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Integrals of certain Dirichlet series

Integrales de ciertas series de Dirichlet

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Abstract. We compute in closed form the integrals of certain expressions involving a class of Dirichlet series. This is a generalization of a formula of Jonathan Borwein to a problem stated (and solved) by A. Ivić.

Key words and phrases. Dirichlet Series, Integrals.

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Resumen. Calculamos en forma cerrada integrales de ciertas expresiones que involucran una clase de series de Dirichlet. Esto es una generalización de una fórmula de Jonathan Borwein a un problema enunciado (y resuelto) por A. Ivić.

Palabras y frases clave. Integrales, Series de Dirichlet.

1. Evaluation of certain integrals

We write as usual $s = \sigma + it$ with σ, t real numbers and let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ be the Riemann zeta function. A. Ivić [3] proved the following remarkable integral evaluation:

$$\int_0^{\infty} \frac{\left\{3 - 2\sqrt{2}\cos(t\ln 2)\right\} |\zeta(1/2+it)|^2}{t^2+1/4} dt = \pi \ln 2.$$

Inspired by this J. Borwein proved that the integrals of expressions involving a certain class of Dirichlet series could be evaluated [1]. If one sets

$$\lambda(s) := \sum_{1}^{\infty} \frac{\lambda_n}{n^s},$$

he proved the following result.

Theorem 1.1. (J. Borwein) *Assume that the above Dirichlet series is absolutely convergent for fixed* $\sigma > 0$ *and* λ_n *are real numbers. Then*

$$\int_0^\infty \left|\frac{\lambda(s)}{s}\right|^2 dt = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty}\frac{\Lambda_n^2-\Lambda_{n-1}^2}{n^{2\sigma}}, \tag{1}$$

where $\Lambda_n := \sum_{k=1}^{n}\lambda_k$, $\Lambda_0 := 0$.

The aim of this note is to extend (a bit) the last two results.

Theorem 1.2. *Assume that the above Dirichlet series is absolutely convergent for fixed* $\sigma > 0$ *and* λ_n *are real. Then*

$$j_\lambda(\sigma) := \int_0^\infty \left|\frac{\lambda(s)}{s^2}\right|^2 dt =$$

$$= \frac{\pi}{2\sigma^2}\sum_{n=1}^{\infty}\frac{1}{n^{2\sigma}}\left\{\frac{\Lambda_n^2-\Lambda_{n-1}^2}{2\sigma}+\lambda_n\Lambda_n\ln n-\lambda_n\sum_{m=1}^{n}\lambda_m\ln m\right\}. \tag{2}$$

Proof. Set

$$\lambda_N(s) := \sum_1^N \frac{\lambda_n}{n^s},$$

and recall that if a, b are real and positive numbers then ([2] Formula 3729 p. 410)

$$\int_0^\infty \frac{\cos ax}{(b^2+x^2)^2}dx = \frac{\pi}{4b^3}(1+ab)e^{-ab}.$$

Then

$$\begin{aligned}\int_0^\infty \frac{|\lambda_N(s)|^2}{(\sigma^2+t^2)^2}dt &= \int_0^\infty \frac{\sum_{0<n,m\le N}\lambda_n\lambda_m n^{-\sigma+it}m^{-\sigma-it}}{(\sigma^2+t^2)^2}dt\\ &= 2\sum_{0<m<n\le N}\frac{\lambda_n\lambda_m}{(nm)^\sigma}\int_0^\infty\frac{\cos(t\ln n/m)}{(\sigma^2+t^2)^2}dt+\sum_{n=1}^{N}\frac{\lambda_n^2}{n^{2\sigma}}\int_0^\infty\frac{1}{(\sigma^2+t^2)^2}dt\\ &= \frac{\pi}{2\sigma^3}\sum_{0<m<n\le N}\frac{\lambda_n\lambda_m}{(nm)^\sigma}\frac{(1+\sigma\ln n/m)}{(n/m)^\sigma}+\frac{\pi}{4\sigma^3}\sum_{n=1}^{N}\frac{\lambda_n^2}{n^{2\sigma}}.\end{aligned}$$

Now, the first sum in the last expression is equal to

$$\begin{aligned}\sum_{0<m<n\le N}\frac{\lambda_n\lambda_m}{(nm)^\sigma}\frac{(1+\sigma\ln n/m)}{(n/m)^\sigma} &= \sum_{n=1}^{N}\frac{\lambda_n}{n^{2\sigma}}\left\{\sum_{m=1}^{n}\lambda_m(1+\sigma\ln n/m)-\lambda_n\right\}\\ &= \sum_{n=1}^{N}\frac{\lambda_n(\Lambda_n-\lambda_n)}{n^{2\sigma}}+\sigma\sum_{n=1}^{N}\frac{\lambda_n}{n^{2\sigma}}\sum_{m=1}^{n}\lambda_m\ln n/m\\ &= \sum_{n=1}^{N}\frac{\lambda_n(\Lambda_n-\lambda_n)}{n^{2\sigma}}+\sigma\sum_{n=1}^{N}\frac{\lambda_n\Lambda_n\ln n}{n^{2\sigma}}-\sigma\sum_{n=1}^{N}\frac{\lambda_n}{n^{2\sigma}}\sum_{m=1}^{n}\lambda_m\ln m.\end{aligned}$$

Inserting the last expression into the above formula and observing that $\lambda_n\Lambda_n-\lambda_n^2/2=\frac{\Lambda_n^2-\Lambda_{n-1}^2}{2}$ yields

$$\int_0^\infty\frac{|\lambda_N(s)|^2}{(\sigma^2+t^2)^2}dt$$

$$=\frac{\pi}{2\sigma^3}\sum_{n=1}^{N}\frac{\lambda_n(\Lambda_n-\lambda_n)}{n^{2\sigma}}+\frac{\pi}{2\sigma^2}\sum_{n=1}^{N}\frac{\lambda_n\Lambda_n\ln n}{n^{2\sigma}}-\frac{\pi}{2\sigma^2}\sum_{n=1}^{N}\frac{\lambda_n}{n^{2\sigma}}\sum_{m=1}^{n}\lambda_m\ln m+\frac{\pi}{4\sigma^3}\sum_{n=1}^{N}\frac{\lambda_n^2}{n^{2\sigma}}$$
$$=\frac{\pi}{4\sigma^3}\sum_{n=1}^{N}\frac{\Lambda_n^2-\Lambda_{n-1}^2}{n^{2\sigma}}+\frac{\pi}{2\sigma^2}\sum_{n=1}^{N}\frac{\lambda_n\Lambda_n\ln n}{n^{2\sigma}}-\frac{\pi}{2\sigma^2}\sum_{n=1}^{N}\frac{\lambda_n}{n^{2\sigma}}\sum_{m=1}^{n}\lambda_m\ln m.$$

The last expression tends to $j_\lambda(\sigma)$ as $N\to\infty$ and yields the result. ☑

As an example if one takes $\lambda(s)=\zeta(s)$, that is $\lambda_n=1,\Lambda_n=n$, then $(1<\sigma)$

$$\int_0^\infty\left|\frac{\zeta(s)}{s^2}\right|^2dt=\frac{\pi}{2\sigma^2}\sum_{n=1}^{\infty}\frac{1}{n^{2\sigma}}\left\{\frac{2n-1}{2\sigma}+\ln\left(\frac{n^n}{n!}\right)\right\}.$$

2. A remark

We note the following result which is the analogue of Ivić's integral.

Lemma 2.1. *Set*

$$S:=\frac{\ln\{1/2\}}{2.3}+\frac{\ln\{1.3/2.4\}}{4.5}+\frac{\ln\{1.3.5/2.4.6\}}{6.7}+\cdots.$$

Then the following evaluation holds:

$$I:=\int_0^\infty\left\{3-2\sqrt{2}\cos(t\ln 2)\right\}\frac{|\zeta(1/2+it)|^2}{(1/4+t^2)^2}dt=2\pi\left(\ln 2+S\right).$$

Proof. We use $(1-2^{1-s})\zeta(s) = \sum_1^\infty (-1)^{n-1} n^{-s}$ $(0 < \sigma)$. Now if $1/2 < \sigma$ then

$$\begin{aligned}
2I = \int_{-\infty}^{\infty} \left| \frac{(1-2^{1-s})\zeta(s)}{s^2} \right|^2 dt &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} (mn)^{-\sigma} \int_{-\infty}^{\infty} \left(\frac{m}{n}\right)^{it} \frac{dt}{(\sigma^2+t^2)^2} \\
&= \frac{\pi}{2\sigma^3}\zeta(2\sigma) + 2\sum_{m=1}^{\infty} \sum_{n<m} (-1)^{m+n} (mn)^{-\sigma} \int_{-\infty}^{\infty} \frac{\cos(t \ln m/n) dt}{(\sigma^2+t^2)^2} \\
&= \frac{\pi}{2\sigma^3} \left\{ \zeta(2\sigma) + 2\sum_{m=1}^{\infty} (-1)^m m^{-\sigma} \sum_{n=1}^{m-1} (-1)^n n^{-\sigma} (1+\sigma \ln m/n) e^{-\sigma \ln m/n} \right\} \\
&= \frac{\pi}{2\sigma^3} \left\{ \zeta(2\sigma) + 2\sum_{m=1}^{\infty} (-1)^m m^{-2\sigma} \sum_{n=1}^{m-1} (-1)^n (1+\sigma \ln m/n) \right\} \\
&= \frac{\pi}{2\sigma^3} \left\{ \zeta(2\sigma)(1-2^{1-2\sigma}) + 2\sigma \sum_{m=1}^{\infty} (-1)^m m^{-2\sigma} \sum_{n=1}^{m-1} (-1)^n \ln m/n \right\}
\end{aligned}$$

The desired result follows letting $\sigma \to 1/2$ observing that $\lim_{\sigma\to 1/2} \zeta(2\sigma)(1 - 2^{1-2\sigma}) = \ln 2$ and

$$S = \sum_{m=1}^{\infty} (-1)^m m^{-1} \sum_{n=1}^{m-1} (-1)^n \ln m/n.$$

The last equality follows observing that the inner sum in the last equation with $m = 1$ is zero and

$$\begin{aligned}
\sum_{m=2}^{3} (-1)^m m^{-1} \sum_{n=1}^{m-1} (-1)^n \ln m/n &= \frac{\ln\{1/2\}}{2.3}, \\
\sum_{m=4}^{5} (-1)^m m^{-1} \sum_{n=1}^{m-1} (-1)^n \ln m/n &= \frac{\ln\{1.3/2.4\}}{4.5},
\end{aligned}$$

and so on. The proof is completed. ☑

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