New analytical method for solving nonlinear time-fractional reaction-diffusion-convection problems

Nuevo método analítico para resolver problemas no lineales fraccionados por tiempo reacción-difusión-convección

ALI KHALOUTA$^{\boxtimes}$, ABDELOUAHAB KADEM

Ferhat Abbas Sétif University 1, 19000 Sétif, Algeria

ABSTRACT. In this paper, we propose a new analytical method called generalized Taylor fractional series method (GTFSM) for solving nonlinear time-fractional reaction-diffusion-convection initial value problems. Our obtained results are given in the form of a new theorem. The advantage of the proposed method compared with the existing methods is, that method solves the nonlinear problems without using linearization and any other restriction. The accuracy and efficiency of the method is tested by means of two numerical examples. Obtained results interpret that the proposed method is very effective and simple for solving different types of nonlinear fractional problems.

Key words and phrases. Nonlinear time-fractional reaction-diffusion-convection problems, Caputo fractional derivative, generalized Taylor fractional series method.

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RESUMEN. En este artículo, proponemos un nuevo método analítico denominado método generalizado de la serie fraccional de Taylor (MGSFT) para resolver problemas de valor inicial no lineales fraccionales en el tiempo de reacción-difusión-convección. Nuestros resultados obtenidos se dan en la forma de un nuevo teorema. La ventaja del método propuesto en comparación con los métodos existentes es que ese método resuelve los problemas no lineales sin utilizar la linealización y cualquier otra restricción. La precisión y la eficiencia del método se prueban mediante dos ejemplos numéricos. Los resultados obtenidos interpretan que el método propuesto es muy eficaz y simple para resolver diferentes tipos de problemas fraccionarios no lineales.

1. Introduction

In recent years, there has been a rapid development in the concept of fractional calculus and its applications [3, 4, 7, 9]. The fractional calculus which deals with derivatives and integrals of arbitrary orders [8] plays a vital role in many fields of applied science and engineering. Recently, nonlinear partial differential equations with fractional order derivatives have been successfully applied to many mathematical models in mathematical biology, aerodynamics, rheology, diffusion, electrostatics, electrodynamics, control theory, fluid mechanics, analytical chemistry and so on. In all these scientific fields, it is important to obtain exact or approximate solutions of nonlinear fractional partial differential equations (NFPDEs). But in general, there exists no method that gives an exact solution for NFPDEs and most of the obtained solutions are only approximations.

Various analytical and numerical methods have been proposed to solve the NFPDEs. The most commonly used ones are: Adomian decomposition method (ADM) [11], variational iteration method (VIM) [6], fractional difference method (FDM) [8], generalized differential transform method (GDTM) [2], homotopy analysis method (HAM) [12], homotopy perturbation method (HPM) [1].

The main objective of this paper is to conduct a new analytical method called generalized Taylor fractional series method (GTFSM) to study the solution of nonlinear time-fractional reaction-diffusion-convection initial value problems described by

$$\begin{cases} D_t^\alpha u = (a(u)u_x)_x + b(u)u_x + c(u), \\ \quad\quad u(x,0) = f_0(x), x \in \mathbb{R}, \end{cases} \tag{1}$$

where D_t^α is the Caputo fractional derivative operator of order α, $0 < \alpha \leq 1$ and $0 < t < R < 1$. $u = u(x,t)$ is an unknown function, and the arbitrary smooth functions $a(u), b(u)$ and $c(u)$ denote the diffusion term, the convection term and the reaction term respectively. The reaction-diffusion-convection problems are very useful mathematical models in applied sciences such as biology modeling, physics, chemistry, astrophysics, hydrology, medicine and engineering.

The paper is organized as follows. In Section 2, we give some necessary definitions and properties of the fractional calculus theory. In Section 3, we introduce our results to solve the nonlinear time-fractional reaction-diffusion–convection initial value problems (1) using the GTFSM. In Section 4, we present two examples to show the efficiency and effectiveness of this method. In Section 5, we discuss our obtained results represented by figures and tables. These results were verified with Matlab (version R2016a). Section 6, is devoted to the conclusions on the work.

2. Basic Definitions

In this section, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper. For more details see [8].

Definition 2.1. A real function $u(x,t), x \in I \subset \mathbb{R}, t > 0$, is considered to be in the space $C_\mu(I \times \mathbb{R}^+)$, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, so that $u(x,t) = t^p v(x,t)$, where $v \in C(I \times \mathbb{R}^+)$, and it is said to be in the space C_μ^n if $u^{(n)} \in C_\mu(I \times \mathbb{R}^+), n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of $u \in C_\mu(I \times \mathbb{R}^+), \mu \geq -1$ is defined as follows

$$I_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} u(x,\xi)d\xi, & \alpha > 0, x \in I, t > \xi \geq 0, \\ u(x,t) & \alpha = 0. \end{cases} \tag{2}$$

Definition 2.3. The Caputo time-fractional derivative operator of order $\alpha > 0$ of $u \in C_{-1}^n(I \times \mathbb{R}^+), n \in \mathbb{N}$ is defined as follows

$$D_t^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} u^{(n)}(x,\xi)d\xi, & n-1 < \alpha < n, \\ u^{(n)}(x,t) & \alpha = n. \end{cases} \tag{3}$$

For this definition we have the following properties

1)
$$D_t^\alpha(c) = 0, \text{ where } c \text{ is a constant.}$$

2)
$$D_t^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha} & \text{if } \beta > n-1, \\ 0 & \text{if } \beta \leq n-1. \end{cases}$$

Definition 2.4. The Mittag-Leffler function is defined as follows

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+1)}, \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{4}$$

A further generalization of (4) is given in the form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha+\beta)}, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0. \tag{5}$$

For $\alpha = 1$, $E_\alpha(z)$ reduces to e^z.

3. Analysis of the Method

Method 3.1. *Consider the nonlinear time-fractional reaction-diffusion-convection initial value problems in the form (1).*

Then, by GTFSM the solution of (1) is given in the form of an infinite series which converges rapidly to the exact solution as follows

$$u(x,t) = \sum_{i=0}^{\infty} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)},$$

where $c_i(x)$ are the coefficients of the series.

Proof. In order to achieve our goal, we consider the following nonlinear reaction-diffusion-convection initial value problems in the form (1).

Assume that the solution takes the following infinite series form

$$u(x,t) = \sum_{i=0}^{\infty} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}. \tag{6}$$

Consequently, the approximate solution to (1), can be written in the form of

$$u_n(x,t) = \sum_{i=0}^{n} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} = c_0(x) + \sum_{i=1}^{n} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}. \tag{7}$$

By applying the operator D_t^α on Eq. (7), and using the properties (1) and (2), we obtain the formula

$$D_t^\alpha u_n(x,t) = \sum_{i=0}^{n-1} c_{i+1}(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}. \tag{8}$$

Next, we substitute both (7) and (8) in (1). Therefore, we have the following recurrence relations

$$
\begin{aligned}
0 = &\sum_{i=0}^{n-1} c_{i+1}(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \\
&- \left(a \left(\sum_{i=0}^{n} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right) \left(\sum_{i=0}^{n} c_i'(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right) \right)_x \\
&- b \left(\sum_{i=0}^{n} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right) \left(\sum_{i=0}^{n} c_i'(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right) \\
&- c \left(\sum_{i=0}^{n} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right).
\end{aligned}
$$

We follow the same analogue used in obtaining the Taylor series coefficients. In particular, to determine the function $c_n(x), n = 1, 2, 3, ..$, we have to solve the following

$$D_t^{(n-1)\alpha} \{L(x, t, \alpha, n)\}_{t=0} = 0,$$

where

$$L(x, t, \alpha, n) = \sum_{i=0}^{n-1} c_{i+1}(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}$$

$$- \left(a \left(\sum_{i=0}^{n} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right) \left(\sum_{i=0}^{n} c_i'(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right) \right)_x$$

$$- b \left(\sum_{i=0}^{n} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right) \left(\sum_{i=0}^{n} c_i'(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right)$$

$$- c \left(\sum_{i=0}^{n} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} \right).$$

Now, we determine the first terms of the sequence $\{c_n(x)\}_1^N$. For $n = 1$ we have

$$L(x, t, \alpha, 1) = c_1(x)$$

$$- \left(a \left(c_0(x) + c_1(x) \frac{t^{\alpha}}{\Gamma(\alpha + 1)} \right) \left(c_0'(x) + c_1'(x) \frac{t^{\alpha}}{\Gamma(\alpha + 1)} \right) \right)_x$$

$$- b \left(c_0(x) + c_1(x) \frac{t^{\alpha}}{\Gamma(\alpha + 1)} \right) \left(c_0'(x) + c_1'(x) \frac{t^{\alpha}}{\Gamma(\alpha + 1)} \right)$$

$$- c \left(c_0(x) + c_1(x) \frac{t^{\alpha}}{\Gamma(\alpha + 1)} \right).$$

Solving $L(x, 0, \alpha, 1) = 0$, yields

$$c_1(x) = \left(a \left(c_0(x) \right) c_0'(x) \right)_x + b \left(c_0(x) \right) c_0'(x) + c \left(c_0(x) \right).$$

To determine $c_2(x)$, we consider $L(x, t, \alpha, 2)$ and we solve

$$D_t^{\alpha} \{L(x, t, \alpha, 2)\} \downarrow_{t=0} = 0.$$

To determine $c_3(x)$, we consider $L(x, t, \alpha, 3)$ and we solve

$$D_t^{2\alpha} \{L(x, t, \alpha, 3)\} \downarrow_{t=0} = 0,$$

and so on.

In general, to obtain the coefficient function $c_k(x)$ we solve

$$D_t^{(k-1)\alpha} \{L(x, t, \alpha, k)\} \downarrow_{t=0} = 0.$$

Finally, the solution of (1), can be expressed by

$$u(x,t) = \sum_{i=0}^{\infty} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}.$$

The proof is complete.   ☑

4. Numerical Examples

In this section, we test the validity of the proposed method to solve some nonlinear Caputo time-fractional reaction-diffusion-convection problems.

We define E_n to be the absolute error between the exact solution u and the approximate solution u_n, as follows

$$E_n(x,t) = |u(x,t) - u_n(x,t)|, n = 0, 1, 2, 3, ...$$

Example 4.1. Consider the following initial value nonlinear problem

$$\begin{cases} D_t^{\alpha} u = u_{xx} + uu_x + u - u^2, \ 0 < \alpha \leq 1, \\ u(x,0) = 1 + e^x \quad\quad\quad\quad x \in \mathbb{R}. \end{cases} \quad (9)$$

By applying the steps involved in GTFSM as presented in Section 3, we have the solution of the problem (9) is in the form

$$u(x,t) = \sum_{i=0}^{\infty} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}, t \in [0, R), x \in \mathbb{R}. \quad (10)$$

and

$$c_i(x) = e^x, \text{ for } i = 1, 2, 3, ...$$

Therefore, the solution of (9), can be expressed by

$$u(x,t) = 1 + e^x \left(1 + \frac{t^{\alpha}}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + ... \right)$$

$$(11) \quad\quad = 1 + e^x \sum_{n=0}^{\infty} \frac{(t^{\alpha})^n}{\Gamma(n\alpha + 1)} = 1 + e^x E_{\alpha}(t^{\alpha}),$$

where $E_{\alpha}(t^{\alpha})$ is the Mittag-Leffler function, defined by Eq. (4).

Taking $\alpha = 1$ in (11), the solution of (9) has the general pattern form which is coinciding with the following exact solution in terms of infinite series

$$u(x,t) = 1 + e^x \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ... \right).$$

So, the exact solution of (9) in a closed form of elementary function will be

$$u(x,t) = 1 + e^{x+t},$$

which is exactly the same solution obtained by HAM [5].

Example 4.2. Consider the following initial value nonlinear problem

$$\begin{cases} D_t^\alpha u = (uu_x)_x + 3uu_x + 2(u - u^2), 0 < \alpha \le 1, \\ u(x,0) = 2\sqrt{e^x - e^{-4x}} \qquad\qquad x \in \mathbb{R}. \end{cases} \tag{12}$$

By applying the steps involved in GTFSM as presented in Section 3, we have the solution of problem (12) in the form

$$u(x,t) = \sum_{i=0}^{\infty} c_i(x) \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)}, t \in [0, R), x \in \mathbb{R}. \tag{13}$$

and

$$c_i(x) = 2^i \sqrt{e^x - e^{-4x}}, \text{ for } i = 1, 2, 3, ...$$

Therefore, the solution of (12), can be expressed by

$$\begin{aligned} u(x,t) &= 2\sqrt{e^x - e^{-4x}} \left(1 + \frac{2t^\alpha}{\Gamma(\alpha + 1)} + \frac{2^2 t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{2^3 t^{3\alpha}}{\Gamma(3\alpha + 1)} + ... \right) \\ (14) \qquad &= 2\sqrt{e^x - e^{-4x}} \sum_{n=0}^{\infty} \frac{(2t^\alpha)^n}{\Gamma(n\alpha + 1)} = 2\sqrt{e^x - e^{-4x}} E_\alpha (2t^\alpha), \end{aligned}$$

where $E_\alpha (2t^\alpha)$ is the Mittag-Leffler function, defined by Eq. (4).

Taking $\alpha = 1$ in (14), the solution of (12) has the general pattern form which is coinciding with the following exact solution in terms of infinite series

$$u(x,t) = 2\sqrt{e^x - e^{-4x}} \left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + ... \right).$$

So, the exact solution of (12) in a closed form of elementary function will be

$$u(x,t) = 2e^{2t} \sqrt{e^x - e^{-4x}},$$

which is exactly the same solution obtained by HAM [10].

5. Numerical Results and Discussion

In this section the numerical results for both Examples 4.1 and 4.2 are presented. Figures 1 and 3 represent the surface graph of the exact solution and the approximate solution $u_4(x,t)$ at $\alpha = 0.6, 0.8, 1$. Figures 2 and 4 represents the behavior of the exact solution and the approximate solution $u_4(x,t)$ at $\alpha = 0.7, 0.8, 0.9, 1$. Tables 1 and 2 show the absolute errors between the exact solution and the approximate solution $u_4(x,t)$ at $\alpha = 1$ for different values of x and t. The numerical results afirm that when α approaches 1, our obtained results by the GTFSM approach the exact solution.
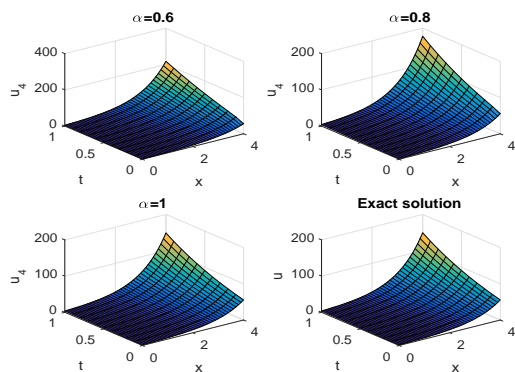
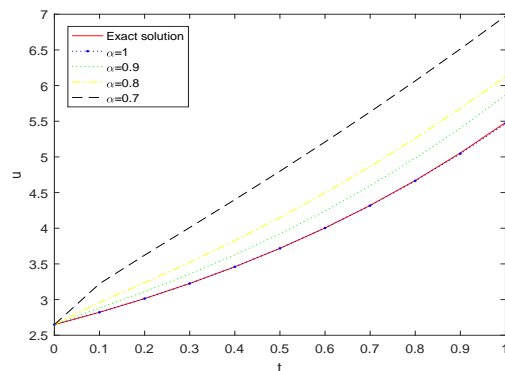FIGURE 1. The surface graph of the exact solution u and the approximate solution u_4 by GTFSM for different values of α for Example 4.1.

FIGURE 2. The behavior of the exact solution u and the approximate solution u_4 by GTFSM for different values of α for Example 4.1 when $x = 0.5$.

t/x	0.1	0.3	0.5	0.7
0.1	9.3655×10^{-8}	1.1439×10^{-7}	1.3972×10^{-7}	1.7065×10^{-7}
0.3	2.3549×10^{-5}	2.8762×10^{-5}	3.5130×10^{-5}	4.2908×10^{-5}
0.5	3.1362×10^{-4}	3.8305×10^{-4}	4.6786×10^{-4}	5.7144×10^{-4}
0.7	1.7482×10^{-3}	2.1353×10^{-3}	2.6081×10^{-3}	3.1855×10^{-3}
0.9	6.3720×10^{-3}	7.7828×10^{-3}	9.5059×10^{-3}	1.1611×10^{-2}

TABLE 1. Comparison of the absolute errors for the approximate solution $u_4(x,t)$ and the exact solution for Example 4.1, when $\alpha = 1$.
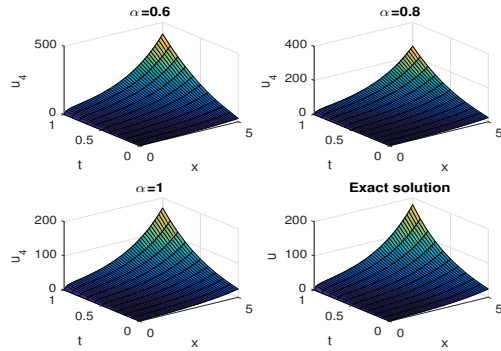
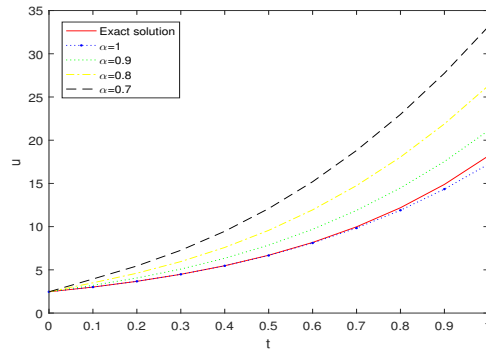FIGURE 3. The surface graph of the exact solution u and the approximate solution u_4 by GTFSM for different values of α for Example 4.2.

FIGURE 4. The behavior of the exact solution u and the approximate solution u_4 by GTFSM for different values of α for Example 4.2 when $x = 0.5$.

t/x	0.1	0.3	0.5	0.7
0.1	3.6376×10^{-6}	5.6490×10^{-6}	6.7862×10^{-6}	7.7089×10^{-6}
0.3	9.4800×10^{-4}	1.4722×10^{-3}	1.7685×10^{-3}	2.0090×10^{-3}
0.5	1.3121×10^{-2}	2.0375×10^{-2}	2.4477×10^{-2}	2.7806×10^{-2}
0.7	7.6230×10^{-2}	1.1838×10^{-1}	1.4221×10^{-1}	1.6155×10^{-1}
0.9	2.9048×10^{-1}	4.5109×10^{-1}	5.4190×10^{-1}	6.1558×10^{-1}

TABLE 2. Comparison of the absolute errors for the approximate solution $u_4(x, t)$ and the exact solution for Example 4.2, when $\alpha = 1$.

6. Conclusion

In this paper, a new analytical method called generalized Taylor fractional series method (GTFSM) is presented for finding the solution of the nonlinear time-fractional reaction-diffusion-convections problems. The method was applied to two numerical examples. The results show that the GTFSM is an efficient and easy to use technique for finding approximate and analytic solutions for these problems. The obtained approximate solutions using the suggested method is in excellent agreement with the analytic solution. This confirms our belief that the efficiency of our technique gives it much wider applicability for general classes of nonlinear fractional problems.

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Laboratory of Fundamental and Numerical Mathematics
Department of Mathematics, Faculty of Sciences
Ferhat Abbas Sétif University 1
19000 Sétif, Algeria
e-mail: nadjibkh@yahoo.fr
e-mail: abdelouahabk@yahoo.fr