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On $\mathcal{I}$-$\oplus$-supplemented modules

Sobre módulos $\mathcal{I}$-$\oplus$-suplementados

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Abstract. In this note we introduce $\mathcal{I}$-$\oplus$-supplemented modules as a proper generalization of $\oplus$-supplemented modules. A module M is called $\mathcal{I}$-$\oplus$-supplemented if for every $\phi \in End_R(M)$, there exists a direct summand L of M such that $\text{Im}\phi + L = M$ and $\text{Im}\phi \cap L \ll L$. It is shown that if M is a $\mathcal{I}$-$\oplus$-supplemented module with D_3 condition, then every direct summand of M is $\mathcal{I}$-$\oplus$-supplemented. We prove that if $M = M_1 \oplus M_2$ is $\mathcal{I}$-$\oplus$-supplemented such that M_1 and M_2 are relative projective, then M_1 and M_2 are $\mathcal{I}$-$\oplus$-supplemented. We study some rings whose modules are $\mathcal{I}$-$\oplus$-supplemented.

Key words and phrases. Supplemented module, $\mathcal{I}$-$\oplus$-supplemented module, dual Rickart module, endomorphism ring, V-ring.

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Resumen. En esta nota nosotros introducimos los módulos $\mathcal{I}$-$\oplus$-suplementados, una generalizacion de los módulos $\oplus$-suplementados. Un módulo M se dice $\mathcal{I}$-$\oplus$-suplementado si para cada $\phi \in End_R(M)$, existe un sumando directo L de M tal que $\text{Im}\phi + L = M$ y $\text{Im}\phi \cap L \ll L$. Se demuestra que si M es un módulo $\mathcal{I}$-$\oplus$-suplementado con la condición D_3, entonces cada sumando directo de M es $\mathcal{I}$-$\oplus$-suplementado. Demostramos que si $M = M_1 \oplus M_2$ es $\mathcal{I}$-$\oplus$-suplementado tal que M_1 y M_2 son proyectivos relativos, entonces M_1 y M_2 son $\mathcal{I}$-$\oplus$-suplementados. Estudiamos algunos anillos cuyos modulos son $\mathcal{I}$-$\oplus$-suplementados.

Palabras y frases clave. Módulo suplementado, módulo $\mathcal{I}$-$\oplus$-suplementado, módulo Rickart dual, endomorfismo de anillos, V-anillo.

1. Introduction

Throughout this paper, R will denote an arbitrary associative ring with identity, M is a unitary right R-module and $S = End_R(M)$ is the ring of all R-endomorphisms of M. We will use the notation $N \ll M$ to indicate that

N is small in M (*i.e.* $\forall L \lneq M, L+N \neq M$). The notation $N \leq^{\oplus} M$ means that N is a direct summand of M. $N \trianglelefteq M$ means that N is a fully invariant submodule of M (*i.e.*, $\forall \phi \in End_R(M),\ \phi(N) \subseteq N$).

Let K and N be submodules of M. K is called a *supplement* of N in M if $M = K + N$ and K is minimal with respect to this property, or equivalently, $M = K + N$ and $K \cap N \ll K$. A module M is called *supplemented* if every submodule of M has a supplement in M. The module M is called *amply supplemented* if for any submodules K and N of M with $M = K + N$, there exists a supplement P of K such that $P \leq N$. A module M is called *lifting* if for every $A \leq M$, there exists a direct summand B of M such that $B \subseteq A$ and $A/B \ll M/B$ [7].

It is well known that the module M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand (see [4, 7]). As a generalization of lifting modules, Mohamed and Müller [7] called a module M $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand of M.

Many generalizations of the concept of $\oplus$-supplemented modules have been introduced and studied by several authors [2, 3, 5].

In [1], we introduced $\mathcal{I}$-lifting modules as a generalization of lifting modules. Following [1], a module M is called $\mathcal{I}$-*lifting* if for every $\phi \in S$ there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq \mathrm{Im}\phi$ and $M_2 \cap \mathrm{Im}\phi \ll M_2$.

In this paper, we define $\mathcal{I}$-$\oplus$-supplemented modules as a proper generalization of $\oplus$-supplemented modules and $\mathcal{I}$-*lifting* modules. A module M is called $\mathcal{I}$-$\oplus$-*supplemented* if for every $\phi \in End_R(M)$, there exists a direct summand L of M such that $\mathrm{Im}\phi + L = M$ and $\mathrm{Im}\phi \cap L \ll L$. Clearly, $\oplus$-supplemented modules, $\mathcal{I}$-lifting modules and dual Rickart modules are $\mathcal{I}$-$\oplus$-supplemented.

In Section 2, we investigate $\mathcal{I}$-$\oplus$-supplemented modules. The relative $\mathcal{I}$-$\oplus$-supplemented condition is introduced and it is used to obtain a characterization of an $\mathcal{I}$-$\oplus$-supplemented module. It is shown that if M is $\mathcal{I}$-$\oplus$-supplemented with D_3, then every direct summand of M is $\mathcal{I}$-$\oplus$-supplemented. We prove that if $M = M_1 \oplus M_2$ is $\mathcal{I}$-$\oplus$-supplemented such that M_1 and M_2 are relative projective, then M_1 and M_2 are $\mathcal{I}$-$\oplus$-supplemented. It is shown that every finite direct sum of copies of a dual Rickart module is $\mathcal{I}$-$\oplus$-supplemented.

In Section 3, we study some rings whose modules are $\mathcal{I}$-$\oplus$-supplemented.

2. $\mathcal{I}$-$\oplus$-supplemented modules

A module M is called $\mathcal{I}$-$\oplus$-*supplemented* if for every $\phi \in End_R(M)$, there exists $L \leq^{\oplus} M$ such that $\mathrm{Im}\phi + L = M$ and $\mathrm{Im}\phi \cap L \ll L$.

It is clear that every $\oplus$-supplemented module is $\mathcal{I}$-$\oplus$-supplemented.

The following example exhibits an $\mathcal{I}$-$\oplus$-supplemented module which is not $\oplus$-supplemented.

Example 2.1. The $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers is $\mathcal{I}$-$\oplus$-supplemented since every endomorphism of a $\mathbb{Z}$-module $\mathbb{Q}$ is an automorphism, but it is not $\oplus$-supplemented.

Let M and N be R-modules. A module M is called an N-$\mathcal{I}$-$\oplus$-*supplemented* module if for every homomorphism $\phi : M \to N$, there exists $L \leq^{\oplus} N$ such that $\mathrm{Im}\phi + L = N$ and $\mathrm{Im}\phi \cap L \ll L$.

In view of the above definition, a right module M is $\mathcal{I}$-$\oplus$-supplemented if and only if M is M-$\mathcal{I}$-$\oplus$-supplemented.

A module M is called *dual Rickart* (for short *d-Rickart*) if for every $\phi \in End_R(M)$, $\mathrm{Im}\phi \leq^{\oplus} M$. Recall that a module M is called N-*dual Rickart* if for every homomorphism $\phi : M \to N$, $\mathrm{Im}\phi \leq^{\oplus} N$ (see [6]). It is clear that if M is N-dual Rickart, then M is N-$\mathcal{I}$-$\oplus$-supplemented. Obviously, if N is a semisimple module, then M is N-$\mathcal{I}$-$\oplus$-supplemented for any R-module M.

It is clear that if M is d-Rickart, then M is $\mathcal{I}$-$\oplus$-supplemented, while the converse in not true (the $\mathbb{Z}$-module $\mathbb{Z}_4$ is $\mathcal{I}$-$\oplus$-supplemented but it is not d-Rickart).

Theorem 2.2. *Let M and N be right modules. Then M is N-$\mathcal{I}$-$\oplus$-supplemented if and only if for all direct summands $M' \leq^{\oplus} M$ and $N' \trianglelefteq^{\oplus} N$, M' is N'-$\mathcal{I}$-$\oplus$-supplemented.*

Proof. Let $M' = eM$ for some $e^2 = e \in End_R(M)$, $N' \trianglelefteq^{\oplus} N$ and $\psi \in \mathrm{Hom}\,(M', N')$. Since $\psi eM = \psi M' \subseteq N' \subseteq N$ and M is N-$\mathcal{I}$-$\oplus$-supplemented, there exists a direct summand K of N such that $\psi eM + K = N$ and $\psi eM \cap K \ll K$. Then $\psi eM + (K \cap N') = N'$. As $N' \trianglelefteq N$, $K \cap N' \leq^{\oplus} N'$. Thus $\psi eM \cap (K \cap N') \ll (K \cap N')$. Therefore M' is N'-$\mathcal{I}$-$\oplus$-supplemented. The converse follows easily. ☑

Next, we characterize $\mathcal{I}$-$\oplus$-supplemented modules.

Corollary 2.3. *The following conditions are equivalent for a module M:*

(1) *M is an $\mathcal{I}$-$\oplus$-supplemented module.*

(2) *For any fully invariant direct summand N of M, every direct summand L of M is N-$\mathcal{I}$-$\oplus$-supplemented.*

Proposition 2.4. *Let M be a module with $Rad(M) = 0$. Then M is $\mathcal{I}$-$\oplus$-supplemented if and only if M is d-Rickart.*

Proof. Let M be an $\mathcal{I}$-$\oplus$-supplemented module and let $\phi \in S$. Then there exists a direct summand K of M such that $M = \mathrm{Im}\phi + K$ and $\mathrm{Im}\phi \cap K \ll K$. So $\mathrm{Im}\phi \cap K \subseteq Rad(M) = 0$. Hence $M = \mathrm{Im}\phi \oplus K$, this means that M is d-Rickart. The converse is clear. ☑

Recall that a ring R is said to be a *right V-ring* if every simple right R-module is injective.

Corollary 2.5. *Let R be a right V-ring and M be an R-module. Then M is $\mathcal{I}$-$\oplus$-supplemented if and only if M is d-Rickart.*

Proof. By [10, 23.1], for every right R-module M, $Rad(M) = 0$. Thus, by Proposition 2.4, every $\mathcal{I}$-$\oplus$-supplemented R-module is d-Rickart. ☑

Corollary 2.6. *Let R be a commutative regular ring and M be an R-module. Then M is $\mathcal{I}$-$\oplus$-supplemented if and only if M is d-Rickart.*

Proof. It is clear by Corollary 2.5 and [10, 23.5(2)]. ☑

An R-module M is called $\mathcal{T}$- *noncosingular* if, $\forall \phi \in End_R(M), Im\phi \ll M$ implies that $\phi = 0$ [8, 9].

Proposition 2.7. *The following conditions are equivalent for an $\mathcal{T}$- noncosingular module M:*

(1) *M is an indecomposable $\mathcal{I}$-$\oplus$-supplemented module;*

(2) *Every non-zero endomorphism $\phi \in S$ is an epimorphism.*

Proof. Let M be an indecomposable $\mathcal{I}$-$\oplus$-supplemented module. Assume that $0 \neq \phi \in End_R(M)$. Then there exists $L \leq^{\oplus} M$ such that $\mathrm{Im}\phi + L = M$ and $\mathrm{Im}\phi \cap L \ll L$. Since M is indecomposable, $L = M$ or $L = 0$. If $L = M$, then $\mathrm{Im}\phi \ll M$. By $\mathcal{T}$- noncosingularity, $\phi = 0$, a contradiction. Thus $L = 0$ and so ϕ is an epimorphism. The converse follows easily. ☑

Corollary 2.8. *Let M be an indecomposable module. Then M is d-Rickart if and only if M is $\mathcal{I}$-$\oplus$-supplemented and $\mathcal{T}$- noncosingular.*

Proof. Let M be d-Rickart, then it is clear that M is $\mathcal{I}$-$\oplus$-supplemented and $\mathcal{T}$- noncosingular. The converse follows from Proposition 2.7 and [6, Proposition 4.4]. ☑

Recall that a module M is said to be *retractable* if, for every $0 \neq N \leq M$, $\exists 0 \neq \phi \in End_R(M)$ with $\phi(M) \subseteq N$.

Corollary 2.9. *Let M be an indecomposable $\mathcal{T}$- noncosingular retractable module. If M is $\mathcal{I}$-$\oplus$-supplemented, then M is simple.*

Proof. Let M be an indecomposable retractable $\mathcal{T}$- noncosingular $\mathcal{I}$-$\oplus$- supplemented module and let N be any non-zero submodule of M. Since M is retractable there exists $0 \neq \phi \in End_R(M)$ such that $\phi(M) \subseteq N$. As ϕ is an epimorphism by Proposition 2.7, $N = M$. Therefore M is a simple module. ☑

Proposition 2.10. *Let M be an $\mathcal{I}$-$\oplus$-supplemented module. Let N be a submodule of M such that M/N is projective. Then M is N-$\mathcal{I}$-$\oplus$-supplemented.*

Proof. Let ϕ be any homomorphism from M to N. Consider the endomorphism $i\phi : M \to M$, where i is the inclusion map from N to M. Since M is $\mathcal{I}$-$\oplus$-supplemented, there exists a direct summand K of M such that $M = \mathrm{Im}\phi + K$ and $\mathrm{Im}\phi \cap K \ll K$. Thus $N = \mathrm{Im}\phi + (N \cap K)$. By [5, Lemma 2.3], $N \cap K$ is a direct summand of M. So $N \cap K$ is a direct summand of N and K. By [7, Lemma 4.2], $\mathrm{Im}\phi \cap K \ll N \cap K$. Therefore M is N-$\mathcal{I}$-$\oplus$-supplemented. ☑

Corollary 2.11. *Let $M = M_1 \oplus M_2$ be an $\mathcal{I}$-$\oplus$-supplemented module and M_2 be projective. Then M_1 is $\mathcal{I}$-$\oplus$-supplemented.*

Proof. By Proposition 2.10, M is M_1-$\mathcal{I}$-$\oplus$ - supplemented. So M_1 is $\mathcal{I}$-$\oplus$-supplemented by Theorem 2.2. ☑

Theorem 2.12. *Let M_1, M_2 and N be right R-modules. Assume that N is M_i-$\mathcal{I}$-$\oplus$-supplemented for $i = 1, 2$, then N is $M_1 \oplus M_2$-$\mathcal{I}$-$\oplus$-supplemented, if for every homomorphism ϕ from N to $M_1 \oplus M_2$ and any projection map π of $M_1 \oplus M_2$ we have $\pi(Im\phi) = Im\phi \cap Im\pi$. The converse is true if every M_i is fully invariant in $M_1 \oplus M_2$.*

Proof. Suppose that N is M_i-$\mathcal{I}$-$\oplus$-supplemented for $i = 1, 2$. We will prove that N is $M_1 \oplus M_2$-$\mathcal{I}$-$\oplus$-supplemented. Let $\phi = (\pi_1\phi, \pi_2\phi)$ be any homomorphism from N to $M_1 \oplus M_2$, where π_i is the projection map from $M_1 \oplus M_2$ to M_i for $i = 1, 2$. Since N is M_i-$\mathcal{I}$-$\oplus$-supplemented, there exists a direct summand K_i of M_i such that $\pi_i\phi N + K_i = M_i$ and $\pi_i\phi N \cap K_i \ll K_i$, for $i = 1, 2$. Let $K = K_1 \oplus K_2$, it is clear that K is a direct summand of $M_1 \oplus M_2$. As $\pi_1\phi(N) = \pi_1(\phi(N) + M_2) = (\phi(N) + M_2) \cap M_1$ and $\pi_2\phi(N) = \pi_2(\phi(N) + K_1) = (\phi(N) + K_1) \cap M_2$ we have $M_1 \leq \phi(N) + M_2 + K_1$ and $M_2 \leq \phi(N) + K_1 + K_2$. Thus $M_1 \oplus M_2 = \phi(N) + K_1 + K_2 = \phi(N) + K$. Moreover, $M_1 \oplus M_2 = \pi_1\phi N + \pi_2\phi N + K_1 + K_2 = \phi N + K$. Since $\phi N \cap (K_1 + K_2) \leq (\phi N + K_1) \cap K_2 + (\phi N + K_2) \cap K_1$, we have $\phi N \cap (K_1 + K_2) \leq (\phi N + M_1) \cap K_2 + (\phi N + M_2) \cap K_1$. As $\phi N + M_1 = \pi_2\phi N \oplus M_1$ and $\phi N + M_2 = \pi_1\phi N \oplus M_2$, thus $\phi N \cap K \subseteq (\pi_2\phi N \cap K_2) + (\pi_1\phi N \cap K_1)$. Since $\pi_i\phi N \cap K_i \ll K_i$ for $i = 1, 2$, $\phi N \cap K \ll K_1 + K_2 = K$. Hence N is $M_1 \oplus M_2$-$\mathcal{I}$-$\oplus$-supplemented. The converse is clear by Theorem 2.2. ☑

Corollary 2.13. *Let $\{M_i\}_{i=1}^n$ be right R-modules. Let $\oplus_{i=1}^n M_i$ be M_j-$\mathcal{I}$-$\oplus$-supplemented, for $j = 1, 2, \cdots, n$. Then $\oplus_{i=1}^n M_i$ is $\mathcal{I}$-$\oplus$-supplemented. The converse is true if every M_i is fully invariant in $\oplus_{i=1}^n M_i$.*

Corollary 2.14. *Let $\{M_i\}_{i=1}^n$ be right R-modules. Let M_i be M_j-d-Rickart for all $i, j \in \mathcal{I} = \{1, 2, \ldots, n\}$. Then $\oplus_{i \in \mathcal{I}} M_i$ is $\mathcal{I}$-$\oplus$-supplemented.*

Proof. By [6, Corollary 5.4], $\oplus_{i\in\mathcal{I}} M_i$ is M_j-d-Rickart for all $j \in \mathcal{I}$, where $\mathcal{I} = \{1, 2, \ldots, n\}$. Thus $\oplus_{i\in\mathcal{I}} M_i$ is M_j-$\mathcal{I}$-$\oplus$-supplemented. By Corollary 2.13, $\oplus_{i\in\mathcal{I}} M_i$ is $\mathcal{I}$-$\oplus$-supplemented. ☑

Corollary 2.15. *Let M be a d-Rickart module. Then every finite direct sum of copies of M is $\mathcal{I}$-$\oplus$-supplemented.*

Theorem 2.16. *Let M_1 and M_2 be right R-modules. Suppose that $M_i \trianglelefteq (M_1 \oplus M_2)$ for $i = 1, 2$. Then $M_1 \oplus M_2$ is an $\mathcal{I}$-$\oplus$-supplemented module if and only if M_i is $\mathcal{I}$-$\oplus$-supplemented for $i = 1, 2$.*

Proof. The necessity follows from Theorem 2.2. Conversely, let M_i be $\mathcal{I}$-$\oplus$-supplemented for $i = 1, 2$. Let $\phi = (\phi_{ij})_{i,j=1,2} \in S = End_R(M_1 \oplus M_2)$ be arbitrary, where $\phi_{ij} \in \mathrm{Hom}\,(M_j, M_i)$. Since $M_i \trianglelefteq M_1 \oplus M_2$ for $i = 1, 2$, $\mathrm{Im}\phi = \mathrm{Im}\phi_{11} \oplus \mathrm{Im}\phi_{22}$. As M_i is $\mathcal{I}$-$\oplus$-supplemented for $i = 1, 2$, there exists direct summand K_i of M_i such that $\mathrm{Im}\phi_{ii} + K_i = M_i$ and $\mathrm{Im}\phi_{ii} \cap K_i \ll K_i$. Let $K = K_1 \oplus K_2$, then K is a direct summand of M. Moreover, $M_1 \oplus M_2 = (\mathrm{Im}\phi_{11} \oplus \mathrm{Im}\phi_{22}) + (K_1 \oplus K_2)$. Since $\mathrm{Im}\phi \cap (K_1 \oplus K_2) \subseteq (\mathrm{Im}\phi + K_1) \cap K_2 + (\mathrm{Im}\phi + K_2) \cap K_1$, we have $\mathrm{Im}\phi \cap (K_1 \oplus K_2) \subseteq (\mathrm{Im}\phi_{11} \cap K_1) + (\mathrm{Im}\phi_{22} \cap K_2) \ll K_1 \oplus K_2$. Hence $M_1 \oplus M_2$ is $\mathcal{I}$-$\oplus$-supplemented. ☑

The module M is called a *duo module*, if every submodule of M is fully invariant.

Corollary 2.17. *Let $M = M_1 \oplus M_2$ be a duo module. Then M is an $\mathcal{I}$-$\oplus$-supplemented module if and only if M_1 and M_2 are $\mathcal{I}$-$\oplus$-supplemented.*

Lemma 2.18. *Let $M = M_1 \oplus M_2$. Then M is M_2-$\mathcal{I}$-$\oplus$-supplemented if and only if for every $\phi \in End_R(M)$ with $Im\phi \geq M_1$, there exists a direct summand K of M such that $K \leq M_2$, $M = K + Im\phi$ and $K \cap Im\phi \ll M$.*

Proof. Let M be M_2-$\mathcal{I}$-$\oplus$-supplemented. Assume that $\phi = (\pi_1\phi, \pi_2\phi)$ is any endomorphism of M with $\mathrm{Im}\phi \geq M_1$, where π_1 is the projection map from M onto M_i for $i = 1, 2$. Since M is M_2-$\mathcal{I}$-$\oplus$-supplemented, there exists a direct summand K of M_2 such that $M_2 = \mathrm{Im}\pi_2\phi + K$ and $\mathrm{Im}\pi_2\phi \cap K \ll K$. It is clear that $\mathrm{Im}\phi \cap M_2 = \mathrm{Im}\pi_2\phi$. Therefore $M = M_1 + M_2 = M_1 + (\mathrm{Im}\phi \cap M_2) + K = \mathrm{Im}\phi + K$ and $\mathrm{Im}\phi \cap K \ll K$.

Conversely, suppose that M has the stated property. Let ϕ be any homomorphism from M to M_2. Consider the endomorphism $\psi = \phi + \pi_1 \in S$, where π_1 is the natural projection of M onto M_1. Since $\mathrm{Im}\psi = \mathrm{Im}\phi \oplus M_1 \geq M_1$, there exists a direct summand K of M such that $K \leq M_2$, $M = K + \mathrm{Im}\psi$ and $K \cap \mathrm{Im}\psi \ll M$ by hypothesis. Then $M_2 = K + \mathrm{Im}\phi$, $K \cap \mathrm{Im}\phi \ll K$ and K is a direct summand of M_2. Therefore M is M_2-$\mathcal{I}$-$\oplus$-supplemented. ☑

Theorem 2.19. *Let $M = M_1 \oplus M_2$ be an $\mathcal{I}$-$\oplus$-supplemented module. Assume that for every direct summand K of M with $M = K + M_2$, $K \cap M_2$ is a direct summand of M. Then M is M_2-$\mathcal{I}$-$\oplus$-supplemented.*

Proof. Let $\phi = (\pi_1\phi, \pi_2\phi)$ be an endomorphism of M with $\text{Im}\phi \geq M_1$, where π_j is the natural projection of M onto M_j for $j = 1, 2$. Consider the endomorphism $i_2\pi_2\phi \in End_R(M)$, where i_2 is the canonical inclusion from M_2 to M. Note that $\text{Im}i_2\pi_2\phi = \text{Im}\phi \cap M_2$. Since M is $\mathcal{I}$-$\oplus$-supplemented, there exists a direct summand K of M such that $M = (\text{Im}\phi \cap M_2) + K$ and $\text{Im}\phi \cap M_2 \cap K \ll K$. It is clear $M = \text{Im}\phi + M_2$. By [4, Lemma 1.2], $M = (K \cap M_2) + \text{Im}\phi$. By hypothesis, $K \cap M_2$ is a direct summand of M as $M = K + M_2$. Therefore M is M_2-$\mathcal{I}$-$\oplus$-supplemented by Lemma 2.18. ☑

Corollary 2.20. *Let M be an $\mathcal{I}$-$\oplus$-supplemented module and N be a direct summand of M such that M/N is N-projective. Then M is an N-$\mathcal{I}$-$\oplus$-supplemented module.*

Proof. Let L be a direct summand of M with $M = L + N$. Note that there exists a submodule N' of M such that $M = N \oplus N'$ as $N \leq^{\oplus} M$. Thus N' is N-projective. By [10, 41.14], there exists $L' \leq L$ such that $M = L' \oplus N$. So $L = L' \oplus (L \cap K)$. This means that $L \cap K \leq^{\oplus} M$. By Theorem 2.19, M is N-$\mathcal{I}$-$\oplus$-supplemented. ☑

Corollary 2.21. *Let $M = M_1 \oplus M_2$ be an $\mathcal{I}$-$\oplus$-supplemented module such that M_1 is M_2-projective. Then M is M_2-$\mathcal{I}$-$\oplus$-supplemented.*

Proof. It is easy to see by Corollary 2.20. ☑

Recall that a module M is said to have D_3 *condition* whenever $M = M_1 + M_2$, where M_1 and M_2 are direct summands of M, then $M_1 \cap M_2$ is a direct summand of M.

Corollary 2.22. (1) *Let $M = M_1 \oplus M_2$ be a module. If for every $\phi \in End_R(M)$ with $Im\phi \geq M_1$, there exists a direct summand K of M such that $K \leq M_2$, $M = K + Im\phi$ and $K \cap Im\phi \ll M$, then M_2 is $\mathcal{I}$-$\oplus$-supplemented.*

(2) *Let M be an $\mathcal{I}$-$\oplus$-supplemented module with D_3 condition. Then every direct summand of M is $\mathcal{I}$-$\oplus$-supplemented.*

(3) *Let $M = M_1 \oplus M_2$ be an $\mathcal{I}$-$\oplus$-supplemented module such that M_1 is M_2-projective. Then M_2 is $\mathcal{I}$-$\oplus$-supplemented.*

Proof. (1) By Lemma 2.18 and Theorem 2.2.

(2) By Theorem 2.19 and Theorem 2.2.

(3) By Corollary 2.21 and Theorem 2.2. ☑

Recall that a module M is said to be *Hopfian* if every epimorphism $\phi \in End_R(M)$ is an isomorphism.

Proposition 2.23. *Let M be an $\mathcal{T}$-noncosingular noetherian $\mathcal{I}$-$\oplus$-supplemented module. Then there exists a decomposition $M = M_1 \oplus \cdots \oplus M_n$, where M_i is an indecomposable noetherian $\mathcal{I}$-$\oplus$-supplemented module with $End_R(M_i)$ a division ring.*

Proof. Since M is noetherian, it has a finite decomposition with indecomposable noetherian direct summands. By Corollary 2.8, [6, Corollary 4.8] and since every noetherian module is Hopfian, each indecomposable direct summand has a division ring. ☑

3. Rings whose modules are $\mathcal{I}$-$\oplus$-supplemented

In this section, we study some rings whose modules are $\mathcal{I}$-$\oplus$-supplemented.

Theorem 3.1. *Consider the following conditions for a ring R:*

(1) *R is a semisimple artinian ring.*

(2) *Every right R-module is an $\mathcal{I}$-$\oplus$-supplemented module.*

(3) *Every free (projective) right R-module is an $\mathcal{I}$-$\oplus$-supplemented module.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *Moreover, if $Rad(R) = 0$, then* (3) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (1) Let I be a right ideal of R. There exists a free R-module F_R and an epimorphism $\phi : F_R \to I$. Note that $\phi(F_R) = I$. By (3), there exists a direct summand K of F_R such that $F_R = K + I$ and $K \cap I \ll K$. Since $Rad(R) = 0$, $K \cap I = 0$. Thus $I \leq^{\oplus} F_R$ and so $I \leq^{\oplus} R_R$. Therefore R is a semisimple artinian ring. ☑

Recall that an epimorphism $f : P \to M$ with P projective, is called a *projective cover* of M if $\mathrm{Ker} f \ll P$.

Proposition 3.2. *Let M be a projective module. Then the following are equivalent:*

(1) *M is an $\mathcal{I}$-$\oplus$-supplemented module.*

(2) *$M/Im\phi$ has a projective cover for every $\phi \in S$.*

Proof. (1) $\Rightarrow$ (2) Let M be an $\mathcal{I}$-$\oplus$-supplemented module and $\phi \in S$. There exists a direct summand $K \leq M$ with $M = \mathrm{Im}\phi + K$ and $\mathrm{Im}\phi \cap K \ll K$. Note that K is projective. Consider the epimorphism $f : K \to M \to M/\mathrm{Im}\phi$. Thus $\mathrm{Ker} f = \mathrm{Im}\phi \cap K \ll K$. Therefore $M/\mathrm{Im}\phi$ has a projective cover.

(2) $\Rightarrow$ (1) Let $\phi \in S$ and $f : P \to M/\mathrm{Im}\phi$ be a projective cover. Then there exists a homomorphism $g : M \to P$ such that $fg = \pi$, where $\pi : M \to$

$M/\mathrm{Im}\phi$ is the canonical epimorphism. Note that g is surjective. Thus g splits. Hence there exists a homomorphism $h : P \to M$ such that $gh = id_P$ and so $f = fgh = \pi h$. Thus $\mathrm{Im}\phi + h(P) = M$ and $\mathrm{Im}\phi \cap h(P) \ll h(P)$. Therefore M is $\mathcal{I}$-$\oplus$-supplemented. ☑

Corollary 3.3. *For a ring R the following statements are equivalent:*

(1) R_R *is $\mathcal{I}$-$\oplus$-supplemented.*

(2) $R_R/Im\phi$ *has a projective cover for every* $\phi \in End_R(R_R)$.

A ring R is called *perfect* if, every R-module has a projective cover.

Proposition 3.4. *Let R be a ring. Consider the following conditions:*

(1) *R is right perfect;*

(2) *Every projective right R-module is $\mathcal{I}$-$\oplus$-supplemented;*

(3) *Every free right R-module is $\mathcal{I}$-$\oplus$-supplemented.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) by Proposition 3.2.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (2) It follows from Corollary 2.11. ☑

A module M is called *lifting* if for all $N \leq M$, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M$.

Obviously, we have the following implications:

lifting $\Rightarrow$ $\oplus$-supplemented $\Rightarrow$ $\mathcal{I}$-$\oplus$-supplemented.

A ring R is called *f-semiperfect* if, every finitely presented R-module has a projective cover. A module M is said to be *principally $\oplus$-supplemented* if, for all cyclic submodule N of M, there exists a direct summand X of M such that $M = N + X$ and $N \cap X \ll X$.

The following theorem gives a characterization of f-semiperfect rings.

Theorem 3.5. *The following are equivalent for a ring R:*

(1) *R is f-semiperfect.*

(2) R_R *is finitely supplemented.*

(3) *Every cyclic right ideal has a supplement in* R_R.

(4) R_R *is principally lifting.*

(5) R_R *is principally $\oplus$-supplemented.*

(6) R_R *is $\mathcal{I}$-$\oplus$-supplemented.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) by [10, 42.11].

(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (3) are clear.

(5) $\Rightarrow$ (6) It is obvious because Imϕ is cyclic for every $\phi \in End_R(R_R)$.

(6) $\Rightarrow$ (5) Assume that $I = aR$ is any principal right ideal of R. Consider the R-homomorphism $\phi : R_R \to R_R$ defined by $\phi(r) = ar$, where $r \in R$. Then Im$\phi = I$. By (6), there exists a direct summand K of R_R such that Im$\phi + K = R_R$ and Im$\phi \cap K \ll K$. So R_R is principally $\oplus$-supplemented. ☑

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