(i) The functor $\mathcal{F}$ induces a bijection $\widehat{\mathcal{F}}$ between the isomorphism classes of simple torsion-free $U(I_\lambda)$-modules to the set of isomorphism classes of simple $\mathbb{A}$-modules;

(1) The inverse of the bijection from (i) is the map that sends a simple $\mathbb{A}$-module N to its $U(I_\lambda)$-socle $\mathrm{soc}_{U(I_\lambda)} N$.

Theorem 3.12. [4, Proposition 3] *Let M be a simple torsion-free $U(I_\lambda)$-module, them $M \cong U(I_\lambda)/(U(I_\lambda) \cap \mathbb{A}\alpha)$, for some $\alpha \in U(I_\lambda)$ which is irreducible as an element of $\mathbb{A}$.*

Many examples of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules have been introduced, see e.g [17, 16, 19]. We will highlight those of them for which we can decide if those modules are graded or not.

Let us define a family of $U(I_\lambda)$-modules modules, as follows. Given two polynomials $p(t), g(t) \in \mathbb{C}[t]$, we set

$$M(p(t), g(t), \lambda) := U(I_\lambda)/U(I_\lambda)(g(h)x + p(h)$$

and

$$M'(p(t), g(t), \lambda) := U(I_\lambda)/U(I_\lambda)(g(h)y + p(h).$$

Theorem 3.13. [17, Theorem 6.50] *Let $\lambda \in \mathbb{C}$, and $g(t), p(t)$ be non-zero polynomials in $\mathbb{C}[t]$, such that if $r \in \mathbb{C}$ is a root of $p(t)$ then*

(1) $r + n$ is not a root for $g(t)$ for all $n \in \mathbb{Z}$.

(2) $(\lambda + 1)^2 \neq (r + n + 1)^2$ for all $n \in \mathbb{Z}$.

Then the $U(I_\lambda)$-modules $M(p(t), g(t), \lambda)$ and $M'(p(t), g(t), \lambda)$ are simple. □

The so called Whittaker modules are a special case of Theorem 3.13. They are defined as follows:

Definition 3.14. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{C}$, then the *Whittaker modules* are the modules $M_\alpha = U(I_\lambda)/U(I_\lambda)(1 - \alpha x) = U(I_\lambda)/U(I_\lambda)(1 - \frac{\alpha}{2}B - \frac{\alpha}{2}C)$.

A full description of torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 (over $\mathbb{C}[h]$) was given in [19].

Definition 3.15. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$. Let us define an $\mathfrak{sl}_2(\mathbb{C})$-module $N(\alpha, \beta)$ as a vector space $\mathbb{C}[h]$ equipped with the following action: for $f(h) \in \mathbb{C}[h]$

$$\begin{aligned} h.f(h) &= hf(h), \\ x.f(h) &= \alpha(\tfrac{h}{2} + \beta)f(h-2), \\ y.f(h) &= -\tfrac{1}{\alpha}(\tfrac{h}{2} - \beta)f(h+2). \end{aligned} \tag{17}$$

Note that $N(\alpha, \beta)$ is simple if and only if $2\beta \notin \mathbb{N}_0$, see [19].

Definition 3.16. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) \geq -\frac{1}{2}$. Let us define an $\mathfrak{sl}_2(\mathbb{C})$-module $N'(\alpha, \beta)$ as a vector space $\mathbb{C}[h]$ equipped with the following action: for $f(h) \in \mathbb{C}[h]$

$$\begin{aligned} h.f(h) &= hf(h), \\ x.f(h) &= \alpha f(h-2), \\ y.f(h) &= -\tfrac{1}{\alpha}(\tfrac{h}{2} + \beta + 1)(\tfrac{h}{2} - \beta)f(h+2). \end{aligned} \tag{18}$$

Definition 3.17. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$, with $\mathrm{Re}(\beta) \geq -\frac{1}{2}$. Let us define an $\mathfrak{sl}_2(\mathbb{C})$-module $\bar{N}(\alpha, \beta)$ as a vector space $\mathbb{C}[h]$ equipped with the following action: for $f(h) \in \mathbb{C}[h]$

$$\begin{aligned} h.f(h) &= -hf(h), \\ x.f(h) &= \tfrac{1}{\alpha}(\tfrac{h}{2} + \beta + 1)(\tfrac{h}{2} - \beta)f(h+2), \\ y.f(h) &= -\alpha f(h-2). \end{aligned} \tag{19}$$

Note that the Whittaker modules are torsion-free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 with type $N'(\frac{1}{\alpha}, \frac{\lambda}{2})$.

Theorem 3.18. [19, Theorem 9, Lemma 12] *Each simple torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module of rank 1 is isomorphic to one of the following (pairwise non-isomorphic) modules:*

(1) $N(\alpha, \beta)$ *for some* $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $\beta \in \mathbb{C}$ *with* $2\beta \notin \mathbb{N}_0$.

(2) $N'(\alpha, \beta)$ *for some* $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $\beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta) \geq -\frac{1}{2}$.

(3) $\bar{N}(\alpha, \beta)$ *for some* $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $\beta \in \mathbb{C}$ *with* $\mathrm{Re}(\beta) \geq -\frac{1}{2}$.

3.5. Gradings on the weight modules

3.5.1. Gradings on simple finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules

It is obvious that every simple finite-dimensional module of $\mathfrak{sl}_2(\mathbb{C})$ is a weight module, i.e., it decomposes as the direct sum of weight spaces and this decomposition is a grading compatible with the Cartan grading on $\mathfrak{sl}_2(\mathbb{C})$. In [12], the authors show that the finite-dimensional simple modules with even highest weight have a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, while those ones with the odd highest weight do not. Here we will give an explicit construction of the grading in the even case.

Let $V = V(n)$ be a simple $\mathfrak{sl}_2(\mathbb{C})$-module with an even highest weight $n = 2m$ and basis $\{v_0, v_1, ..., v_n\}$. To construct a $\mathbb{Z}_2^2$-grading on V, we first define a new basis of V as follows. Set

$$e_i = v_i + v_{n-i} \text{ for all } i = 0, 1, \ldots, m,$$

and

$$d_i = v_i - v_{n-i} \text{ for all } i = 0, 1, \ldots, m-1.$$

Then $\{e_0, e_1, \ldots, e_m, d_0, d_1, \ldots, d_{m-1}\}$ is a basis of V and the module action is given as follows.

$$\begin{aligned}
h.e_i &= (n-2i)d_i \text{ for all } i = 0, 1, \ldots, m; \\
B.e_i &= \begin{cases} (n-i+1)e_{i-1} + (i+1)e_{i+1}, & \text{if } i = 0, 1, \ldots, m-1; \\ 2(m+1)e_m, & \text{if } i = m; \end{cases} \\
C.e_i &= \begin{cases} (n-i+1)d_{i-1} - (i+1)d_{i+1}, & \text{if } i = 0, 1, \ldots, m-1; \\ 2(m+1)d_{m-1}, & \text{if } i = m; \end{cases} \\
h.d_i &= (n-2i)e_i \text{ for all } i = 0, 1, \ldots, m-1; \\
B.d_i &= (n-i+1)d_{i-1} + (i+1)d_{i+1} \text{ if } i = 0, 1, \ldots, m-1; \\
C.d_i &= (n-i+1)e_{i-1} - (i+1)e_{i+1} \text{ if } i = 0, 1, \ldots, m-1.
\end{aligned}$$

Let $V_{(0,0)} = \langle e_i \mid i \text{ even}\rangle$, $V_{(0,1)} = \langle e_i \mid i \text{ odd}\rangle$, $V_{(1,0)} = \langle d_i \mid i \text{ even }\rangle$, and $V_{(1,1)} = \langle d_i \mid i \text{ odd}\rangle$. One now easily checks the following.

Proposition 3.19. *The above formulas provide a $\mathbb{Z}_2^2$-grading*

$$\Gamma : V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$$

on the highest weight module $V = V(n)$, n even, which is compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$. □

3.5.2. Gradings on Verma $\mathfrak{sl}_2(\mathbb{C})$-modules

As we mentioned above, any weight $\mathfrak{sl}_2(\mathbb{C})$-module has a grading compatible with the Cartan grading on $\mathfrak{sl}_2(\mathbb{C})$ via the weight decomposition. As a special case, we will explicitly describe the Cartan gradings on the Verma modules.

Let $\{v_0, v_1, \ldots, v_k, \ldots\}$ be a basis of $V(\lambda)$, as described in Subsection 3.3.2. Consider the canonical basis $\{x, y, h\}$ of $\mathfrak{sl}_2(\mathbb{C})$ with the Cartan grading by $\mathbb{Z}$, that is, $\deg(x) = 1$, $\deg(y) = -1$, $\deg(h) = 0$. The action of $\mathfrak{sl}_2(\mathbb{C})$ on V is the following:

$\cdot$	v_0	v_1	v_2	$\ldots$	v_k	$\ldots$
h	λv_0	$(\lambda-2)v_1$	$(\lambda-4)v_2$	$\ldots$	$(\lambda-2k)v_k$	$\ldots$
x	0	λv_0	$(\lambda-1)v_1$	$\ldots$	$(\lambda-k+1)v_{k-1}$	$\ldots$
y	v_1	$2v_2$	$3v_3$	$\ldots$	$(k+1)v_{k+1}$	$\ldots$

Let $V_{-k} = \langle v_k\rangle$ for $k = 0, 1, 2, ...$, and $V_k = \{0\}$ for $k = 1, 2, \ldots$, then the grading $V = \bigoplus_{k=0}^{\infty} V_{-k}$ makes V a graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Theorem 3.20. *Let V be a Verma $\mathfrak{sl}_2(\mathbb{C})$-module with highest weight $\lambda \in \mathbb{C} \setminus 2\mathbb{N}_0$. Then V is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Let $V = \bigoplus\limits_{\mu\in\mathbb{C}} V_\mu$, with a maximal vector $v_0 \in V_\lambda$. Then V has a basis $\{v_0, v_1, v_2, \ldots\}$ given in Subsection 3.3.2. Assume that V has a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, so it can written as $V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. Now let $V^0 = V_{(0,0)} \oplus V_{(1,0)}$, and $V^1 = V_{(0,1)} \oplus V_{(1,1)}$. The modules V^0 and V^1 are thus h-invariant, with the action of B sending V^0 to V^1 and vice versa. By Lemma 3.6, V^0 and V^1 are spanned by weight vectors. Since $V_\lambda = \mathbb{C}v_0$, we must have either $v_0 \in V^0$ or $v_0 \in V^1$.

Without loss of generality, suppose $v_0 \in V^0$ (otherwise apply the shift of grading), then $V^1 \ni B.v_0 = v_1$, so $v_1 \in V^1$. Hence $V^0 \ni B.v_1 = \lambda v_0 + 2v_2$. Since $v_0 \in V^0$ we get $v_2 \in V^0$. Again $V^1 \ni B.v_2 = (\lambda - 1)v_1 + 3v_3$, which implies $v_3 \in V^1$, and so on. We have shown that V^0 is spanned by the set

$\{v_0, v_2, v_4, \dots\}$ and V^1 by $\{v_1, v_3, v_5, \dots\}$.
Now let $0 \neq v \in V_{(0,0)} \subseteq V^0$. Then v can be written as $v = \alpha_0 v_0 + \alpha_2 v_2 + \cdots + \alpha_{2k} v_{2k}$, for some non-negative integer k, and some $\alpha_i \in \mathbb{C}$. Since $V_{(0,0)}$ is h^2-invariant, the elements

$$\begin{aligned} h^2.v &= \alpha_0 \lambda^2 v_0 + \alpha_2 (\lambda - 4)^2 v_2 + \cdots + \alpha_{2k} (\lambda - 4k)^2 v_{2k}, \\ h^4.v &= \alpha_0 \lambda^4 v_0 + \alpha_2 (\lambda - 4)^4 v_2 + \cdots + \alpha_{2k} (\lambda - 4k)^4 v_{2k}, \\ &\cdots \\ h^{2k}.v &= \alpha_0 \lambda^{2k} v_0 + \alpha_2 (\lambda - 4)^{2k} v_2 + ... + \alpha_{2k} (\lambda - 4k)^{2k} v_{2k} \end{aligned}$$

all belong to $V_{(0,0)}$. In order to use the Vandermonde's argument, we have to show that $\lambda^2, (\lambda-4)^2, \dots, (\lambda-4k)^2$ are all distinct. Assume that we have two different weights, $(\lambda-4n)$ and $(\lambda-4m)$ such that $(\lambda-4n)^2 = (\lambda-4m)^2$. Then $|\lambda - 4n| = |\lambda - 4m|$. Hence either $\lambda - 4n = \lambda - 4m$ or $\lambda - 4n = 4m - \lambda$, the first case being impossible. This means that $\lambda = 2(n+m) \in 2\mathbb{N}_0$, which is a contradiction. Hence,

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda^2 & (\lambda-4)^2 & \dots & (\lambda-4k)^2 \\ \vdots & \vdots & \dots & \vdots \\ \lambda^{2k} & (\lambda-4)^{2k} & \dots & (\lambda-4k)^{2k} \end{vmatrix} \neq 0.$$

It follows that $V_{(0,0)}$ is spanned by the weight vectors, which means that there is $v_s \in V_{(0,0)}$ for some s. Then $h.v_s = (\lambda - 2s) v_s \in V_{(1,0)}$, a contradiction. ☑

Corollary 3.21. *Let V be a Verma $\mathfrak{sl}_2(\mathbb{C})$-module with a non-negative even integer highest weight n. Then V cannot be a $\mathbb{Z}_2^2$-graded module.*

***Proof*.** Assume that V is $\mathbb{Z}_2^2$-graded module. Since the highest weight is an integer number then V is not simple and has a unique maximal submodule $Z(-n-2)$, which therefore must be a graded submodule. But $(-n-2)$ is a negative number, so we get a contradiction with Theorem 3.20. ☑

3.5.3. Gradings on Anti-Verma $\mathfrak{sl}_2(\mathbb{C})$-modules

From what we said above about $\mathbb{Z}$-gradings on the weight modules, it follows that $\overline{Z}(\lambda)$ is a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module. Let $V = \overline{Z}(\lambda)$ with the basis $\{v_0, v_1, \dots, v_k, \dots\}$. Consider the basis $\{x, y, h\}$ of $\mathfrak{sl}_2(\mathbb{C})$ with the Cartan grading by $\mathbb{Z}$, that is, $\deg(x) = 1,\ \deg(y) = -1,\ \deg(h) = 0$. The action of $\mathfrak{sl}_2(\mathbb{C})$ on V is the following:

$\cdot$	v_0	v_1	v_2	$\dots$	v_k	$\dots$
h	λv_0	$(\lambda+2)v_1$	$(\lambda+4)v_2$	$\dots$	$(\lambda+2k)v_k$	$\dots$
x	v_1	v_2	v_3	$\dots$	v_{k+1}	$\dots$
y	0	$-\lambda v_0$	$-2(\lambda+1)v_1$	$\dots$	$-k(\lambda+k-1)v_{k-1}$	$\dots$

Let $V_k = \mathbb{C}v_k$ for $k = 0, 1, 2, ...$, and $V_k = 0$ for $k = -1, -2, \dots$, then the grading $V = \bigoplus_{k=0}^{\infty} V_k$ makes V a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.

Theorem 3.22. *Let V be an anti-Verma $\mathfrak{sl}_2(\mathbb{C})$-module with lowest weight $\lambda \in \mathbb{C}$. Then V cannot be a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Let $V = \bigoplus_{k=0}^{\infty} V_k$ where $V_k = \mathbb{C}v_k$ for $k = 0, 1, 2, ...$, and $\{v_0, v_1, v_2, \dots\}$ be the basis of V. Assume that V has a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, so it can written as $V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. Now let $V^0 = V_{(0,0)} \oplus V_{(1,0)}$, and $V^1 = V_{(0,1)} \oplus V_{(1,1)}$. We have that V^0 and V^1 are thus h-invariant, with the action of B and C sending V^0 to V^1 and vice versa. By Lemma 3.6, V^0 and V^1 are spanned by the weight vectors. Since $V_0 = \mathbb{C}v_0$, we must have $v_0 \in V^0$ or $v_0 \in V^1$.

Without loss of generality, suppose $v_0 \in V^0$ (otherwise apply the shift of grading), then $V^1 \ni B.v_0 = v_1$, so $v_1 \in V^1$. Hence $V^0 \ni B.v_1 = v_2 - \lambda v_0$. Since $v_0 \in V^0$ we get $v_2 \in V^0$. Again $V^1 \ni B.v_2 = v_3 - 2(\lambda+1)v_1$, which implies $v_3 \in V^1$, and so on. We have shown that V^0 is spanned by the set $\{v_0, v_2, v_4, \dots\}$ and V^1 by $\{v_1, v_3, v_5, \dots\}$.

Now let $0 \neq v \in V_{(0,0)} \subseteq V^0$. Then v can be written as $v = \alpha_0 v_0 + \alpha_2 v_2 + \dots + \alpha_{2k} v_{2k}$ for some non-negative integer k and some $\alpha_i \in \mathbb{C}$. But since $V_{(0,0)}$ is h^2-invariant, the elements

$$
\begin{aligned}
h^2.v &= \alpha_0 \lambda^2 v_0 + \alpha_2 (\lambda+4)^2 v_2 + \dots + \alpha_{2k}(\lambda+4k)^2 v_{2k}, \\
h^4.v &= \alpha_0 \lambda^4 v_0 + \alpha_2 (\lambda+4)^4 v_2 + \dots + \alpha_{2k}(\lambda+4k)^4 v_{2k}, \\
&\dots \\
h^{2k}.v &= \alpha_0 \lambda^{2k} v_0 + \alpha_2 (\lambda+4)^{2k} v_2 + \dots + \alpha_{2k}(\lambda+4k)^{2k} v_{2k},
\end{aligned}
$$

all belong to $V_{(0,0)}$.

Now we have two cases:

Case 1 Assume that $-\lambda \notin 2\mathbb{N}_0$. In order to use the Vandermonde's argument, we need to show that $\lambda^2, (\lambda+4)^2, \dots, (\lambda+4k)^2$ are all distinct. Assume that we have two different weights, $(\lambda+4n)$ and $(\lambda+4m)$ such that $(\lambda+4n)^2 = (\lambda+4m)^2$. Then $|\lambda+4n| = |\lambda+4m|$. Hence either $\lambda+4n = \lambda+4m$ or $\lambda+4n = -4m-\lambda$, but the first case is impossible. Therefore

$-\lambda = 2(n+m) \in 2\mathbb{N}_0$, a contradiction. Hence,

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda^2 & (\lambda+4)^2 & \dots & (\lambda+4k)^2 \\ \vdots & \vdots & \dots & \vdots \\ \lambda^{2k} & (\lambda+4)^{2k} & \dots & (\lambda+4k)^{2k} \end{vmatrix} \neq 0.$$

It follows that $V_{(0,0)}$ is spanned by the weight vectors, which means that there is $v_s \in V_{(0,0)}$ for some s. Note that $h.v_s = (\lambda+2s)v_s \in V_{(1,0)}$, which is a contradiction.

Case 2 Assume that $-\lambda \in 2\mathbb{N}_0$. Then V is not simple and has a unique maximal submodule $\overline{Z}(-\lambda+2)$. If V is graded by $\mathbb{Z}_2^2$, then the unique maximal submodule of V must be graded. However, this contradicts Case 1 since $(-(-\lambda+2)) \notin 2\mathbb{N}_0$. ☑

3.5.4. Gradings on dense $\mathfrak{sl}_2(\mathbb{C})$-modules

As usual, the weight modules are graded by $\mathbb{Z}$. Let $\xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$, and let $V = V(\xi,\tau)$ with basis $\{v_\mu \mid \mu \in \xi\}$ as in Definition 3.3.4, and consider the basis $\{x,y,h\}$ of $\mathfrak{sl}_2(\mathbb{C})$ with a Cartan grading by $\mathbb{Z}$, that is, $\deg(x) = 1,\ \deg(y) = -1,\ \deg(h) = 0$. Now, since $\xi \in \mathbb{C}/2\mathbb{Z}$ then $\xi = \lambda + 2\mathbb{Z}$ for some $\lambda \in \mathbb{C}$ and hence, for any $\mu \in \xi$, $\mu = \lambda + 2i$ for some $i \in \mathbb{Z}$. Let $V_i = \mathbb{C}v_{\lambda+2i}$, $i \in \mathbb{Z}$, then the grading $V = \bigoplus_{i\in\mathbb{Z}} V_i$ makes V a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module with $\deg(V_i) = i$.

As for the grading by $\mathbb{Z}_2^2$, some of the dense modules can be graded while some others can not.

Let us study the case where $\xi = \bar{0}$.

Proposition 3.23. *Let $\tau \in \mathbb{C}$ be such that the module $V = V(\bar{0},\tau)$ is simple, then V can be made a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

***Proof*.** Since $\xi = \bar{0}$, we can choose $\lambda = 0 \in \xi$. Then $V = \bigoplus_{i\in\mathbb{Z}} V_i$, where $V_i = \mathbb{C}v_{2i}$, being $\{v_{2i} \mid i \in \mathbb{Z}\dots\}$ the basis of V. We set $e_0 = v_0, e_{-1} = 0$ and $e_k = \frac{1}{4^k}(\prod_{j=0}^{k}(\tau-(2j-1)^2))v_{2k}+v_{-2k}$, and also $d_0 = 0$ and $d_k = \frac{1}{4^k}(\prod_{j=0}^{k}(\tau-(2j-1)^2))v_{2k}-v_{-2k}$, for $k \in \mathbb{N}$. Since V is simple, the set $\{e_0, e_1, \dots, d_1, d_2, \dots\}$ is a basis for V with a module action given by:

$$\begin{aligned} h.e_k &= 2kd_k, \\ h.d_k &= 2ke_k, \\ B.e_k &= e_{k+1} + \tfrac{1}{4}(\tau-(2k-1)^2)e_{k-1}, \\ B.d_k &= d_{k+1} + \tfrac{1}{4}(\tau-(2k-1)^2)d_{k-1}, \\ C.e_k &= d_{k+1} - \tfrac{1}{4}(\tau-(2k-1)^2)d_{k-1}, \\ C.d_k &= e_{k+1} - \tfrac{1}{4}(\tau-(2k-1)^2)e_{k-1}, \end{aligned} \tag{20}$$

Let $V_{(0,0)} = \langle e_i \mid i \text{ is even}\rangle$, $V_{(0,1)} = \langle e_i \mid i \text{ is odd}\rangle$, $V_{(1,0)} = \langle d_i \mid i \text{ is even }\rangle$, and $V_{(1,1)} = \langle d_i \mid i \text{ is odd}\rangle$. Then $\Gamma : V = \bigoplus_{g\in\mathbb{Z}_2^2} V_g$ is a $\mathbb{Z}_2^2$-grading of V making V a graded $\mathfrak{sl}_2(\mathbb{C})$-module. ☑

Theorem 3.24. *Let $\bar{0} \neq \xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$ be such that the module $V = V(\xi,\tau)$ is simple. Then V is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

***Proof*.** If $\lambda \in \xi$ then $V = \bigoplus_{k\in\mathbb{Z}} V_k$, where $V_k = \mathbb{C}v_{\lambda+2k}$, $\{v_{\lambda+2i} \mid i \in \mathbb{Z}\ldots\}$ being the basis of V given in Definition 3.3.4. Assume that V has a grading compatible with the Pauli grading on $\mathfrak{sl}_2(\mathbb{C})$, so it can written as $V = V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(1,1)}$. Now let $V^0 = V_{(0,0)} \oplus V_{(1,0)}$, and $V^1 = V_{(0,1)} \oplus V_{(1,1)}$. Then V^0 and V^1 are thus h-invariant, with the action of B and C sending V^0 to V^1 and vice versa. By Lemma 3.6, V^0 and V^1 are spanned by the weight vectors. Since $V_\lambda = \mathbb{C}v_\lambda$, we must have $v_\lambda \in V^0$ or $v_\lambda \in V^1$.

Without loss of generality, suppose $v_\lambda \in V^0$ (otherwise apply the shift of grading). Then $V^1 \ni B.v_\lambda = \frac{1}{4}(\tau - (\lambda+1)^2)v_{\lambda+2} + v_{\lambda-2}$ and $V^1 \ni C.v_\lambda = \frac{1}{4}(\tau-(\lambda+1)^2)v_{\lambda+2} - v_{\lambda-2}$. Since V is simple, we have $(\tau-(\lambda+1)^2 \neq 0$ and hence $v_{\lambda+2}, v_{\lambda-2} \in V^1$. Now $B.v_{\lambda+2} = \frac{1}{4}(\tau - (\lambda+3)^2)v_{\lambda+4} + v_\lambda$ and $B.v_{\lambda-2} = \frac{1}{4}(\tau - (\lambda-1)^2)v_\lambda + v_{\lambda-4}$ are both in V^0. Since V is simple and $v_{\lambda \in V^0}$ then $v_{\lambda+4}, v_{\lambda-4} \in V^0$. Apply B again to $v_{\lambda+4}, v_{\lambda-4}$ to get that $v_{\lambda+6}, v_{\lambda-6} \in V^1$, and so on. We have shown that V^0 is spanned by the set $\{\ldots, v_{\lambda-8}, v_{\lambda-4}, v_\lambda, v_{\lambda+4}, v_{\lambda+8}, \ldots\}$ and V^1 by $\{\ldots, v_{\lambda-6}, v_{\lambda-2}, v_{\lambda+2}, v_{\lambda+6}, \ldots\}$.

Now let $0 \neq v \in V_{(0,0)} \subseteq V^0$. Then v can be written as $v = \alpha_{-m}v_{\lambda-4m} + \cdots + \alpha_{-1}v_{\lambda-4} + \alpha_0 v_\lambda + \cdots + \alpha_n v_{\lambda+4n}$ for some non-negative integers m, n and some $\alpha_i \in \mathbb{C}$. Since $V_{(0,0)}$ is h^2-invariant, the elements

$$
\begin{aligned}
&h^2.v = \alpha_{-m}(\lambda-4m)^2 v_{\lambda-4m} + \cdots + \alpha_0\lambda^2 v_\lambda + \cdots + \alpha_n(\lambda+4n)^2 v_{\lambda+4n},\\
&h^4.v = \alpha_{-m}(\lambda-4m)^4 v_{\lambda-4m} + \cdots + \alpha_0\lambda^4 v_\lambda + \cdots + \alpha_n(\lambda+4n)^4 v_{\lambda+4n},\\
&\qquad\qquad\cdots\\
&h^{2(m+n)}.v = \alpha_{-m}(\lambda-4m)^{2(m+n)} v_{\lambda-4m} + \cdots + \alpha_0\lambda^{2(m+n)} v_\lambda + \cdots\\
&\ + \alpha_n(\lambda+4n)^{2(m+n)} v_{\lambda+4n}
\end{aligned}
$$

are in $V_{(0,0)}$. Now, to use the Vandermonde's determinant, we have to show that $(\lambda-4m)^2, \ldots, \lambda^2, (\lambda+4)^2, \ldots, (\lambda+4n)^2$ are all distinct. Assume that we have two different weights, $(\lambda+4k_1)$ and $(\lambda+4k_2)$, such that $(\lambda+4k_1)^2 = (\lambda+4k_2)^2$, that is, $|\lambda+4k_1| = |\lambda+4k_2|$. Hence either $\lambda+4k_1 = \lambda+4k_2$ or $\lambda+4k_1 = -4k_2-\lambda$, but the first one is impossible. This means that $\lambda = -2(k_1+k_2) \in 2\mathbb{Z}$, which is not the case since $\xi \neq \bar{0}$. Hence,

$$\begin{vmatrix} 1 & \dots & 1 & \dots & 1 \\ (\lambda - 4m)^2 & \dots & \lambda^2 & \dots & (\lambda + 4n)^2 \\ \vdots & \vdots & \dots & \vdots & \\ (\lambda - 4m)^{2(m+n)} & \dots & \lambda^{2(m+n)} & \dots & (\lambda + 4n)^{2(m+n)} \end{vmatrix} \neq 0.$$

It follows that $V_{(0,0)}$ is spanned by weight vectors, which means that there is $v_{\lambda+4s} \in V_{(0,0)}$ for some $s \in \mathbb{Z}$. At the same time, $h.v_s = (\lambda + 4s)v_s \in V_{(1,0)}$, which is a contradiction. ☑

Corollary 3.25. *Let $\bar{0} \neq \xi \in \mathbb{C}/2\mathbb{Z}$ and $\tau \in \mathbb{C}$. Then the module $V = V(\xi, \tau)$ cannot be a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Theorem 3.24 covers the case where V is simple, so it is enough to prove this fact when V is non-simple. Suppose that V is a non-simple $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module; then V has a unique maximal Verma submodule (see e.g. [17, Theorem 3.29]), which has to be graded; this is a contradiction since Verma modules cannot be a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules. ☑

3.6. Gradings on torsion-free modules

Let V be a G-graded vector space, U a G-graded subspace of V, then V/U is canonically G-graded with $V/U = \bigoplus_{g\in G}(V/U)_g$, where $(V/U)_g = (V_g + U)/U$.

Lemma 3.26. *Let V be a G-graded vector space, U a subspace of V such that V/U is canonically G-graded. Then U is graded.*

Proof. Let $V = \bigoplus_{g\in G} V_g$ be G-graded, and $V/U = \bigoplus_{g\in G}(V/U)_g$, where $(V/U)_g = V_g + U/U$. Now let $u \in U$, then u can be written as $u = v_1 + v_2 + \cdots + v_m$, where $v_i \in V_{g_i}$ for some $g_i \in G$. Now

$$\begin{aligned} U &= u + U = (v_1 + v_2 + \cdots v_m) + U \\ &= (v_1 + U) + (v_2 + U) + \cdots + (v_m + U), \end{aligned}$$

but $\bar{v_i} = (v_i + U) \in (V/U)_{g_i}$, so in the algebra (V/U)

$$\overline{v_1} + \bar{v_2} + \cdots + \bar{v_m} = \bar{0},$$

and since the sum is direct, then $\overline{v_i} = \bar{0}$ for all $1 \leq i \leq m$, and hence $v_i \in U$ for all $1 \leq i \leq m$, showing that U is graded. ☑

We will study now the canonical gradings of the modules described in Theorem 3.13. These gradings depend on the degree of the polynomial $p(t)$. Since the gradings of $M(p(t), g(t), \lambda)$ and $M'(p(t), g(t), \lambda)$ are similar, we will study only one of them.

Theorem 3.27. *Let $M(p(t), g(t), \lambda)$ be as in Theorem 3.13, with $p(t) = \mu$ a non-zero constant. Then $M(p(t), g(t), \lambda)$ is not a canonically $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Suppose that the left ideal $I = U(I_\lambda)(g(h)x + \mu)$ is graded by $\mathbb{Z}_2^2$, then the element $g(h)x+\mu = g(h)(\frac{B+C}{2})+\mu = g(h)\frac{B}{2}+g(h)\frac{C}{2}+\mu$ belongs to I. The polynomial $g(h)$ has a linear combination of elements of degrees $(0,0)$ or $(1,0)$, so the term $g(h)\frac{B}{2}$ is a linear combination of elements of degrees $(0,1)$ or $(1,1)$; similarly $g(h)\frac{C}{2}$ is a linear combination of elements of degrees $(0,1)$ or $(1,1)$. Since only μ has degree $(0,0)$ it follows that $\mu \in I$ or, in other words, $1 \in I$, which means that $I = U(I_\lambda)$, so $M(p(t), g(t), \lambda)$ is trivial, a contradiction. As a result, I is not graded. Using Lemma 3.26, we conclude that $M(p(t), g(t), \lambda)$ is not canonically graded. ☑

Theorem 3.28. *Let $M(p(t), g(t), \lambda)$ be as in Theorem 3.13, with $p(t) = \mu$, a non-zero constant. Then $M(p(t), g(t), \lambda)$ is not a canonically $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Suppose that the left ideal $I = U(I_\lambda)(g(h)x + \mu)$ is graded by $\mathbb{Z}$, so that the element $g(h)x + \mu \in I$. Then the polynomial $g(h)$ has degree 0, so the term $g(h)x$ has degree 1. As before, μ is the only element of degree 0, which implies that $\mu \in I$, a contradiction. Using Lemma 3.26 again, we can see that $M(p(t), g(t), \lambda)$ is not canonically graded by $\mathbb{Z}$. ☑

Theorem 3.29. *Let $M(p(t), g(t), \lambda)$ be as in Theorem 3.13, with* $\deg p(t) \geq 1$. *Then $M(p(t), g(t), \lambda)$ is not a canonically $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Suppose that the left ideal $I = U(I_\lambda)(g(h)x + \mu)$ is graded by $\mathbb{Z}$. Since $g(h)x$ has degree 1, $p(h)$ is the only term of degree 0, which implies $p(h) \in I$, so for any nonzero generator v of $M(p(t), g(t), \lambda)$, $p(h).v = 0$. Now let we have that $p(h) = (h-\beta_1)(h-\beta_2)\cdots(h-\beta_k)$, and let $(h-\beta_j)$ be the last term with $(h-\beta_{j+1})(h-\beta_{j+2})\cdots(h-\beta_k).v \neq 0$, then $(h-\beta_j)$ annihilates a nonzero vector and hence h has an eigenvector, which implies that $M(p(t), g(t), \lambda)$ has a weight vector. So $M(p(t), g(t), \lambda)$ now is a simple $\mathbb{Z}$-graded *weight* module, which is a contradiction. ☑

Theorem 3.30. *Let $M(p(t), g(t), \lambda)$ be as in Theorem 3.13, with* $\deg p(t) \geq 1$. *Then $M(p(t), g(t), \lambda)$ is not a canonically $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Since $g(h)x$ is a linear combination of elements of degrees $(0,0)$ and $(1,1)$, and $p(h)$ is a linear combination of elements of degrees $(0,0)$ and $(1,0)$, it follows that $p(h) \in I = U(I_\lambda)(g(h)x + p(h)$. We can factor the polynomial $p(t) = (t-\beta_1)(t-\beta_2)\cdots(t-\beta_k)$, for a generator $u \in M(p(t), g(t), \lambda)$, $p(h).u = (h-\beta_1)(h-\beta_2)\cdots(h-\beta_k).u = 0$. Now let $(h-\beta_j)$ be the last term with $(h-\beta_{j+1})(h-\beta_{j+2})\cdots(h-\beta_k).u \neq 0$, which implies that $h-\beta_j$ annihilates a nonzero vector and hence h has an eigenvector, which means that $M(p(t), g(t), \lambda)$ is a weight module of $\mathfrak{sl}_2(\mathbb{C})$, a contradiction. ☑

Now we will study the gradings of the torsion-free modules of rank 1.

Lemma 3.31. *Let M be a G-graded torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module, $p(h) \in \mathbb{C}[h]$ a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$, and $v \in M$ a non-homogeneous element. Then the element $p(h).v \in M$ is non-homogeneous.*

Proof. Since $p(h)$ is homogeneous, then $p(h) \in (U(\mathfrak{sl}_2(\mathbb{C})))_g$ for some $g \in G$. Since v is non-homogeneous, then $v = v_{g_1} + v_{g_2} + \cdots + v_{g_k}$ for some $k > 1$ and $g_1, g_2, \ldots, g_k$ are distinct in G, where $v_{g_i} \in M_{g_i}$, with at least two of them non-zero (say v_{g_1}, v_{g_2} are non-zero). Now $p(h).v = p(h).v_{g_1} + p(h).v_{g_2} + \cdots + p(h).v_{g_k}$, where $p(h).v_{g_i} \in M_{g_i+g}$. But $g_1 + g, g_2 + g, \ldots, g_k + g$ are distinct in G. Since M is torsion-free, $p(h).v_{g_1}, p(h).v_{g_2}$ are non-zero, which means that $p(h).v$ is non-homogeneous. ☑

Theorem 3.32. *Torsion free $\mathfrak{sl}_2(\mathbb{C})$-modules of rank 1 cannot be $\mathbb{Z}$ or $\mathbb{Z}_2^2$-graded.*

We will prove this theorem for every kind of torsion-free module of rank 1 separately. A useful property is the following.

Proposition 3.33. *Let M be a torsion-free $\mathfrak{sl}_2(\mathbb{C})$-module, and $0 \neq v \in M$. Then one of $x.v$ or $y.v$ is non-zero.*

Proof. Assume that $x.v = 0$ and $y.v = 0$, then $0 = (xy - yx).v = h.v$, which means that $h.v = 0$, a contradiction. ☑

Proposition 3.34. *The module $N(\alpha, \beta)$, as in Definition 3.15, is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Assume that $N = N(\alpha, \beta)$ is a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module, so that $N = N_{(0,0)} + N_{(1,0)} + N_{(0,1)} + N_{(1,1)}$. Given a non-zero homogeneous element $f(h) \in N$, we define $\overline{f}(h)$ to be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h)$ can be written as the sum of a linear combination of monomials in h^{2k+1}, for $k = 0, 1, 2, \ldots$, and a linear combination of the monomials h^{2k}, for $k = 0, 1, 2, \ldots$, of degrees $(1, 0)$ and $(0, 0)$, respectively. As a result, $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2)$ of degree 0 with respect to the $\mathbb{Z}_2$-grading on $U(\mathfrak{sl}_2)$ given by

$$U(\mathfrak{sl}_2) = (U(\mathfrak{sl}_2))^0 \oplus (U(\mathfrak{sl}_2))^1,$$

where

$$(U(\mathfrak{sl}_2))^0 = (U(\mathfrak{sl}_2))_{(0,0)} \oplus (U(\mathfrak{sl}_2))_{(1,0)}$$

and

$$(U(\mathfrak{sl}_2))^1 = (U(\mathfrak{sl}_2))_{(0,1)} \oplus (U(\mathfrak{sl}_2))_{(1,1)}.$$

Since $f(h)$ is homogeneous with respect to the $\mathbb{Z}_2^2$-grading, it will be homogeneous in the coarsening grading over $\mathbb{Z}_2$, where $N = N^0 \oplus N^1$, being $N^0 = N_{(0,0)} + N_{(1,0)}$ and $N^1 = N_{(0,1)} + N_{(1,1)}$. Thus either $f(h) \in N^0$ or $f(h) \in N^1$. But $\overline{f}(h).1 = f(h)$. Since $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2(\mathbb{C}))$, with respect to the $\mathbb{Z}_2$-grading, and $f(h)$ is homogeneous in N with respect to the $\mathbb{Z}_2$-grading, using Lemma 3.31 we conclude that 1 is homogeneous in N with respect to the $\mathbb{Z}_2$-grading. Now either $1 \in N^0$ or $1 \in N^1$. Without loss of generality assume that $1 \in N^0$, which means that $N = N^0$ and N^1 is trivial. But using Proposition 3.33, we have either $B.1 \neq 0$ or $C.1 \neq 0$. These elements belong to N^1, which provides the desired contradiction. ☑

Proposition 3.35. *The module $N(\alpha, \beta)$ as in Definition 3.15 is not a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Assume that $N = N(\alpha, \beta)$ is a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module, hence $N = \bigoplus_{i \in \mathbb{Z}} N_i$. Let $f(h) \in N$ be a non-zero homogeneous element, define $\overline{f}(h)$ to be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$ of degree 0 with respect to the $\mathbb{Z}$-grading on $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h).1 = f(h)$. Since $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2)$ and $f(h)$ is homogeneous in N, it follows that 1 is homogeneous in N . Hence $1 \in N_k$ for some $k \in \mathbb{Z}$, which means that $N = N_k$ and N^i is trivial for all $i \neq k$. But using Proposition 3.33, we have either $0 \neq x.1 \in N_{k-1}$ or $0 \neq y.1 \in N_{k+1}$, a contradiction in any case. ☑

Proposition 3.36. *The module $N'(\alpha, \beta)$ as in Definition 3.16 is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Assume that $N = N'(\alpha, \beta)$ is a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module. Let $f(h) \in N$ be a non-zero homogeneous element, and define $\overline{f}(h)$ to be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2)$. It follows that $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$ of degree 0 with respect to the $\mathbb{Z}_2$-grading on $U(\mathfrak{sl}_2(\mathbb{C}))$. Then $f(h)\overline{f}(h).1 = f(h)$ is homogeneous with respect to the coarsening grading by $\mathbb{Z}_2$. Either $f(h) \in N^0$ or $f(h) \in N^1$. But $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2(\mathbb{C}))$ with respect to the $\mathbb{Z}_2$-grading, and $f(h)$ is homogeneous in N with respect to the $\mathbb{Z}_2$-grading. Using Theorem 3.31, it follows that 1 is homogeneous in N with respect to the $\mathbb{Z}_2$-grading. Hence either $1 \in N^0$ or $1 \in N^1$. Without loss of generality assume that $1 \in N^0$, which means that $N = N^0$ and N^1 is trivial. But using Proposition 3.33, we have either $0 \neq B.1 \in N^1$ or $0 \neq C.1 \in N^1$, a contradiction in both cases. ☑

Proposition 3.37. *The module $N'(\alpha, \beta)$, as in Definition 3.16, is not a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Assume that $N = N'(\alpha, \beta)$ is a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module. Let $f(h) \in N$ be a non-zero homogeneous element, and let $\overline{f}(h)$ be the same as $f(h)$ but computed in the algebra $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h)$ is a homogeneous element in $U(\mathfrak{sl}_2(\mathbb{C}))$ of degree 0 with respect to the $\mathbb{Z}$-grading on $U(\mathfrak{sl}_2(\mathbb{C}))$. Now $\overline{f}(h).1 = f(h)$, and since $\overline{f}(h)$ is homogeneous in $U(\mathfrak{sl}_2(\mathbb{C}))$ and $f(h)$ is homogeneous in N then 1 is homogeneous in N. Hence $1 \in N_k$ for some $k \in \mathbb{Z}$, which means that $N = N_k$ and N^i is trivial for all $i \neq k$. But using Proposition 3.33, we have either $0 \neq x.1 \in N_{k-1}$ or $0 \neq y.1 \in N_{k+1}$, which is a contradiction in any case. ☑

Proposition 3.38. *The module $\bar{N}(\alpha, \beta)$ is not a $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Use the arguments from the proof of Propositions 3.34 and 3.36. ☑

Proposition 3.39. *The module $\bar{N}(\alpha, \beta)$ is not a $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-module.*

Proof. Use the arguments from the proof of Propositions 3.35 3.37. ☑

In view of Theorem 3.18, the above propositions complete the proof of Theorem 3.32.

Corollary 3.40. *The Whittaker modules cannot be $\mathbb{Z}$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules.*

Corollary 3.41. *The Whittaker modules cannot be $\mathbb{Z}_2^2$-graded $\mathfrak{sl}_2(\mathbb{C})$-modules.*

3.7. Transition to graded-simple modules

In conclusion, we remark that it is easy to construct a graded $U(I_\lambda)$-module. For example one might consider the module $M = U(I_\lambda)/U(I_\lambda)\alpha$ for some homogeneous element $0 \neq \alpha \in U(I_\lambda)$. For instance, one could take $\alpha = C$, in which case also $U(I_\lambda)\alpha \neq U(I_\lambda)$. Of course, such modules need not be simple. At the same time, using Theorem 3.5, one can construct the series

$$\{0\} \neq U(I_\lambda)\alpha = J_0 \subset J_1 \subset \cdots \subset J_n = U(I_\lambda)$$

of graded left ideals, where the quotients J_{i+1}/J_i are graded-simple $\mathfrak{sl}_2$-modules. The technique developed for the study of graded-simple modules (the *loop construction*) is provided in the next section of this paper. It describes the connection between graded-simple and simple graded modules.

4. Graded-simple modules via the loop construction

Let G be an abelian group and let R be a G-graded unital associative algebra, for example, $R = U(L)$, where L is a G-graded Lie algebra. In this section, we review the relation between simple R-modules and graded-simple R-modules given by the so-called loop construction. Under some restrictions, this construction reduces the classification of graded-simple R-modules to that of gradings by certain quotient groups of G on simple R-modules.

4.1. Loop algebras and loop modules

Let $\pi : G \to \overline{G}$ be an epimorphism of abelian groups and let H be the kernel of π. Any G-graded vector space W (in particular, a G-graded algebra or module) over a field $\mathbb{F}$ can be regarded as $\overline{G}$-graded using the grading induced by π, i.e., $W_{\overline{g}} = \bigoplus_{g\in\pi^{-1}(\overline{g})} W_g$ for any $\overline{g} \in \overline{G}$, and this gives us a 'forgetful' functor from the category of G-graded vector spaces (respectively, algebras or modules) to the category of $\overline{G}$-graded vector spaces (respectively, algebras or modules). The loop construction, defined as follows, is the right adjoint of this functor (see [14, Remark 3.3]). For a given $\overline{G}$-graded vector space V, consider the tensor product $V \otimes \mathbb{F}G$, where $\mathbb{F}G$ denotes the group algebra of G with coefficients in $\mathbb{F}$. Define $L_\pi(V)$ as the following subspace of $V \otimes \mathbb{F}G$:

$$L_\pi(V) := \bigoplus_{g\in G} V_{\pi(g)} \otimes g,$$

which is naturally G-graded: $L_\pi(V)_g = V_{\pi(g)} \otimes g$.

If A is a $\overline{G}$-graded algebra (not necessarily associative) then $L_\pi(A)$ is a G-graded algebra with respect to the usual product on $A \otimes \mathbb{F}G$, defined by $(a_1 \otimes g_1)(a_2 \otimes g_2) := a_1a_2 \otimes g_1g_2$. If $\mathbb{F}$ is infinite, then $L_\pi(A)$ belongs to a given variety of algebras (for example, associative or Lie) if and only if so does A. A classical example is the so-called *twisted loop algebra* $L(\mathfrak{g}, \Gamma)$ in Lie theory: given a semisimple complex Lie algebra $\mathfrak{g}$ and a $\mathbb{Z}/m\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \bigoplus_{\bar{k}\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_{\bar{k}}$, one defines $L(\mathfrak{g}, \Gamma) := \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\bar{k}} \otimes t^k$, which is a subalgebra of $\mathfrak{g}[t, t^{-1}] := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, so in our notation $L(\mathfrak{g}, \Gamma) = L_\pi(\mathfrak{g})$, where $\pi : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is the natural homomorphism.

Similarly, if R is a G-graded associative algebra and V is a $\overline{G}$-graded left R-module (where we regard R as a $\overline{G}$-graded algebra) then $L_\pi(V)$ is a G-graded left R-module through $r(v \otimes g) := rv \otimes g'g$ for all $g, g' \in G$, $v \in V_{\pi(g)}$, $r \in R_{g'}$.

Moreover, if $\psi : V \to V'$ is a homomorphism of $\overline{G}$-graded vector spaces (respectively, algebras or modules) then the linear map $L_\pi(\psi) : L_\pi(V) \to L_\pi(V)$ that sends $v \otimes g \mapsto \psi(v) \otimes g$, for all $g \in G$ and $v \in V_{\pi(g)}$, is a homomorphism of G-graded vector spaces (respectively, algebras or modules).

If H is finite and $\mathbb{F}$ is sufficiently good then there is an alternative definition of the loop functor as follows. Recall that the group of characters $\widehat{G}$ acts on any G-graded vector space (see Equation (3)). Similarly, a $\overline{G}$-graded vector space V becomes a module over the group algebra $\mathbb{F}(H^\perp)$, where the subgroup

$$H^\perp := \{\chi \in \widehat{G} : \chi(h) = 1 \ \forall h \in H\}$$

is naturally isomorphic to the group of characters of $\overline{G}$. Assume for now that $|H| = n < \infty$ and that $\mathbb{F}$ is algebraically closed and its characteristic does not divide n. Then we have $|\widehat{H}| = n$ and, moreover, any character of H extends to a character of G. Fix such extensions, $\chi_1,\dots,\chi_n$, for all characters of H, so $\widehat{H}=\{\chi_1|_H,\dots,\chi_n|_H\}$. Then $\{\chi_1,\dots,\chi_n\}$ is a transversal of $H^\perp$ in $\widehat{G}$ (i.e., a set of coset representatives of $H^\perp$ in $\widehat{G}$), hence $\mathbb{F}\widehat{G}=\chi_1\mathbb{F}(H^\perp)\oplus\cdots\oplus\chi_n\mathbb{F}(H^\perp)$.

If V is a $\overline{G}$-graded vector space then we can consider the induced $\mathbb{F}\widehat{G}$-module,

$$I_\pi(V):=\operatorname{Ind}_{H^\perp}^{\widehat{G}}(V)=\mathbb{F}\widehat{G}\otimes_{\mathbb{F}(H^\perp)}V=\chi_1\otimes V\oplus\cdots\oplus\chi_n\otimes V,$$

which is clearly $\overline{G}$-graded, with the homogeneous component of degree $\bar g$ being $\chi_1\otimes V_{\bar g}\oplus\cdots\oplus\chi_n\otimes V_{\bar g}$. In fact, this $\overline{G}$-grading on $I_\pi(V)$ can be refined to a G-grading:

$$I_\pi(V)_g:=\{x\in\chi_1\otimes V_{\pi(g)}\oplus\cdots\oplus\chi_n\otimes V_{\pi(g)}:\chi\cdot x=\chi(g)x\ \forall\chi\in\widehat{G}\}.$$

Now, if A is a $\overline{G}$-graded algebra then $I_\pi(A)$ is a G-graded algebra with multiplication defined by $(\chi_i\otimes a')(\chi_j\otimes a''):=\delta_{ij}\chi_i\otimes a'a''$ for $1\le i,j\le n$ and $a',a''\in A$, so each of the direct summands $\chi_j\otimes A$ is a $\overline{G}$-graded ideal isomorphic to A as a $\overline{G}$-graded algebra. If R is a G-graded associative algebra and V is a $\overline{G}$-graded left R-module then $I_\pi(V)$ is a G-graded left R-module by means of

$$r(\chi_j\otimes v)=\chi_j(g')^{-1}\chi_j\otimes rv\quad\forall r\in R_{g'},\ v\in V_{\pi(g)},\ g,g'\in G,\ 1\le j\le n.\tag{21}$$

Note that the direct summands $\chi_j\otimes V$ are $\overline{G}$-graded R-submodules, but they are not necessarily isomorphic. In fact, Equation (21) tells us that, as a left R-module, $\chi_j\otimes V$ is isomorphic to V twisted by $\alpha_{\chi_j}^{-1}$, where α_χ, for any $\chi\in\widehat{G}$, are the automorphisms of R given by the action of $\widehat{G}$, and the twists of a module are defined as follows:

Definition 4.1. Given an automorphism α of R and a left R-module V, we define a new left R-module $V^\alpha=(V,*)$ which equals V as a vector space, but with the new action given by $r*v=\alpha(r)v$. This module V^α is referred to as *V twisted by α*.

It turns out that, under the above assumptions on H and $\mathbb{F}$, $I_\pi(V)$ is isomorphic to $L_\pi(V)$ as a G-graded vector space (respectively, algebra or module). An isomorphism $L_\pi(V)\to I_\pi(V)$ is given by

$$v\otimes g\mapsto\sum_{j=1}^n\chi_j(g)^{-1}\chi_j\otimes v\text{ for all }v\in V_{\pi(g)},\ g\in G,$$

it does not depend on the choice of the transversal $\{\chi_1,\dots,\chi_n\}$, and its inverse $I_\pi(V)\to L_\pi(V)$ is given by

$$\chi_j\otimes v\mapsto\frac{1}{n}\sum_{h\in H}\chi_j(gh)v\otimes gh\text{ for all }v\in V_{\pi(g)},\ g\in G,\ 1\le j\le n$$

(see [14, Proposition 3.8] for the case of R-modules).

4.2. Correspondence Theorem

The loop functor L_π associated to an epimorphism $\pi : G \to \overline{G}$, as described in the previous subsection, can be used to establish a correspondence between, on the one hand, the class $\mathfrak{A}(\pi)$ of $\overline{G}$-graded algebras that are simple and central (disregarding the grading) and, on the other hand, the class $\mathfrak{B}(\pi)$ of G-graded algebras that are graded-simple and whose centroid is isomorphic to $\mathbb{F}H$ as a graded algebra, where H is the kernel of π. This correspondence was established in [1, Theorem 7.1.1] over an arbitrary field $\mathbb{F}$, but the result is easier to state if $\mathbb{F}$ is algebraically closed (thanks to [1, Lemmas 4.3.8 and 6.3.4(v)]). Then the above condition on the centroid is equivalent to its identity component being $\mathbb{F}$ (i.e., the algebra being graded-central) and its support being H, while the Correspondence Theorem says that L_π is a functor $\mathfrak{A}(\pi) \to \mathfrak{B}(\pi)$ that gives a bijection between the isomorphism classes in these categories. (Under some restrictions, the surjectivity was already established in [3, Theorem 7].) Thus, the classification of G-graded-central-simple algebras reduces to the classification of gradings on central simple algebras by the quotient groups of G.

A similar approach works for graded modules, although with some additional difficulties arising from the fact that the centralizer of a graded-simple module, unlike the centroid of a graded-simple algebra, need not be commutative. The use of the loop construction in this context was started in [18] and the Correspondence Theorem was obtained in [14]. Before we state the result, we need to introduce some terminology and notation.

Let R be a G-graded unital associateive algebra. We denote the centralizer of a left R-module V by $C(V) := \mathrm{End}_R(V)$ and apply the elements of $C(V)$ to the elements of V on the right. Recall that a linear map $W \to W'$ of G-graded vector spaces is said to be *homogeneous of degree* g if it sends W_k to W'_{gk} for all $k \in G$. In particular, for a G-graded left R-module W, let $C(W)_g$ be the set of all elements of $C(W)$ that are homogeneous of degree g. It is clear from the definition that $C^{\mathrm{gr}}(W) := \bigoplus_{g\in G} C(W)_g$ is a G-graded algebra and W is a G-graded right $C^{\mathrm{gr}}(W)$-module. Moreover, if W is graded-simple then $C^{\mathrm{gr}}(W) = C(W)$ (see [14, Proposition 2.1]).

Note that if V is a $\overline{G}$-graded left R-module then $C^{\mathrm{gr}}(V)$ is a $\overline{G}$-graded algebra, so $L_\pi(C^{\mathrm{gr}}(V))$ is a G-graded algebra, which acts naturally on the G-graded left R-module $L_\pi(V)$:

$$(v \otimes g)(\delta \otimes g') := v\delta \otimes gg' \quad \forall v \in V_{\pi(g)},\, \delta \in C(V)_{\pi(g')},\, g, g' \in G,$$

and this action centralizes that of R. Thus, we can identify $L_\pi(C^{\mathrm{gr}}(V))$ with a G-graded subalgebra of $C^{\mathrm{gr}}(L_\pi(V))$.

The classical Schur's Lemma, which says that the centralizer of a simple module is a division algebra, has a graded analog: the centralizer of a graded-simple module is a graded-division algebra (see, for instance, [11, Lemma

2.4]), and hence the module is free over its centralizer. Commutative graded-division algebras are called *graded-fields* (not to be confused with fields that are graded!).

A module V is called *central* (or *Schurian*) if $C(V) = \mathbb{F}1$, i.e., $C(V)$ consists of the scalar multiples of the identity map. Similarly, a graded module W is called *graded-central* if $C(W)_e = \mathbb{F}1$.

We need one more concept, which is a generalization of G-grading and is called *G-pregrading* or *G-covering* (see [21] and [6]).

Definition 4.2. Let V be a left R-module.

(1) A family of subspaces $\Sigma = \{V_g : g \in G\}$ is called a *G-pregrading* on V if $V = \sum_{g\in G} V_g$ and $R_g V_k \subset V_{gk}$ for all $g, k \in G$.

(2) Given two pregradings $\Sigma^i = \{V_g^i : g \in G\}$, $i = 1, 2$, Σ^1 is said to be a *refinement* of Σ^2 (or Σ^2 a *coarsening* of Σ^1) if $V_g^1 \subset V_g^2$ for all $g \in G$. If at least one of these inclusions is strict, the refinement is said to be *proper*.

(3) A G-pregrading Σ is called *thin* if it admits no proper refinement.

Example 4.3. Let S be a subgroup of G and suppose $V = \bigoplus_{\bar{g}\in G/S} V_{\bar{g}}$ is a G/S-graded left R-module. Then the family $\Sigma := \{V'_g : g \in G\}$, where $V'_g = V_{gS}$ for all $g \in G$, is a G-pregrading on V, which will be referred to as the *G-pregrading associated to the given G/S-grading on V*.

The importance of thin coverings in our context stems from the next result:

Proposition 4.4 ([18, Lemma 27])**.** *Let $\pi : G \to G/S$ be the natural homomorphism and let V be a G/S-graded left R-module. The following are equivalent:*

(i) *$L_\pi(V)$ is G-graded-simple;*

(ii) *V is G/S-graded-simple and the G-pregrading on V associated to its G/S-grading is thin.* ☑

The Correspondence Theorem we are about to state relates the following two categories.

Definition 4.5. Fix a subgroup S of G and let $\pi : G \to \overline{G} = G/S$ be the natural homomorphism.

(1) $\mathfrak{M}(\pi)$ is the category whose objects are the simple, central, $\overline{G}$-graded left R-modules such that the G-pregrading associated to the $\overline{G}$-grading is thin, and whose morphisms are the isomorphisms of $\overline{G}$-graded modules.

(2) $\mathfrak{N}(\pi)$ is the category whose objects are the pairs $(W, \mathcal{F})$, where W is a G-graded-simple left R-module and $\mathcal{F}$ is a maximal graded-subfield of $C(W)$, which is isomorphic to the group algebra $\mathbb{F}S$ as a G-graded algebra, and the morphisms $(W, \mathcal{F}) \to (W', \mathcal{F}')$ are the isomorphism of G-graded modules $\phi : \mathcal{W} \to \mathcal{W}'$ such that $\phi\mathcal{F}\phi^{-1} = \mathcal{F}'$.

Theorem 4.6. [14, Proposition 4.5 and Theorem 4.14] *If V is an object of $\mathfrak{M}(\pi)$ then $\big(L_\pi(\mathcal{V}), L_\pi(\mathbb{F}1)\big)$ is an object of $\mathfrak{N}(\pi)$, and if $\varphi : V \to V'$ is a morphism in $\mathfrak{M}(\pi)$, then $L_\pi(\varphi)$ is a morphism in $\mathfrak{N}(\pi)$, so we have the loop functor $L_\pi : \mathfrak{M}(\pi) \to \mathfrak{N}(\pi)$. This functor has the following properties:*

(i) L_π is faithful, i.e., injective on the set of morphisms $V \to V'$, for any objects V and V' in $\mathfrak{M}(\pi)$.

(ii) L_π is essentially surjective, i.e., any object $(W, \mathcal{F})$ in $\mathfrak{N}(\pi)$ is isomorphic to $\big(L_\pi(V), L_\pi(\mathbb{F}1)\big)$ for some object V in $\mathfrak{M}(\pi)$.

(iii) If V and V' are objects in $\mathfrak{M}(\pi)$ such that their images under L_π are isomorphic in $\mathfrak{N}(\pi)$, then there is a character $\chi \in \widehat{S}$ such that V' is isomorphic to V^χ in $\mathfrak{M}(\pi)$. ☑

The definition of the twisted module V^χ, $\chi \in \widehat{S}$, is technical (see [14, Definition 4.10], which is analogous to [1, Definition 6.3.1]), but if χ can be extended to a character of G (which is guaranteed if $\mathbb{F}$ is algebraically closed) then V^χ is isomorphic to V^{α_χ}, where α_χ is the automorphism of R given by the action of the extended χ (see [14, Proposition 4.11]).

If $\mathbb{F}$ is algebraically closed, this Correspondence Theorem gives a classification of G-graded-central-simple R-modules up to isomorphism as follows. The centralizer of any such module contains a maximal graded-subfield $\mathcal{F}$ isomorphic to $\mathbb{F}S$ for some subgroup S of G (see [14, Proposition 3.5]). We partition all G-graded-central-simple modules according to the graded isomorphism class of their centralizer and, for each class, make a choice of $\mathcal{F}$ (equivalently, of S) and let $\pi : G \to \overline{G} = G/S$ be the natural homomorphism. Then for every G-graded-central-simple W with a fixed centralizer, there exists a simple, central, $\overline{G}$-graded module V such that $W \simeq L_\pi(V)$, and this V is unique up to isomorphism of $\overline{G}$-graded modules and twisting by the action of $\widehat{G}$ on R. Thus, we can obtain the classification of G-graded-central-simple modules if we know the classification of gradings on central-simple modules by the quotient groups of G. Finally, we observe that, assuming $\mathbb{F}$ is algebraically closed, the condition of graded-centrality is automatic for graded-simple modules whose dimension (as a vector space) is less than the cardinality of $\mathbb{F}$ (see [18, Theorem 14]).

4.3. Graded Brauer invariants of graded-simple modules with a semisimple finite-dimensional centralizer

The Brauer invariants that we are going to define belong to the graded version of Brauer group introduced in [20]. Given a field $\mathbb{F}$ and an abelian group G, the group $B_G(\mathbb{F})$ consists of the equivalence classes of finite-dimensional associative $\mathbb{F}$-algebras that are central, simple, and G-graded, where $A_1 \sim A_2$ if and only if there exist finite-dimensional G-graded $\mathbb{F}$-vector spaces V_1 and V_2 such that $A_1 \otimes \mathrm{End}_{\mathbb{F}}(V_1) \simeq A_2 \otimes \mathrm{End}_{\mathbb{F}}(V_2)$ as G-graded algebras. Here, unlike for some more general versions of the graded Brauer group, $A \otimes B$ denotes the usual (untwisted) tensor product of $\mathbb{F}$-algebras, equipped with the natural G-grading: $(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1a_2 \otimes b_1b_2$ and $\deg(a \otimes b) := \deg(a)\deg(b)$ for nonzero homogeneous $a \in A$ and $b \in B$. This tensor product induces a group structure on the set of equivalence classes: $[A][B] := [A \otimes B]$.

Every class $[A]$ contains a unique graded-division algebra (up to isomorphism). Indeed, recall that there exist a graded-division algebra $\mathcal{D}$ and a graded right $\mathcal{D}$-module $\mathcal{V}$ such that A is isomorphic to $\mathrm{End}_{\mathcal{D}}(\mathcal{V})$ as a G-graded algebra, where $\mathcal{D}$ is unique up to graded isomorphism and $\mathcal{V}$ up to graded isomorphism and shift of grading. Pick a $\mathcal{D}$-basis $\{v_1, \dots, v_k\}$ of $\mathcal{V}$ that consists of homogeneous elements. Let $\widetilde{\mathcal{V}} = \mathbb{F}v_1 \oplus \cdots \oplus \mathbb{F}v_k$. Then $\widetilde{\mathcal{V}}$ is a G-graded vector space, and the map

$$\widetilde{\mathcal{V}} \otimes \mathcal{D} \to \mathcal{V},\ v \otimes d \mapsto vd,$$

is a graded isomorphism. Thus we can assume $\mathcal{V} = \widetilde{\mathcal{V}} \otimes \mathcal{D}$ and hence identify

$$\mathrm{End}_{\mathcal{D}}(\mathcal{V}) \simeq \mathrm{End}_{\mathbb{F}}(\widetilde{\mathcal{V}}) \otimes \mathcal{D}.$$

Now the isomorphism $A \simeq \mathrm{End}_{\mathbb{F}}(\widetilde{\mathcal{V}}) \otimes \mathcal{D}$ implies that $\mathcal{D}$ is central simple and that $[A] = [\mathcal{D}]$, while the uniqueness of $\mathcal{D}$ mentioned above implies that $[\mathcal{D}_1] = [\mathcal{D}_2]$ if and only if $\mathcal{D}_1 \simeq \mathcal{D}_2$ as graded algebras.

In general, the graded Brauer group $B_G(\mathbb{F})$ can be complicated because it contains the classical Brauer group $B(\mathbb{F})$ as the classes of central division algebras with trivial G-grading. But if $\mathbb{F}$ is algebraically closed then, for any abelian group G, $B_G(\mathbb{F})$ is isomorphic to the group of alternating continuous bicharacters of the pro-finite group $\widehat{G_0}$, where G_0 is the torsion subgroup of G if $\mathrm{char}\,\mathbb{F} = 0$ and the p'-torsion subgroup of G if $\mathrm{char}\,\mathbb{F} = p > 0$ (i.e., the set of all elements whose order is finite and coprime with p) —see [12, §2]. By means of duality, each such bicharacter corresponds to a pair (T, β) where T is a finite subgroup of G and $\beta : T \times T \to \mathbb{F}^\times$ is a nondegenerate alternating bicharacter. This pair is connected with the corresponding unique graded-division algebra $\mathcal{D}$ as follows: T is the support of $\mathcal{D}$ and β is defined by Equation (4).

From now on, we assume that $\mathbb{F}$ is algebraically closed and restrict our attention to G-graded-simple left R-modules W such that $\dim C(W)$ is finite and not divisible by $\mathrm{char}\,\mathbb{F}$. This is necessary and sufficient to guarantee that

$\mathcal{D} := C(W)$ contains a maximal graded-subfield $\mathcal{F}$ isomorphic to $\mathbb{F}S$ where $|S|$ is finite and not divisible by char $\mathbb{F}$; it also implies that W is semisimple as an ungraded module (see [14, Corollary 5.4]). Let T be the support of $\mathcal{D}$ and let $\beta : T \times T \to \mathbb{F}^\times$ be the alternating bicharacter defined by Equation (4). It is not necessarily nondegenerate: its radical is precisely the support of the center of $\mathcal{D}$, which we denote by H. The subgroup S is a maximal isotropic subgroup of T (i.e., a maximal subgroup with the property $\beta|_{S\times S} = 1$), and it contains H (see [14, Proposition 5.3], where our H is denoted by Z and our S by H; here we follow the notation of [12]).

Definition 4.7. Assume that W is a G-graded-simple left R-module such that $\dim C(W)$ is finite and not divisible by char $\mathbb{F}$.

(1) The *inertia group* of W is $K_W := H^\perp \subset \widehat{G}$, where H is the support of the center of $\mathcal{D} := C(W)$.

(2) The *(graded) Brauer invariant* of W is the class of the G/H-graded-division algebra $\mathcal{D}\varepsilon$ in $B_{G/H}(\mathbb{F})$, where ε is any primitive central idempotent of $\mathcal{D}$.

(3) The *(graded) Schur index* of W is the degree of the matrix algebra $\mathcal{D}\varepsilon$.

We note that $\mathcal{D}\varepsilon$ is a G/H-graded-division algebra that is central simple (disregarding the grading), so $[\mathcal{D}\varepsilon]$ is indeed an element of $B_{G/H}(\mathbb{F})$, and this element does not depend on the choice of ε (see [14, Theorem 5.7]). It corresponds to the pair (T', β'), where $T' = T/H$ and β' is the nondegenerate bicharacter $T' \times T' \to \mathbb{F}^\times$ induced by β (i.e., $\beta'(sH, tH) := \beta(s,t)$ for all $s, t \in T$). The Schur index equals $|S/H| = \sqrt{|T/H|}$ and has the meaning of the multiplicity of any simple constituent of W. The number of non-isomorphic simple constituents is $|H|$, they form an orbit under the action of $\widehat{G}$ on the isomorphism classes of R-modules by twisting, and the inertia group K_W is the stabilizer of each point in this orbit (see [14, Proposition 5.12]). By the Correspondence Theorem, $W \simeq L_\pi(V) \simeq I_\pi(V)$ for some object V of $\mathfrak{M}(\pi)$, where $\pi : G \to G/S$ is the natural homomorphism. Disregarding the G/S-grading, V is isomorphic to a simple constituent of W. In fact, any of these constituents can serve as V, since they are twists of each other.

4.4. Finite-dimensional graded-simple modules

We have already seen that the inertia group of a G-graded-simple left R-module W can be expressed in terms of any (ungraded) simple constituent V of W: $K_W = K_V$, where

$$K_V := \{\chi \in \widehat{G} : V^{\alpha_\chi} \text{ is isomorphic to } V\}.$$

If W is finite-dimensional then also its Brauer invariant can be expressed in terms of V. In fact, this is the way Brauer invariants were defined in [12] (for the case $R = U(L)$, where L is a semisimple finite-dimensional Lie algebra equipped with a G-grading). We continue assuming that $\mathbb{F}$ is algebraically closed.

Theorem 4.8. [14, Corollary 6.4] *Let W be a finite-dimensional G-graded-simple left R-module such that* $\operatorname{char}\mathbb{F}$ *does not divide the dimension of $C(W)$. Let V be a simple (ungraded) submodule of W and let $\varrho_V : R \to \operatorname{End}_{\mathbb{F}}(V)$ be the associated representation. Let H be the support of the center of $C(W)$. Then there is a unique G/H-grading on* $\operatorname{End}_{\mathbb{F}}(V)$ *that makes ϱ_V a homomorphism of G/H-graded algebras. With respect to this grading, the class of* $\operatorname{End}_{\mathbb{F}}(V)$ *is precisely the Brauer invariant of W.* ☑

The G-graded-simple module W can be reconstructed from V if we compute the pair (T', β') corresponding to the unique G/H-graded-division algebra $\mathcal{D}'$ in $[\operatorname{End}_{\mathbb{F}}(V)] \in B_{G/H}(\mathbb{F})$. As mentioned in the previous subsection, the support T and bicharacter $\beta : T \times T \to \mathbb{F}^{\times}$ of the G-graded-division algebra $\mathcal{D} := C(W)$ are given by $T = (\pi')^{-1}(T')$ and $\beta = \beta' \circ (\pi' \times \pi')$, where $\pi' : G \to G/H$ is the natural homomorphism. In fact, $\mathcal{D} \simeq L_{\pi'}(\mathcal{D}')$ by [14, Remark 5.10]. Now fix any maximal isotropic subgroup S' of T' (with respect to β'), then $S := (\pi')^{-1}(S')$ is a maximal isotropic subgroup of T (with respect to β), so $\mathcal{F} := \bigoplus_{s \in S} \mathcal{D}_s$ is a maximal graded-subfield of $\mathcal{D}$ isomorphic to $\mathbb{F}S$. Hence, it follows from the Correspondence Theorem that V admits a structure of G/S-graded R-module such that V becomes an object in $\mathfrak{M}(\pi)$ and $W \simeq L_\pi(V)$, where $\pi : G \to G/S$ is the natural homomorphism.

***Remark* 4.9.** All G/S-gradings that make V a graded R-module are shifts of each other.

***Proof*.** Suppose we have two such gradings, Γ and Γ'. Since R acts on V through ϱ_V and the simple, G/S-graded algebra $\operatorname{End}_{\mathbb{F}}(V)$ admits a unique G/S-simple-graded module up to isomorphism and shift, there exist $g \in G$ and an isomorphism of G/S-graded modules $f : (V, \Gamma)^{[g]} \to (V, \Gamma')$. Forgetting the gradings, f is an element of $\operatorname{End}_R(V)$, so f is a scalar multiple of the identity map and thus $\Gamma' = \Gamma^{[g]}$. ☑

***Remark* 4.10.** W can be obtained from V by a two-step loop construction: first we get the G/H-graded module $W' := L_{\pi''}(V)$, where $\pi'' : G/H \to G/S$ is the natural homomorphism (so $\pi = \pi'' \circ \pi'$), and then $W \simeq L_{\pi'}(W')$ (see [14, p. 83]). The centralizer of W' is isomorphic to $\mathcal{D}'$ (the Brauer invariant) as a G/H-graded algebra, and V is the only simple constituent of W', with multiplicity equal to the Schur index. This two-step approach was taken in [12].

There remains the question which simple R-modules appear as simple constituents of G-graded-simple modules. Assume $\operatorname{char}\mathbb{F} = 0$.

Theorem 4.11. [14, Theorem 7.1] *A finite-dimensional simple left R-module V is isomorphic to a simple submodule of a finite-dimensional G-graded-simple left R-module if and only if the index $[\widehat{G} : K_V]$ is finite.* ☑

Thus, the loop functor gives a bijection between, on the one hand, the classes of finite-dimensional G-graded-simple R-modules under isomorphism and shift and, on the other hand, the finite $\widehat{G}$-orbits of isomorphism classes of finite-dimensional simple R-modules. (Note that W and $W^{[g]}$ are isomorphic if and only if $g \in T$.)

Knowing the structure of G-graded-simple modules allows us to determine which semisimple modules admit a G-grading that makes them graded modules because, with such a grading, the module must be isomorphic to a direct sum of graded-simple modules. Hence, assuming $\mathbb{F}$ is algebraically closed and $\operatorname{char} \mathbb{F} = 0$, a finite-dimensional semisimple R-module M admits a G-grading if and only if, for each of its simple constituents V, the $\widehat{G}$-orbit is finite and all simple modules in the orbit occur in M with the same multiplicity that is divisible by the Schur index of V.

In the case $R = U(L)$, where L is a semisimple finite-dimensional Lie algebra, all orbits are finite because V^α is isomorphic to V for any inner automorphism α of L, and the outer automorphism group is finite. The Brauer invariants of finite-dimensional simple modules for all simple finite-dimensional Lie algebras, endowed with all possible G-gradings, were computed in [12, 13, 8].

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