DOI: https://doi.org/10.15446/recolma.v54n1.89789

Revista Colombiana de Matemáticas
Volumen 54(2020)1, páginas 65-74

On n-th roots of meromorphic maps

Sobre raíces n-ésimas de funciones meromorfas

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Abstract. Let S be a connected Riemann surface and let $\varphi : S \to \widehat{\mathbb{C}}$ be branched covering map of finite type. If $n \geq 2$, then we describe a simple geometrical necessary and sufficient condition for the existence of some n-th root, that is, a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = \psi^n$.

Key words and phrases. Riemann surfaces, holomorphic branched coverings, maps.

2010 Mathematics Subject Classification. 30F10, 57M12.

Resumen. Sea S una superficie de Riemann conexa y $\varphi : S \to \widehat{\mathbb{C}}$ un cubrimiento ramificado holomorfo de tipo finito. Para cada $n \geq 2$ describimos una condición geométrica necesaria y suficiente para la existencia de alguna raíz n-ésima, esto es, una función meromorfa $\psi : S \to \widehat{\mathbb{C}}$ de manera que $\varphi = \psi^n$.

Palabras y frases clave. Superficies de Riemann, cubrimientos ramificados holomorfos, mapas.

1. Introduction

In this paper, S will denote a connected (not necessarily compact or of finite type) Riemann surface and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ will be the Riemann sphere. A holomorphic surjective map $\varphi : S \to \widehat{\mathbb{C}}$ is a holomorphic branched covering if: (i) it has a finite set $B_\varphi \subset \widehat{\mathbb{C}}$ of branching points, (ii) $\varphi : S \setminus \varphi^{-1}(B_\varphi) \to \widehat{\mathbb{C}} \setminus B_\varphi$ is a holomorphic covering map and (iii) around each point $q \in B_\varphi$ there is a open disc Δ_q such that $\varphi^{-1}(\Delta_q)$ consists of a collection of pairwise disjoint discs V_j such that each of the restrictions $\varphi : V_j \to \Delta_q$ is a finite degree $d_{q,j}$ holomorphic map (i.e., there are biholomorphisms $z : V_j \to \mathbb{D}$ and $w : \Delta_q \to \mathbb{D}$, where $\mathbb{D}$ is the unit disc, such that $w \circ \varphi \circ z^{-1}(z) = z^{d_{q,j}}$). For each $q \in B_\varphi$, let $M_q \subset \{1, 2, \ldots\}$ be the set of local degrees of φ at the points in the fiber $\varphi^{-1}(p)$. We say that φ is of *finite type* if the sets M_q are finite. This condition permits to define the *branch order* of $q \in B_\varphi$ as the minimum common multiple of the values in M_q.

Let $\varphi : S \to \widehat{\mathbb{C}}$ a holomorphic branched covering of finite type. If $n \geq 2$, then a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$ such that $\varphi = \psi^n$ is called an n-th root of φ (the others n-th roots of φ are of the form $e^{2k\pi i/n}\psi$, where $k = 0, 1, \ldots, n-1$).

The existence of an n-th root of φ necessarily implies that: (a) $\infty, 0 \in B_\varphi$ and (b) the branch orders of both 0 and ∞ are multiples of n. These two conditions are not sufficient for φ to have an n-root. For $S = \widehat{\mathbb{C}}$ the existence of an n-th root is equivalent for each zero and each pole of the rational map φ to have degree a multiple of n (which in particular asserts conditions (a) and (b)). But, for other Riemann surfaces, the above is not sufficient in general.

In [3] there was provided a simple geometrical necessary and sufficient condition for φ to have a 2-th root. In this paper, we generalize such a description for every $n \geq 2$ (Theorem 2.3).

In the final section we generalize some of the tools in the proof of the main result to the context of Kleinian groups of higher dimension.

***Remark* 1.1** (A connection to Fuchsian groups)**.** Let K be a finitely generated Fuchsian group, acting on the hyperbolic plane $\mathbb{H}^2$, such that $\mathbb{H}^2/K$ is an orbifold of genus zero (so its underlying Riemann surface structure is $\widehat{\mathbb{C}}$) and let $k_1, \ldots, k_r$ be the orders of its cone points. Let Γ be a subgroup of K and let S be the underlying Riemann surface structure associated to the hyperbolic orbifold $\mathbb{H}^2/\Gamma$ (if Γ is assumed to be torsion free, then this orbifold has no cone points). It is well known that S is of finite type if and only if Γ is finitely generated (in particular, by taking infinitely generated subgroups we obtain examples of surfaces of infinite type). The inclusion $\Gamma \leq K$ induces a holomorphic branched covering of finite type $S = \mathbb{H}^2/\Gamma \to \widehat{\mathbb{C}}$. The local degree of φ at each point of S is either 1 (the generic case) or a divisor of some k_j (if Γ is torsion free, then the local degree at each point over a cone point of order k_j is also k_j). Conversely, the uniformization theorem asserts that, for a connected Riemann surface S of hyperbolic type, each branched covering of finite type $\varphi : S \to \widehat{\mathbb{C}}$ is obtained in such a way for suitable choices of Γ and K.

2. Main results

Before stating our main result we need some definitions.

2.1. Admissible arcs and n-Z-orientability

Let us recall that a *map* on an orientable and connected surface X is a 2-cell decomposition of it, induced by the embedding of a connected graph $\mathcal{H}$ for which each of its vertices has a finite degree and each face (i.e., the connected components of $X \setminus \mathcal{H}$) are finite-sided polygons.

An *admisible arc* for a branched covering of finite type $\varphi : S \to \widehat{\mathbb{C}}$, with $0, \infty \in B_\varphi$, is a simple arc $\delta \subset \widehat{\mathbb{C}}$ whose end points are 0 and ∞, and $B_\varphi \setminus \{0, \infty\} \subset \delta$. For such an admissible arc, the graph $\varphi^{-1}(\delta) = \mathcal{G}_\delta \subset S$ defines a map $\mathcal{F}_\delta$ on S; each of it faces is a polygon with $2(r-1)$ sides, $r = \#B_\varphi$. We say that $\mathcal{F}_\delta$ is *n-Z-orientable* if we may label its faces with numbers inside $\{1, 2, \ldots, n\}$ (see Figure 1 at the end for the case $n = 3$), such that the following two properties hold:

(1) around each vertex $q \in \varphi^{-1}(0)$ (respectively, $q \in \varphi^{-1}(\infty)$), following the counterclockwise (respectively, the clockwise) orientation, the labelling is a finite consecutive sequence of the ordered tuple $(1, 2, \ldots, n)$ (these correspond, respectively, to the first two figures at the left of Figure 1),

(2) around each vertex $q \notin \varphi^{-1}\{0, \infty\}$, following the counterclockwise orientation we see a finite sequence of a same tuple $(i, i+1)$, if $i = 1, \ldots, n-1$, or of the tuple $(1, n)$ (i.e., we see as labelling $i, i+1, i, i+1, \ldots, i, i+1$, or we see $1, n, 1, n, \ldots, 1, n$) (these correspond to the last three figures at the left of Figure 1).

***Remark* 2.1.** For instance, if we let $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ to be $\varphi(z) = z^3$, then it can be seen that the positive real line is an admissible arc. As in this case $\mathcal{F}_\delta$ has exactly three faces (cyclicly permuted by the rotation $A(z) = e^{2\pi i/3} z$), this cannot be n-Z-orientable for $n \neq 3$.

The 2-Z-orientable definition was introduced by Zapponi in [9, 10, 11] (he used the term "orientable") in order to decide if a given Strebel quadratic meromorphic form $\mathcal{Q}$ [8] on a closed Riemann surface has an square root (the 2-cell decomposition is obtained from the graph whose vertices are the zeroes of $\mathcal{Q}$ and the edges are its non-compact horizontal trajectories).

***Remark* 2.2.** If δ_1 and δ_2 are admissible arcs for φ, then there is an orientation-preserving homeomorphism $h : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ fixing the points 0 and ∞ and such that $h(\delta_1) = \delta_2$.

2.2. n-Z-orientability is a necessary and sufficient condition

The following generalizes the results in [3] done for the case $n = 2$.

Theorem 2.3. *Let S be a connected Riemann surface. Let $\varphi : S \to \widehat{\mathbb{C}}$ be a holomorphic branched covering map of finite type with $0, \infty \in B_\varphi$ and each one with branch order a multiple of $n \geq 2$. If δ is an admissible arc for φ, then the existence of n-th roots of φ is equivalent for the map $\mathcal{F}_\delta$ to be n-Z-orientable.*

In terms of Fuchsian groups, Theorem 2.3 asserts the following.

Corollary 2.4. *Let $K < \mathrm{PSL}_2(\mathbb{R})$ be a co-compact Fuchsian group acting on the hyperbolic plane $\mathbb{H}^2$ such that $\mathbb{H}^2/K$ has genus zero. Let $n \geq 2$ be an integer and assume that there are two cone points $p, q \in \mathbb{H}^2/K$ whose cone orders are multiples of n. Let $\delta \subset \mathbb{H}^2/K$ be a simple arc whose end points are p and q and containing all other cone points in its interior. Let $\mathcal{F}_\delta$ be the map of $\mathbb{H}^2$ induced by the lifting of δ to $\mathbb{H}^2$. Then the existence of a normal subgroup Γ_n, of index $n \geq 2$, in K such that $K/\Gamma_n \cong \mathbb{Z}_n$ and $\mathbb{H}^2/\Gamma_n$ has genus zero, is equivalent for $\mathcal{F}_\delta$ to be n-Z-orientable.*

In order to see the above, we identify $\mathbb{H}^2/K$ with $\widehat{\mathbb{C}}$ and $p = 0$, $q = \infty$. Then in Theorem 2.3 we set $S = \mathbb{H}^2$ and take $\varphi : \mathbb{H}^2 \to \widehat{\mathbb{C}}$ a branched regular covering with K as its deck group.

2.3. Case of compact Riemann surfaces

If S is a compact Riemann surface, then every non-constant meromorphic map $\varphi : S \to \widehat{\mathbb{C}}$ is a holomorphic branched covering of finite type. In this way, Theorem 2.3 can be rewritten as follows.

Corollary 2.5. *Let S be a compact Riemann surface. Let $\varphi : S \to \widehat{\mathbb{C}}$ be a non-constant meromorphic map with $0, \infty \in B_\varphi$ and each one with branch order a multiple of $n \geq 2$. If δ is an admissible arc for φ, then the existence of n-th roots of φ is equivalent for the associated map $\mathcal{F}_\delta$ to be n-Z-orientable.*

The compact Riemann surface S can be defined by a complex projective algebraic curve inside $\mathbb{P}^n$ and the meromorphic map $\varphi : S \to \widehat{\mathbb{C}}$ can be described by a rational map.

Let us assume S is defined as the zero locus of the homogeneous polynomials $P_1, \dots, P_r \in \mathbb{C}[x_1, \cdots, x_{n+1}]$ and that φ corresponds to the quotient Q_1/Q_2, where $Q_1, Q_2 \in \mathbb{C}[x_1, \dots, x_{n+1}]$ are homogeneous polynomials of the same degree. If $\sigma \in \mathrm{Gal}(\mathbb{C})$, the group of field automorphisms of $\mathbb{C}$, then we set S^σ (respectively, φ^σ) the projective algebraic curve defined by the polynomials $P_1^\sigma, \dots, P_r^\sigma$ (respectively, Q_1^σ/Q_2^σ), where P_j^σ (respectively, Q_j^σ) is obtained from P_j (respectively, Q_j) by applying σ to all of its coefficients. It can be checked that S^σ is again a compact Riemann surface and that $\varphi^\sigma : S^\sigma \to \widehat{\mathbb{C}}$ is a holomorphic branched covering map of finite type.

If (S_1, φ_1) and (S_2, φ_2) are isomorphic (i.e., there is an isomorphism $\psi : S_1 \to S_2$ such that $\varphi_1 = \varphi_2 \circ \psi$), then for every $\sigma \in \mathrm{Gal}(\mathbb{C})$ it holds that the two new pairs $(S_1^\sigma, \varphi_1^\sigma)$ and $(S_2^\sigma, \varphi_2^\sigma)$ are still isomorphic (by ψ^σ). This process provides of an action of $\mathrm{Gal}(\mathbb{C})$ on (equivalence classes) of pairs (S, φ). As the property of having an n-th root is a $\mathrm{Gal}(\mathbb{C})$-invariant, the above asserts the following.

Corollary 2.6. *Let S be a compact Riemann surface. Let $\varphi : S \to \widehat{\mathbb{C}}$ be a non-constant meromorphic map with $0, \infty \in B_\varphi$ and each one with branch order* *a multiple of* $n \geq 2$. *Then the n-Z-orientability property of a pair* (S, φ) *is a* $\mathrm{Gal}(\mathbb{C})$*-invariant.*

Belyi's theorem [2] asserts that a compact Riemann surface S can be defined by a curve over the field $\overline{\mathbb{Q}}$ of algebraic numbers if and only if there is a non-constant meromorphic map, called a Belyi map, $\beta : S \to \widehat{\mathbb{C}}$ whose branching points are contained inside $\{\infty, 0, 1\}$. On S there is a 2-cell decomposition $\mathcal{D}_\beta$, called a dessin d'enfant [6], whose underlying graph $\beta^{-1}([0,1])$ is bipartite (the black vertices are $\beta^{-1}(1)$ and the white ones are $\beta^{-1}(0)$). Corollary 2.5 provides a geometrical condition for the new Belyi map $\varphi = \beta/(\beta - 1)$ to have n-square roots. Such a geometrical condition, in terms of the dessin $\mathcal{D}_\beta$, is that its faces can be labeled using numbers in $\{1, 2, \ldots, n\}$ such that around each black vertex (respectively, white vertex), following the counterclockwise orientation, we see a finite consecutive sequence of the tuple $(1, 2, \ldots, n)$ (respectively, $(n, n-1, \ldots, 2, 1)$). This condition provides a Galois invariant for the dessin $\mathcal{D}_\beta$. In [4], for $n = 2$, it was observed that this is a new Galois invariant on dessins d'enfants. We expect (but we have no explicit evidence) that for each $n \geq 3$ it provides a new Galois invariant.

3. Proof of Theorem 2.3

3.1.

Let $r \geq 2$ be the cardinality of B_φ and let $\delta \subset \widehat{\mathbb{C}}$ an admissible arc for φ, starting at the branch point $p_1 = 0$, ending at the branch point $p_r = \infty$. We label the rest of the branch points of φ as $p_2, \ldots, p_{r-1}$, such that p_j is between p_{j-1} and p_{j+1}. Let us denote by k_j be the branch order of p_j. We are assuming that k_1 and k_r are both multiples of $n \geq 2$. Set $\mathbb{X}$ to be either $\widehat{\mathbb{C}}$, $\mathbb{C}$ or $\mathbb{H}^2$ depending on if

$$\sum_{j=1}^{r} (1 - k_j^{-1}) - 2$$

is negative, zero or positive, respectively.

3.2.

Let K be a discrete group of isometries of $\mathbb{X}$ such that $\mathbb{X}/K = \widehat{\mathbb{C}}$ and whose cone points are $p_1, \ldots, p_r$, with respective cone orders $k_1, \ldots, k_r$. Let $\pi_K : \mathbb{X} \to \widehat{\mathbb{C}}$ be a regular holomorphic branched covering with K as deck group.

The arc δ defines a fundamental domain P_δ for K (see Figure 2 at the end), with $2(r-1)$ sides, and set of side pairings $\mathcal{A}_{P_\delta} = \{C_1, \ldots, C_{r-1}\}$, such that

$$K = \langle C_1, \ldots, C_{r-1} : C_1^{k_1} = (C_1^{-1} C_2)^{k_2} = \cdots = (C_{r-2}^{-1} C_{r-1})^{k_{r-1}} = C_{r-1}^{k_r} = 1 \rangle.$$

The K-translates of P_δ produces a 2-cell decomposition $\mathcal{T}_{K, P_\delta}$ of $\mathbb{X}$, that is, a map on $\mathbb{X}$. As k_1 and k_r are multiples of n, the following produces a surjective homomorphism

$$\theta_0 : K \to G = \langle \sigma \rangle \cong \mathbb{Z}/n\mathbb{Z} : C_j \mapsto \sigma,\ j = 1, \dots, r-1.$$

As $\ker(\theta_0)$ is the group generated by the K-conjugates of the elements

$$C_1^n, C_1^{-1}C_2, C_1C_2^{-1}, \dots, C_1^{-1}C_{r-1}, C_1C_{r-1}^{-1},$$

it follows that adjacent faces of the map $\mathcal{T}_{K,P_\delta}$ have different labels.

Let $x_1, \dots, x_r \in \mathbb{X}$ be the fixed points of the elements $C_1, C_1^{-1}C_2, C_2^{-1}C_3, \dots, C_{r-2}^{-1}C_{r_1}$ and C_{r-1}, respectively. Then, $\pi_K(x_j) = p_j$, for $j = 1, \dots, r$.

***Remark* 3.1.** The map $\mathcal{T}_{K,P_\delta}$ is n-Z-orientable. To see this, for each $T \in K$, we label the T-translated of P_δ by the element $\theta_0(T) \in G$. Now, in order to be consequent with our definiton of n-Z-orientability as in the introduction, we make the identification of σ^j with the integer $j+1$, for $j = 0, 1, \dots .n-1$.

The orbifold $\mathbb{X}/\ker(\theta_0)$ can be identified with the Riemann sphere $\widehat{\mathbb{C}}$ with exactly $n(r-2)+2$ cone points, these being of orders $k_1/n, k_2, \overset{n}{\dots}, k_2, \dots, k_{r-1}, \overset{n}{\dots}, k_{r-1}, k_r/n$. The pair $(K, \ker(\theta_0))$ induces a Möbius transformation A, of order n, and (by using the above identification) a degree n meromorphic map $\eta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, whose deck group is $\langle A \rangle \cong \mathbb{Z}/n\mathbb{Z}$, branched at the end points of δ, i.e., 0 and ∞. Up to conjugation by a suitable Möbius transformation, we may assume that $A(z) = e^{2\pi i/n}z$ and $\eta(z) = z^n$.

The set $\eta^{-1}(\delta)$ is a collection of n simple arcs (containing all the cone points of $\mathbb{X}/\ker(\theta_0)$ and whose end points are the two fixed points of A, i.e., 0 and ∞) which are cyclically permuted by A. This provides a n-Z-orientable map $\mathcal{F}_0$ on $\widehat{\mathbb{C}}$ and it is induced by the map $\mathcal{T}_{K,P_\delta}$.

***Remark* 3.2.** Each index normal subgroup Γ_0 of K, such that that $K/\Gamma_0 \cong \mathbb{Z}/n\mathbb{Z}$, is given as the kernel of a surjective homomorphism $\theta : K \to \mathbb{Z}/n\mathbb{Z}$. Let us assume that $\mathbb{X}/\Gamma_0$ has genus zero. In this case, the inclusion $\Gamma_0 \lhd K$ induces a regular holomorphic branched covering $\pi_{\Gamma_0} : \mathbb{X} \to \widehat{\mathbb{C}}$ with Γ_0 as its deck group, such that $\pi_K = R \circ \pi_{\Gamma_0}$, where $R(z) = z^n$. Let us restrict to those Γ_0 such that $\pi_{\Gamma_0}(x_1) = 0$ and $\pi_{\Gamma_0}(x_r) = \infty$. (Note that if each of the k_j, where $j = 2, \dots, r-1$, are not a multiple of n, then this is the only possibility). In this case, for $j = 2, \dots, r-1$, $\pi_{\Gamma_0}(x_j)$ is a cone point of order k_j (since that point is not critical point of R). This asserts that $C_{j-1}^{-1}C_j \in \Gamma_0$. If we set $\sigma = \theta(C_1)$, which is a generator of $\mathbb{Z}/n\mathbb{Z}$, the previous asserts that $\theta(C_j) = \sigma$, for every $j = 1, \dots, r-1$. In other words, up to post-composing by an automorphism of $\mathbb{Z}/n\mathbb{Z}$, we obtain θ_0, i.e., Γ_0 is uniquely determined.

3.3.

As a consequence of the uniformization theorem, there is a proper subgroup Γ of K such that S is the Riemann surface structure of the orbifold $\mathbb{X}/\Gamma$, a regular holomorphic (possible branched) covering $\pi_\Gamma : \mathbb{X} \to S$, with deck group Γ, such that $\pi_K = \varphi \circ \pi_\Gamma$ (i.e., φ is induced by the inclusion $\Gamma < K$).

Let us observe that the π_Γ-image of the n-Z-orientable map $\mathcal{T}_{K,P_\delta}$ on $\mathbb{X}$ is the map $\mathcal{F}_\delta$ (which might or not be n-Z-orientable in principle).

Now, the existence of a meromorphic map $\psi : S \to \widehat{\mathbb{C}}$, such that $\varphi = \eta \circ \psi = \psi^n$ is equivalent to have that $\Gamma \leq \ker(\theta_0)$ (see Remark 3.2). By lemma 3.3 (whose arguments follow the same ideas as in [3] for $n = 2$), the previous is equivalent for the n-Z-orientability of $\mathcal{F}_\delta$. This will provide the desired result.

Lemma 3.3. *The map $\mathcal{F}_\delta$ is n-Z-orientable if and only if* $\Gamma \leq \ker(\theta_0)$.

Proof. Let us assume $\Gamma \leq \ker(\theta_0)$. The idea is to push-down the (n-Z-orientable) labelling on the faces of $\mathcal{T}_{K,P_\delta}$ to the faces of $\mathcal{F}_\delta$. Two faces F_1 and F_2 of $\mathcal{T}_{K,P_\delta}$ are projected to the same face if and only if there is some $T \in \Gamma$ such that $F_2 = T(F_1)$. As we are assuming $\Gamma \leq \ker(\theta_0)$, the induced labelling is well defined. It is not difficult to observe that induced labelling on the map $\mathcal{F}_\delta$ satisfies the condition for being n-Z-orientable.

In the other direction, let us assume we have a labelling for $\mathcal{F}_\delta$ satisfying the n-Z-orientability. By the connectivity of S, we may construct a fundamental (connected) domain $\mathcal{Q}$ for Γ by gluing some copies K-translated of P_δ (as many as the index of Γ in K). The projection of those copies of P_δ, used in the construction of $\mathcal{Q}$, projects under π_Γ exactly to the faces of $\mathcal{F}_\delta$. Now, lift the labelling of the n-Z-orientable map $\mathcal{F}_\delta$ to obtain labelling of these copies of P_δ included in $\mathcal{Q}$. Use the group K to translate these labels to the rest of K-translates of P_δ. This provides a labelling on $\mathcal{T}_{K,P_\delta}$ satisfying the n-Z-orientability property. By Remark 3.2, this can be assumed to be the labelling provided by θ_0. As the above procedure of pulling-down the labelling from $\mathcal{T}_{K,P_\delta}$ to $\mathcal{F}_\delta$ induces the given labelling, it follows from the first part that $\Gamma \leq \ker(\theta_0)$. ☑

4. A remark: θ-Zapponi-orientability of Kleinian groups

In the previous section we have considered a Fuchsian group K, a fundamental polygon P, the set $\mathcal{A}_P \subset K$ of its side pairings, and a surjective homomorphism $\theta_0 : K \to G = \mathbb{Z}/n\mathbb{Z}$ such that $\ker(\theta_0) \cap \mathcal{A}_P = \varnothing$. The homomorphism θ_0 permitted to label each of the faces of the map $\mathcal{T}_{K,P}$, using as labelling the elements of G, and such that adjacent faces have different labels. This procedure can be generalized for any Kleinian group as follows.

Let K be a discrete group of isometries of $\mathbb{X}^m$, where $\mathbb{X}^m$ is either the m-dimensional hyperbolic $\mathbb{H}^m$ or the m-dimensional Euclidian space $\mathbb{E}^m$ or the m-dimensional sphere $\mathbb{S}^m$. Let $P \subset \mathbb{X}^m$ be a fundamental polyhedron of K and let $\mathcal{A}_P$ the subset of K consisting of the side-pairings of P. It is well known that $\mathcal{A}_P$ is a set of generators for K and that a complete set of relations is provided by how the sides of P are glued by these side-pairings (Poincaré Polyhedron

Theorem, see [1, 5, 7]). The K-translates of P provides a n-tessellation $\mathcal{T}_{K,P}$ of $\mathbb{X}^m$.

4.1. (K, P)-admissible homomorphisms

Let $\theta : K \to G$, where G is a finite group, be a surjective homomorphism. For each $T \in K$ we proceed to label the n-face $T(P)$ using the element $\theta(T) \in G$. If adjacent faces have different labels, then we say that θ is (K, P)-*admissible*.

Lemma 4.1. *θ is (K, P)-admissible if and only if $\mathcal{A}_P \cap \ker(\theta) = \varnothing$.*

Proof. This follows from the fact that, for $T_1, T_2 \in K$, one has that $T_1(P)$ and $T_2(P)$ are adjacent if and only if there is some $L \in \mathcal{A}_P$ such that $T_2 = T_1L$. ☑

Remark **4.2.** For every surjective homomorphism $\theta : K \to G$, it is possible to find a fundamental polyhedron P for K such that θ is (K, P)-admissible.

Example 4.3. (1) If K_P is the subgroup of K generated by all the elements of the form AB, where $A, B \in \mathcal{A}_P$, then either $K_P = K$ or has index two in K. If $\theta : K \to G = \mathbb{Z}_2$ is any homomorphism, then $K_P \leq \ker(\theta)$. It follows that θ is (K, P)-admissible if and only if $K \neq K_P = \ker(\theta)$ (in particular, there is at most one (K, P)-admissible homomorphism onto $\mathbb{Z}_2$). (2) Let $n, r \geq 2$ and let us consider a Fuchsian group, acting in the hyperbolic plane $\mathbb{H}^2$, with the following presentation

$$K = \langle C_1, \ldots, C_{r-1} : C_1^{k_1} = (C_1^{-1}C_2)^{k_2} = \cdots = (C_{r-2}^{-1}C_{r-1})^{k_{r-1}} = C_{r-1}^{k_r} = 1\rangle.$$

Let P be a fundamental domain of K as shown in Figure 2. Its set of side-pairings is $\mathcal{A}_P = \{C_1, \ldots, C_{r-1}\}$. Let $G = \langle\sigma\rangle \cong \mathbb{Z}/n\mathbb{Z}$. If k_1 and k_r are both multiples of n, then we may consider the surjective homomorphism $\theta_0 : K \to G$, defined by $\theta_0(C_j) = \sigma$, for every $j = 1, \ldots, r-1$. As $\ker(\theta_0)$ is the group generated by the conjugates of the elements $C_1^n, C_1^{-1}C_2, C_1C_2^{-1}, \ldots, C_1^{-1}C_{r-1}, C_1C_{r-1}^{-1}$, it follows that θ_0 is (K, P)-admissible. The induced labelling of $\mathcal{T}_{K,P}$ by θ_0 satisfies to be n-Z-orientable.

4.2. θ-Z-orientable subgroups

Let $\theta : K \to G$ be a (K, P)-admissible homomorphism. If Γ is a proper subgroup of K, then the tessellation $\mathcal{T}_{K,P}$ induces an m-dimensional tessellation $\mathcal{T}_{K,P,\Gamma}$ on the geometric orbifold $\mathcal{O}_\Gamma = \mathbb{X}^m/\Gamma$. The labelling on the faces of $\mathcal{T}_{K,P}$, provided by the (K, P)-admissible homomorphism θ, induces a labelling of the faces of the tessellation $\mathcal{T}_{K,P,\Gamma}$. It is not difficult to see that the adjacent m-faces of this last tessellation have different labels if and only if $\Gamma \leq \ker(\theta)$. If this is the situation, we say that (K, P, Γ) is *θ-Z-orientable*.

We summarize all the above in the following.

Lemma 4.4. *Let K be a discrete group of isometries of $\mathbb{X}^m$, $P \subset \mathbb{X}^m$ be a fundamental polyhedron for it and $\mathcal{A}_P \subset K$ be the set of side-pairings of P. Let $\theta : K \to G$ be a (K,P)-admissible homomorphism onto a finite group G (equivalently,* $\ker(\theta) \cap \mathcal{A}_P = \varnothing$*). Then (K,P,Γ) is θ-Z-orientable if and only if* $\Gamma \leq \ker(\theta)$.

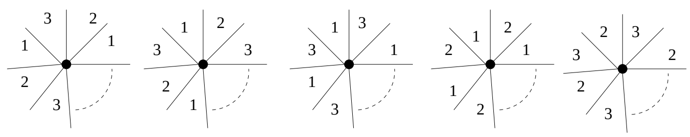

FIGURE 1. The labelling for 3-Z-orientability

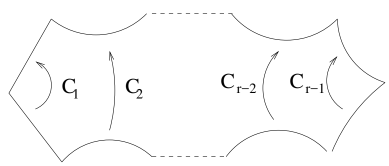

FIGURE 2. The fundamental polygon P_δ

Acknowledgment. We thank the anonymous referee for her/his comments and suggestions which contributed to improving this manuscript.

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(Recibido en diciembre de 2019. Aceptado en abril de 2020)

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