DOI: https://doi.org/10.15446/recolma.v55n1.99097

Revista Colombiana de Matemáticas
Volumen 55(2021)1, páginas 21-41

Boundedness of the Maximal Function of the Ornstein-Uhlenbeck semigroup on variable Lebesgue spaces with respect to the Gaussian measure and consequences

Acotación de la Función Maximal del Semigrupo de Ornstein-Uhlenbeck en Espacios de Lebesgue Variables y sus consecuencias

Jorge Moreno[1], Ebner Pineda[2], Wilfredo Urbina[3,✉]

[1]Universidad Centro Occidental Lisandro Alvarado, Barquisimeto, Venezuela

[2]Escuela Superior Politécnica del Litoral, Guayaquil, Ecuador

[3]Roosevelt University, Chicago, USA

Abstract. The main result of this paper is the proof of the boundedness of the Maximal Function T^* of the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ in $\mathbb{R}^d$, on Gaussian variable Lebesgue spaces $L^{p(\cdot)}(\gamma_d)$, under a condition of regularity on $p(\cdot)$ following [5] and [8]. As an immediate consequence of that result, the $L^{p(\cdot)}(\gamma_d)$-boundedness of the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ in $\mathbb{R}^d$ is obtained. Another consequence of that result is the $L^{p(\cdot)}(\gamma_d)$-boundedness of the Poisson-Hermite semigroup and the $L^{p(\cdot)}(\gamma_d)$- boundedness of the Gaussian Bessel potentials of order $\beta > 0$.

Key words and phrases. Gaussian harmonic analysis, variable Lebesgue spaces, Ornstein-Uhlenbeck semigroup.

2020 Mathematics Subject Classification. Primary 42B25, 42B35 ; Secondary 46E30, 47G10 .

Resumen. El principal resultado de este artículo es la prueba de la acotación de la Función Maximal T^* del semigrupo de Ornstein-Uhlenbeck $\{T_t\}_{t\geq 0}$ en

$\mathbb{R}^d$ sobre espacios de Lebesgue variables respecto de la medida Gaussiana $L^{p(\cdot)}(\gamma_d)$, asumiendo una condición de regularidad en $p(\cdot)$ siguiendo [5] y [8]. Como consecuencia inmediata de éste resultado se obtiene la acotación-$L^{p(\cdot)}(\gamma_d)$ del semigrupo de Ornstein-Uhlenbeck $\{T_t\}_{t\geq 0}$ en $\mathbb{R}^d$. Otras consecuencias del resultado es la acotación $L^{p(\cdot)}(\gamma_d)$ del semigrupo Poisson-Hermite y la acotación $L^{p(\cdot)}(\gamma_d)$ de los potenciales de Bessel Gaussianos de orden $\beta > 0$.

Palabras y frases clave. Análisis Armónico Gaussiano, espacios de Lebesgue Gaussianos, semigrupo de Ornstein-Uhlenbeck.

1. Introduction and Preliminaries

The Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is the semigroup of operators generated in $L^2(\gamma_d)$ by the Ornstein-Uhlenbeck operator

$$L = \frac{1}{2}\Delta_x - \langle x, \nabla_x\rangle = \sum_{i=1}^{d}\Big[\frac{1}{2}\frac{\partial^2}{\partial x_i^2} - x_i\frac{\partial}{\partial x_i}\Big] \tag{1}$$

as infinitesimal generator, i.e., formally $T_t = e^{-tL}$. In view of the spectral theorem, for $f = \sum_{k=0}^{\infty}\mathbf{J}_k f \in L^2(\mathbb{R}^d, \gamma_d)$ and $t \geq 0$, T_t is defined as

$$T_t f = \sum_{\nu} e^{-t|\nu|}\langle f, \vec{h}_\nu\rangle_{\gamma_d}\vec{h}_\nu = \sum_{k=0}^{\infty} e^{-tk}\sum_{|\nu|=k}\langle f, \vec{h}_\nu\rangle_{\gamma_d}\vec{h}_\nu = \sum_{k=0}^{\infty} e^{-tk}\mathbf{J}_k f, \tag{2}$$

where $\{\vec{h}_\nu\}_\nu$ are the normalized Hermite polynomials in d variables, and

$$\mathbf{J}_k f = \sum_{|\nu|=k}\langle f, \vec{h}_\nu\rangle_{\gamma_d}\vec{h}_\nu$$

is the orthogonal projection of $L^2(\mathbb{R}^d, \gamma_d)$ onto

$$\mathcal{C}_k = \overline{\operatorname{span}\left(\left\{\vec{h}_\nu : |\nu| = k\right\}\right)}^{L^2(\mathbb{R}^d,\gamma_d)}.$$

Using Mehler's formula, it can be proved that the Ornstein-Uhlenbeck semigroup has an integral representation as

$$\begin{aligned} T_t f(x) =& \frac{1}{(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy) \\ =& \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} f(y)dy, \end{aligned} \tag{3}$$

for all $f \in L^1(\mathbb{R}^d, \gamma_d)$. Taking the change of variable $s = 1 - e^{-2t}$, we obtain that

$$T_t f(x) = \frac{1}{(\pi s)^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|y-\sqrt{(1-s)}x|^2}{s}} f(y)dy. \tag{4}$$

The *maximal function of the Ornstein-Uhlenbeck semigroup* is defined by

$$T^*f(x) = \sup_{t>0} |T_t f(x)|,$$

for all $x \in \mathbb{R}^d$.

It is well know that the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ in $\mathbb{R}^d$ is a *Markov operator semigroup* in $L^p(\mathbb{R}^d, \gamma_d), 1 \leq p \leq \infty$, i.e. a positive conservative symmetric diffusion semigroup, strongly L^p-continuous in $L^p(\mathbb{R}^d, \gamma_d), 1 \leq p \leq \infty$; with the Ornstein-Uhlenbeck operator L as its infinitesimal generator, see [2], [1] or [12]. Its properties can be obtained directly from the general theory of Markov semigroups, see [1] or [11]. It is also well known that the maximal function T^* is bounded in $L^p(\mathbb{R}^d, \gamma_d), 1 < p \leq \infty$.

Even thought there are some known results about boundedness of operators on Gaussian variable Lebesgue spaces $L^{p(\cdot)}(\gamma_d)$, see for instance [5], as far as we know, there is not proof in the literature of boundedness of the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$, nor of the boundedness of the maximal function of the semigroup. The main result of this paper is the proof that the maximal function T^* of the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ on $\mathbb{R}^d$ is bounded for Gaussian variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$, under certain conditions of regularity on $p(\cdot)$, that will be determined later (see Definitions 1.1, 1.2, 1.5 and 2.1)

Theorem 1.1. *Let $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \leq p_+ < \infty$. Then there exists a constant $C > 0$ such that*

$$||T^*f||_{p(\cdot),\gamma_d} \leq C||f||_{p(\cdot),\gamma_d}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$.

As a consequence of Theorems 1.1 we obtain,

Corollary 1.2. *Let $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \leq p_+ < \infty$. Then there exists a constant $C > 0$ such that*

$$||T_t f||_{p(\cdot),\gamma_d} \leq C||f||_{p(\cdot),\gamma_d}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ and for all $t \geq 0$.

An important remark is needed here. Observe that from Theorem 1.1 we can not conclude, as in the classical case, that the semigroup $\{T_t\}$ is a contraction semigroup is $L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$. We do not know if that is actually true for this case. Therefore questions like some form of hypercontractivity for the semigroup in this context are unknown.

Additionally, let us consider the *Poisson-Hermite semigroup* as the subordinated semigroup to the Ornstein-Uhlenbeck semigroup, using the *Bochner's subordination formula*, see E. Stein [10], defined then as,

$$\begin{aligned} P_t f(x) =& \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{(t^2/4u)} f(x) du \\ =& \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 t \frac{\exp(t^2/4\log r)}{(-\log r)^{3/2}} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r} f(y) dy. \end{aligned} \tag{5}$$

It is also well known that the Poisson-Hermite semigroup $\{P_t\}_{t\geq 0}$ is a strongly continuous, symmetric, conservative semigroup of positive contractions in $L^p(\gamma_d)$, $1 \leq p \leq \infty$, with infinitesimal generator $(-L)^{1/2}$. Additionally, the *maximal function of the Poisson-Hermite semigroup* is defined by

$$P^* f(x) = \sup_{t>0} |P_t f(x)|,$$

for all $x \in \mathbb{R}^d$.

As a consequence of the boundedness of $\{T_t\}$, we will prove that $\{P_t\}_{t\geq 0}$ is also bounded for Gaussian variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ under the same conditions of regularity on $p(\cdot)$.

Theorem 1.3. *Let $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \leq p_+ < \infty$. Then there exists a constant $C > 0$ such that*

$$||P_t f||_{p(\cdot),\gamma_d} \leq C ||f||_{p(\cdot),\gamma_d}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$.

Finally, the Gaussian Bessel potential of order $\beta > 0$, $\mathscr{J}_\beta$ is defined as

$$\mathscr{J}_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} P_s f(x)\, ds \tag{6}$$

for all $x \in \mathbb{R}^d$.

It can be proved, using P. A. Meyer's multiplier theorem, that the Gaussian Bessel potentials $\mathscr{J}_\beta$ are $L^p(\gamma_d)$-bounded $1 < p < \infty$. Moreover we will see that as consequence of Theorem 1.3 we obtain the boundedness of Gaussian Bessel potential on $L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$.

Theorem 1.4. *Let $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \leq p_+ < \infty$. Then there exists a constant $C > 0$ such that*

$$||\mathscr{J}_\beta f||_{p(\cdot),\gamma_d} \leq C ||f||_{p(\cdot),\gamma_d}$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ and $\beta > 0$.

Now, for completeness, let us introduce some basic background on variable Lebesgue spaces with respect to a Borel measure μ.

Any μ-measurable function $p(\cdot) : \mathbb{R}^d \to [1, \infty]$ is an exponent function; the set of all the exponent functions will be denoted by $\mathcal{P}(\mathbb{R}^d, \mu)$. For $E \subset \mathbb{R}^d$ we set

$$p_-(E) = \operatorname{ess}\inf_{x\in E} p(x) \text{ and } p_+(E) = \operatorname{ess}\sup_{x\in E} p(x).$$

We use the abbreviations $p_+ = p_+(\mathbb{R}^d)$ and $p_- = p_-(\mathbb{R}^d)$.

Definition 1.5. Let $E \subset \mathbb{R}^d$. We say that $\alpha(\cdot) : E \to \mathbb{R}$ is locally log-Hölder continuous, and denote this by $\alpha(\cdot) \in LH_0(E)$, if there exists a constant $C_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{C_1}{log(e + \frac{1}{|x-y|})}$$

for all $x, y \in E$. We say that $\alpha(\cdot)$ is log-Hölder continuous at infinity with base point at $x_0 \in \mathbb{R}^d$, and denote this by $\alpha(\cdot) \in LH_\infty(E)$, if there exist constants $\alpha_\infty \in \mathbb{R}$ and $C_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_2}{log(e + |x - x_0|)}$$

for all $x \in E$. We say that $\alpha(\cdot)$ is log-Hölder continuous, and denote this by $\alpha(\cdot) \in LH(E)$ if both conditions are satisfied. The maximum, $\max\{C_1, C_2\}$ is called the log-Hölder constant of $\alpha(\cdot)$.

Definition 1.6. We denote $p(\cdot) \in \mathcal{P}_d^{log}(\mathbb{R}^d)$, if $\frac{1}{p(\cdot)}$ is log-Hölder continuous and denote by $C_{log}(p)$ or C_{log} the log-Hölder constant of $\frac{1}{p(\cdot)}$.

We will need the following technical result; for its proof see Lemma 3.26 in [4].

Lemma 1.7. *Let $\rho(\cdot) : \mathbb{R}^d \to [0, \infty)$ be such that $\rho(\cdot) \in LH_\infty(\mathbb{R}^d)$, $0 < \rho_\infty < \infty$, and let $R(x) = (e + |x|)^{-N}$, $N > d/\rho_-$. Then there exists a constant C depending on d, N and the LH_∞ constant of $\rho(\cdot)$ such that given any set E and any function F with $0 \leq F(y) \leq 1$ for $y \in E$,*

$$\int_E F^{\rho(y)}(y)dy \leq C \int_E F(y)^{\rho_\infty} dy + \int_E R^{\rho_-}(y)dy, \tag{7}$$

$$\int_E F^{\rho_\infty}(y)dy \leq C \int_E F^{\rho(y)}(y)dy + \int_E R^{\rho^-}(y)dy. \tag{8}$$

Definition 1.8. For a μ-measurable function $f : \mathbb{R}^d \to \mathbb{R}$, we define the modular

$$\rho_{p(\cdot),\mu}(f) = \int_{\mathbb{R}^d\setminus\Omega_\infty} |f(x)|^{p(x)}\mu(dx) + \|f\|_{L^\infty(\Omega_\infty,\mu)}, \tag{9}$$

and the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^d,\mu)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot),\mu}(f/\lambda) \leq 1\right\}. \tag{10}$$

Definition 1.9. The variable exponent Lebesgue space on $\mathbb{R}^d$, $L^{p(\cdot)}(\mathbb{R}^d,\mu)$ consists on those μ-measurable functions f for which there exists $\lambda > 0$ such that $\rho_{p(\cdot),\mu}\left(\frac{f}{\lambda}\right) < \infty$, i.e.,

$$L^{p(\cdot)}(\mathbb{R}^d,\mu)=\left\{f\colon \mathbb{R}^d \to \overline{\mathbb{R}}\colon f \text{ measurable and } \rho_{p(\cdot),\mu}\left(\frac{f}{\lambda}\right)<\infty, \text{ for some } \lambda > 0\right\}.$$

If $\mathcal{B}$ is a family of balls (or cubes) in $\mathbb{R}^d$, we say that $\mathcal{B}$ is N-finite if it has bounded overlappings for N, that is $\sum_{B\in\mathcal{B}} \chi_B(x) \leq N$ for all $x \in \mathbb{R}^d$; in other words, there are at most N balls (resp. cubes) that intersect at the same time.

The following definition was introduced for the first time by Berezhnoĭ in [3], defined for a family of disjoint balls or cubes. In the context of variable spaces, it has been considered in [6], allowing the family to have bounded overlappings.

Definition 1.10. Given an exponent $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, we will say that $p(\cdot) \in \mathcal{G}$, if for every family of balls (or cubes) $\mathcal{B}$ which is N-finite, there is a constant C

$$\sum_{B\in\mathcal{B}} ||f\chi_B||_{p(\cdot)}||g\chi_B||_{p'(\cdot)} \leq C\,||f||_{p(\cdot)}||g||_{p'(\cdot)}$$

for all functions $f \in L^{p(\cdot)}(\mathbb{R}^d)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^d)$. The constant only depends on N.

Lemma 1.11 (Teorema 7.3.22 in [6])**.** *If $p(\cdot) \in LH(\mathbb{R}^d)$, then $p(\cdot) \in \mathcal{G}$.*

As usual, in what follows C represents a constant that is not necessarily the same in each occurrence; also we will use the following notation: given two functions f, g, the symbols $\lesssim$ and $\gtrsim$ denote, that there is a constant c such that $f \leq cg$ and $cf \geq g$, respectively. When both inequalities are satisfied, that is, $f \lesssim g \lesssim f$, we will denote $f \approx g$.

2. Proofs of the main results.

In this section we are going to consider Lebesgue variable spaces with respect to the Gaussian measure γ_d, $L^{p(\cdot)}(\mathbb{R}^d,\gamma_d)$. The next condition was introduced by E. Dalmasso and R. Scotto in [5].

Definition 2.1. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d,\gamma_d)$, we say that $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d)$ if there exist constants $C_{\gamma_d} > 0$ and $p_\infty \geq 1$ such that

$$|p(x) - p_\infty| \leq \frac{C_{\gamma_d}}{|x|^2}, \tag{11}$$

for $x \in \mathbb{R}^d \setminus \{(0,0,\ldots,0)\}$.

Observation 2.2. If $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d)$, then $p(\cdot) \in LH_{\infty}(\mathbb{R}^d)$

Lemma 2.3. *If $1 < p_- \leq p_+ < \infty$, the following statements are equivalent*

(i) $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d)$

(ii) There exists $p_\infty > 1$ such that

$$C_1^{-1} \leq e^{-|x|^2(p(x)/p_\infty - 1)} \leq C_1 \qquad \textit{and} \qquad C_2^{-1} \leq e^{-|x|^2(p'(x)/p'_\infty - 1)} \leq C_2,$$

for all $x \in \mathbb{R}^d$, where $C_1 = e^{C_{\gamma_d}/p_\infty}$ and $C_2 = e^{C_{\gamma_d}(p_-)'/p_\infty}$.

Definition 2.1 with Observation 2.2 and Lemma 2.3 end up strengthening the regularity conditions on the exponent functions $p(\cdot)$ to obtain the boundedness of the maximal function T^*. As a consequence of Lemma 1.11, we have

Corollary 2.4. *If $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$, then $p(\cdot) \in \mathcal{G}$.*

Lemma 2.5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d, \gamma_d)$, then*

$$\|f\|_{p(\cdot),\gamma_d} \approx \|fe^{-|\cdot|^2/p(\cdot)}\|_{p(\cdot)}.$$

Proof. Let

$$A = \left\{\lambda > 0 : \ \rho_{p(\cdot)}\left(\frac{fe^{-|\cdot|^2/p(\cdot)}}{\lambda}\right) \leq 1\right\}$$

and

$$B = \left\{\lambda > 0 : \ \rho_{p(\cdot),\gamma_d}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

We will prove that $\inf(A) \lesssim \inf(B)$ and $\inf(B) \lesssim \inf(A)$. In fact, taking $\lambda \in A$ then

$$\rho_{p(\cdot)}\left(\frac{fe^{-|\cdot|^2/p(\cdot)}}{\lambda}\right) = \int_{\mathbb{R}^d} \left|\frac{f(x)}{\lambda}\right|^{p(x)} e^{-|x|^2} dx \leq 1$$

which implies

$$\rho_{p(\cdot),\gamma_d}\left(\frac{f}{\lambda}\right) \leq \frac{1}{\pi^{d/2}} \leq 1$$

and then $\lambda \in B$. Therefore, $A \subset B$, and then $\inf B \leq \inf A$.

On the other hand, taking $\lambda \in B$ then

$$\rho_{p(\cdot),\gamma_d}\left(\frac{f}{\lambda}\right) = \int_{\mathbb{R}^d} \left|\frac{f(x)}{\lambda}\right|^{p(x)} \frac{e^{-|x|^2}}{\pi^{d/2}} dx \leq 1$$

which implies

$$\rho_{p(\cdot)}\left(\frac{fe^{-|\cdot|^2/p(\cdot)}}{\lambda\pi^{d/2}}\right) = \int_{\mathbb{R}^d}\left|\frac{f(x)e^{-|x|^2/p(x)}}{\lambda\pi^{d/2}}\right|^{p(x)} dx$$
$$\leq \int_{\mathbb{R}^d}\left|\frac{f(x)}{\lambda}\right|^{p(x)}\frac{e^{-|x|^2}}{\pi^{d/2}}dx = \rho_{p(\cdot),\gamma_d}\left(\frac{f}{\lambda}\right) \leq 1,$$

and therefore $\lambda \in \pi^{-d/2}A$. Thus $\inf A \leq \pi^{d/2}\inf B$, and then $\inf(A) \approx \inf(B)$.

Hence, we get

$$\|f\|_{p(\cdot),\gamma_d} \approx \|fe^{-|\cdot|^2/p(\cdot)}\|_{p(\cdot)}.$$

☑

2.1. Boundedness of the maximal function of Ornstein-Uhlenbeck semigroup with the condition $\mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d)$

For $x \in \mathbb{R}^d$ let us consider admissible (or hyperbolic) balls,

$$B_h(x) = \{y \in \mathbb{R}^d : |x-y| \leq d(1 \wedge 1/|x|)\}. \tag{12}$$

It is well known that the Gaussian measure is essentially constant on $B_h(x)$, see [12, Chapter 1].

As it is nowadays a standard technique in Gaussian harmonic analysis, we split T_t into its *local part* and its *global part*,

$$T_tf(x) = T_t^0f(x) + T_t^1f(x),$$

for $x \in \mathbb{R}^d$, where, using the integral representation (4) and the change of variables $s = 1 - e^{-2t}$ we will write

$$T_s^0f(x) := \int_{B_h(x)}\frac{e^{-\frac{|\sqrt{(1-s)}x-y|^2}{s}}}{(\pi s)^{d/2}}f(y)dy,$$

the local part, which is the restriction of T_t to the admissible ball $B_h(x)$, and

$$T_s^1f(x) := \int_{B_h^c(x)}\frac{e^{-\frac{|\sqrt{(1-s)}x-y|^2}{s}}}{(\pi s)^{d/2}}f(y)dy$$

the global part, which is the restriction of T_s to the complement of admissible ball $B_h(x)$.

Therefore the maximal function of the Ornstein-Uhlenbeck semigroup will be bounded by the sum of the operators,

$$T_0^*f(x) := \sup_{0\leq s<1}\left|\int_{B_h(x)}\frac{e^{-\frac{|\sqrt{(1-s)}x-y|^2}{s}}}{(\pi s)^{d/2}}f(y)dy\right|, \tag{13}$$

and

$$T_1^* f(x) := \sup_{0\leq s<1} \Big| \int_{B_h^c(x)} \frac{e^{-\frac{|\sqrt{(1-s)}x-y|^2}{s}}}{(\pi s)^{d/2}} f(y)dy \Big|, \tag{14}$$

which we call the local and global maximal operators respectively.

Next, we will need the following technical lemma to handle the proof of boundedness of the local part, for the proof see [12], for an earlier formulation see also [7].

Lemma 2.6. *Let us define the secuence $x_k = \sqrt{k}$ for $k \in \mathbb{N}$. For this strictly increasing secuence, we obtain a family of disjoint balls B_j^k, for $k \in \mathbb{N}$ and $1 \leq j \leq N_k$ with the following properties:*

(i) *If $\tilde{B_j^k} = 2B_j^k$, the colection $\mathcal{F} = \{B(0,1), \{\tilde{B_j^k}\}_{j,k}\}$ is a covering of $\mathbb{R}^d$;*

(ii) *$\mathcal{F}$ has bounded overlappings;*

(iii) *The center y_j^k of B_j^k, satisfies $|y_j^k| = (x_{k+1} + x_k)/2$;*

(iv) *$diam(B_j^k) = x_{k+1} - x_k = 1/(2|y_j^k|)$;*

(v) *For all ball $B \in \mathcal{F}$, and all $x, y \in B$, $\gamma_d(x) \approx \gamma_d(y)$ with constants independents on B;*

(vi) *There exists a uniform constant, $C_n > 0$, such that, if $x \in B \in \mathcal{F}$, then $B_h(x) \subset C_n B := \hat{B}$. Moreover, the colection $\hat{\mathcal{F}} = \{\hat{B}\}_{B\in\tilde{F}}$, also verifies properties (ii)-(v).*

Now, we present the boundedness of the local maximal operator T_0^*.

Theorem 2.7. *Let $p(\cdot) \in \mathcal{P}_{\gamma_d}^{\infty}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \leq p_+ < \infty$. There exists a constant $C > 0$ such that*

$$||T_0^* f||_{p(\cdot),\gamma_d} \leq C||f||_{p(\cdot),\gamma_d}$$

for all function $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$.

Proof. We follow the proof of Theorem 3.3. in [5]. Without lost of generality let us assume that $f \geq 0$.

$$\begin{aligned} T_s^0 f(x) &:= \int_{B_h(x)} \frac{e^{-\frac{|\sqrt{(1-s)}x-y|^2}{s}}}{(\pi s)^{d/2}} f(y)dy = \int_{\mathbb{R}^d} \frac{e^{-u(s)}}{(\pi s)^{d/2}} f(y)\chi_{B_h(x)}(y)dy \\ &= \int_{\mathbb{R}^d} M(s,x,y) f(y) \chi_{B_h(x)}(y)dy \end{aligned}$$

where $M(s,x,y) = \dfrac{e^{-u(s)}}{(\pi s)^{d/2}}$, and $u(s) = \dfrac{|\sqrt{(1-s)}x - y|^2}{s}$.

Following [8] we obtain that if $y \in B_h(x)$ then $e^{-u(s)} \leq C_d e^{-\frac{|x-y|^2}{s}}$ and therefore

$$M(s,x,y) \leq C_d \frac{e^{-\frac{|x-y|^2}{s}}}{(\pi s)^{d/2}}. \tag{15}$$

Now, given $x \in \mathbb{R}^d$, by Lemma 2.6, there exists $B \in \mathcal{F}$ such that $x \in B$ and $B_h(x) \subset \hat{B}$, so we get,

$$\begin{aligned}\int_{\mathbb{R}^d} M(s,x,y) f(y) \chi_{B_h(x)}(y)dy &\leq \int_{B_h(x)} C_d \frac{e^{-\frac{|x-y|^2}{s}}}{(\pi s)^{d/2}} f(y)dy \\ &\leq C_d \int_{\mathbb{R}^d} \frac{e^{-\frac{|x-y|^2}{s}}}{s^{d/2}} f(y)\chi_{\hat{B}}(y)dy.\end{aligned}$$

Set $\phi_s(z) = \frac{e^{-\frac{|z|^2}{s}}}{s^{d/2}}$ and since $\{\phi_s\}_{s>0}$ is an approximation of identity, we have

$$\begin{aligned}\int_{\mathbb{R}^d} M(s,x,y) f(y) \chi_{B_h(x)}(y)dy &\leq C_d |(\phi_s * f\chi_{\hat{B}})(x)| \\ &\leq C_d M_{H-L}(f\chi_{\hat{B}})(x).\end{aligned}$$

Therefore,

$$T_s^0 f(x) \lesssim M_{H-L}(f\chi_{\hat{B}})(x) = M_{H-L}(f\chi_{\hat{B}})(x)\chi_B(x) \text{ if } x \in B.$$

Thus,

$$T_s^0 f(x) \lesssim \sum_{B \in \mathcal{F}} M_{H-L}(f\chi_{\hat{B}})(x)\chi_B(x), \tag{16}$$

for all $x \in \mathbb{R}^d$. Since, the right hand side is independent of s we immediately get

$$T_0^* f(x) \lesssim \sum_{B \in \mathcal{F}} M_{H-L}(f\chi_{\hat{B}})(x)\chi_B(x), \tag{17}$$

for all $x \in \mathbb{R}^d$. Let $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$. Using the characterization of the norm by duality,

$$||T_0^* f||_{p(\cdot),\gamma_d} \leq 2 \sup_{||g||_{p'(\cdot),\gamma_d} \leq 1} \int_{\mathbb{R}^d} T_0^* f(x)|g(x)|\gamma_d(dx)$$

from (17) and following again [5] we obtain that

$$\int_{\mathbb{R}^d} T_0^* f(x)|g(x)|\gamma_d(dx) \lesssim \sum_{B \in \mathcal{F}} e^{-|c_B|^2} \int_{\mathbb{R}^d} M_{H-L}(f\chi_{\hat{B}})(x)|g(x)|\chi_B(x)dx$$

where c_B is the center of the balls B and $\hat{B}$. Using Hölder's inequality and the boundedness of the maximal operator M_{H-L} on $L^{p(\cdot)}(\mathbb{R}^d)$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} T_0^* f(x)|g(x)|\gamma_d(dx) &\lesssim \sum_{B\in\mathcal{F}} e^{-|c_B|^2\left(\frac{1}{p_\infty}+\frac{1}{p'_\infty}\right)}\|M_{H-L}(f\chi_{\hat{B}})\|_{p(\cdot)}\|g\chi_B\|_{p'(\cdot)}\\ &\lesssim \sum_{B\in\mathcal{F}} e^{-\frac{|c_B|^2}{p_\infty}}\|f\chi_{\hat{B}}\|_{p(\cdot)}e^{-\frac{|c_B|^2}{p'_\infty}}\|g\|_{p'(\cdot)}\end{aligned}$$

since $p(\cdot)\in\mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d)$, by Lemma 2.3, we obtain that

$$e^{-\frac{|c_B|^2}{p_\infty}}\|f\chi_{\hat{B}}\|_{p(\cdot)} \lesssim \|f\chi_{\hat{B}}\|_{p(\cdot),\gamma_d}$$

and

$$e^{-\frac{|c_B|^2}{p'_\infty}}\|g\chi_{\hat{B}}\|_{p'(\cdot)} \lesssim \|g\chi_{\hat{B}}\|_{p'(\cdot),\gamma_d}.$$

By Lemma 2.5, we have that

$$\|f\chi_{\hat{B}}\|_{p(\cdot),\gamma_d} \approx \|f\chi_{\hat{B}}e^{-|\cdot|^2/p(\cdot)}\|_{p(\cdot)} \quad\text{and}\quad \|g\chi_{\hat{B}}\|_{p'(\cdot),\gamma_d} \approx \|g\chi_{\hat{B}}e^{-|\cdot|^2/p'(\cdot)}\|_{p'(\cdot)}$$

and therefore,

$$\int_{\mathbb{R}^d} T_0^* f(x)|g(x)|\gamma_d(dx) \lesssim \sum_{B\in\mathcal{F}}\|f\chi_{\hat{B}}e^{-|\cdot|^2/p(\cdot)}\|_{p(\cdot)}\|g\chi_{\hat{B}}e^{-|\cdot|^2/p'(\cdot)}\|_{p'(\cdot)}.$$

Since the family of balls $\hat{\mathcal{F}}$ has bounded overlaps; applying Corollary 2.4, to the functions $fe^{-|\cdot|^2/p(\cdot)}\in L^{p(\cdot)}(\mathbb{R}^d)$ and $ge^{-|\cdot|^2/p'(\cdot)}\in L^{p'(\cdot)}(\mathbb{R}^d)$ and again applying Lemma 2.5, we get

$$\int_{\mathbb{R}^d} T_0^* f(x)|g(x)|\gamma_d(dx) \lesssim \|f\|_{p(\cdot),\gamma_d}\|g\|_{p'(\cdot),\gamma_d}.$$

Taking supremum on all the functions $g\in L^{p'(\cdot)}(\mathbb{R}^d,\gamma_d)$ with

$\|g\|_{p'(\cdot),\gamma_d}\le 1$, we obtain that

$$\begin{aligned}||T_0^* f||_{p(\cdot),\gamma_d} &\lesssim \sup_{||g||_{p'(\cdot),\gamma_d}\le 1}\int_{\mathbb{R}^d} T_0^* f(x)|g(x)|\gamma_d(dx)\\ &\lesssim \sup_{||g||_{p'(\cdot),\gamma_d}\le 1}\|f\|_{p(\cdot),\gamma_d}\|g\|_{p'(\cdot),\gamma_d} = \|f\|_{p(\cdot),\gamma_d}.\end{aligned}$$

☑

Finally, we will obtain the boundedness of the global maximal operator

Theorem 2.8. *Let $p(\cdot) \in \mathcal{P}^{\infty}_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ con $1 < p_- \leq p_+ < \infty$. Then there exists a constant $C > 0$ such that*

$$||T_1^* f||_{p(\cdot),\gamma_d} \leq C||f||_{p(\cdot),\gamma_d}$$

for all the function $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$.

Proof. Suppose that $f \geq 0$. Again, we follow the proof of Theorem 3.5 in [5].

$$T_s^1 f(x) := \int_{B_h^c(x)} \frac{e^{-\frac{|\sqrt{(1-s)}x-y|^2}{s}}}{(\pi s)^{d/2}} f(y)dy = \int_{B^c(x)} M(s,x,y)f(y)dy.$$

For $x \in \mathbb{R}^d$ fix, set $E_x = \{y : b(x,y) > 0\}$ where $b := b(x,y) = 2\langle x, y\rangle$. Given $y \in B_h^c(x)$, the following inequalities are satisfied:

(i) If $b \leq 0$, then
$$M(s,x,y) \lesssim e^{-|y|^2}. \tag{18}$$

(ii) If $b > 0$, then
$$M(s,x,y) \lesssim \frac{e^{-u_0}}{t_0^{d/2}} \tag{19}$$

where $a = |x|^2 + |y|^2$, $t_0 = 2\sqrt{a^2 - b^2}/(a + \sqrt{a^2 - b^2})$ and $u_0 = \frac{1}{2}(|y|^2 - |x|^2 + |x+y||x-y|)$. For details see [8] or [12, Chapter 4].

Let $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ with $f \geq 0$ and $\|f\|_{p(\cdot),\gamma_d} = 1$. If $b \leq 0$, applying (18) and Hölder's inequality for the exponent p_- we obtain that

$$\begin{aligned}
I &= \int_{\mathbb{R}^d} (T_1^*(f\chi_{E_x^c})(x))^{p(x)} \gamma_d(dx) \\
&= \int_{\mathbb{R}^d} \left(\sup_{0 \leq s < 1} \Big| \int_{B_h^c(x) \cap E_x^c} M(s,x,y)f(y)dy \Big| \right)^{p(x)} \gamma_d(dx) \\
&\lesssim \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (f(y))^{p_-} e^{-|y|^2} dy \right)^{p(x)/p_-} \gamma_d(dx).
\end{aligned}$$

Moreover, $\rho_{p(\cdot),\gamma_d}(f) \leq 1$, implies that,

$$\begin{aligned}
I &\lesssim \int_{\mathbb{R}^d} \left(\int_{|f|>1} (f(y))^{p_-} e^{-|y|^2} dy + \int_{|f|\leq 1} (f(y))^{p_-} e^{-|y|^2} dy \right)^{p(x)/p_-} \gamma_d(dx) \\
&\lesssim \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (f(y))^{p(y)} \gamma_d(dy) + \int_{\mathbb{R}^d} \gamma_d(dy) \right)^{p(x)/p_-} \gamma_d(dx) \\
&\lesssim \int_{\mathbb{R}^d} (2)^{p(x)/p_-} \gamma_d(dx) = C_{d,p}.
\end{aligned}$$

With this we obtain that $\|T_1^*(f\chi_{E^c_{(\cdot)}})\|_{p(\cdot),\gamma_d} \leq C_{d,p}$.

Now, if $b > 0$ by (19) and for all $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ with $\|f\|_{p,\gamma_d} = 1$, we have that

$$
\begin{aligned}
II &= \int_{\mathbb{R}^d} (T_1^*(f\chi_{E_x})(x))^{p(x)}\gamma_d(dx) \\
&= \int_{\mathbb{R}^d} \left(\sup_{0\leq s<1} \Big| \int_{B^c_h(x)\cap E_x} M(s,x,y)f(y)dy \Big| \right)^{p(x)} \gamma_d(dx) \\
&\lesssim \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} \left(\int_{B^c_h(x)\cap E_x} \frac{e^{-u_0}e^{|y|^2/p(y)}e^{-|x|^2/p(x)}}{t_0^{d/2}} f(y)e^{-|y|^2/p(y)}dy \right)^{p(x)} dx.
\end{aligned}
$$

Since $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d)$, we obtain that $e^{|y|^2/p(y)-|x|^2/p(x)} \approx e^{(|y|^2-|x|^2)/p_\infty}$ and by the Cauchy-Schwartz inequality we have, $\big||y|^2 - |x|^2\big| \leq |x+y||x-y|$, for all $x, y \in \mathbb{R}^d$. On the other hand, for $b > 0$, $|x+y||x-y| \geq d$ wherever $y \in B^c_h(x)$. In fact, since $b > 0$

$$|x+y| \geq |x-y| \text{ and } |x+y| > |x|.$$

Now, since $y \in B^c_h(x)$, we consider two cases:

<u>Case 1:</u> If $|x| \leq 1$, applying (2.1), we obtain that $|x-y| \geq d\left(1 \wedge \frac{1}{|x|}\right) = d$ and then

$$|x-y||x+y| \geq d^2 \geq d.$$

<u>Case 2:</u> If $|x| > 1$, applying (2.1), we obtain that $|x-y| \geq d\left(1 \wedge \frac{1}{|x|}\right) = \frac{d}{|x|}$ and then

$$|x-y||x+y| \geq |x-y||x| \geq d.$$

Moreover, $t_0 \approx |x+y||x-y|/(|x|^2+|y|^2)$. Since $|x|^2+|y|^2 = a < a+b = |x+y|^2$, we have that

$$t_0 \geq c\frac{|x+y||x-y|}{|x|^2+|y|^2} \geq c\frac{d}{|x+y|^2}$$

thus

$$\frac{1}{t_0^{d/2}} \lesssim |x+y|^d.$$

Therefore,

$$
\begin{aligned}
&\int_{B^c_h(x)\cap E_x} \frac{e^{-u_0}e^{|y|^2/p(y)}e^{-|x|^2/p(x)}}{t_0^{d/2}} f(y)e^{-|y|^2/p(y)}dy \\
&\qquad\qquad \lesssim \int_{B^c_h\cap E_x} P(x,y)f(y)e^{-|y|^2/p(y)}dy,
\end{aligned}
$$

where

$$P(x,y) = |x+y|^d e^{-\alpha_\infty |x+y||x-y|} \text{ and } \alpha_\infty = \left(\frac{1}{2} - \left|\frac{1}{p_\infty} - \frac{1}{2}\right|\right) > 0.$$

It can be proved that $P(x,y)$ is integrable on each variable (for details see [8]) and the value of each integral is independent on x and y.

Set $A_x = \left\{y : \frac{d}{|x|} < |y-x| < \frac{1}{2}\right\}$ and $C_x = B^c(x,1/2) = \left\{y : \; |y-x| > \frac{1}{2}\right\}$, thus it follows, $B_h^c(x) \subset A_x \cup C_x$. Define

$$J_1 = \int_{A_x \cap E_x} P(x,y) f(y) e^{-|y|^2/p(y)} dy \;\text{ and }\; J_2 = \int_{C_x \cap E_x} P(x,y) f(y) e^{-|y|^2/p(y)} dy.$$

We will estimate J_1 first. Observe that, if $y \in A_x$, $\frac{3}{4}|x| \le |y| \le \frac{5}{4}|x|$ and then $|x| \approx |y|$ hence $|x| \approx |x+y|$, and thus

$$\begin{aligned} J_1 &\lesssim \int_{\frac{d}{|x|} < |x-y|} |x|^d e^{-\alpha_\infty |x||x-y|} f(y) e^{-|y|^2/p(y)} dy \\ &\lesssim M_{H-L}(f e^{-|\cdot|^2/p(\cdot)})(x). \end{aligned}$$

From the hypothesis on $p(\cdot)$ we get

$$\|M_{H-L}(f e^{-|\cdot|^2/p(\cdot)})\|_{p(\cdot)} \lesssim \|f e^{-|\cdot|^2/p(\cdot)}\|_{p(\cdot)} \approx \|f\|_{p(\cdot),\gamma_d} = 1,$$

therefore

$$\rho_{p(\cdot)}\left(M_{H-L}(f e^{-|\cdot|^2/p(\cdot)})\right) \lesssim 1. \tag{20}$$

In order to estimate J_2, we have

$$J_2 \le \|P(x,\cdot)\chi_{C_x}\|_{p'(\cdot)} \le C;$$

for details see [5]. This implies that there exists a constant independent on x such that,

$$J_2 = \int_{C_x \cap E_x} P(x,y) f(y) e^{-|y|^2/p(y)} dy \le C$$

thus

$$\frac{1}{C} \int_{C_x \cap E_x} P(x,y) f(y) e^{-|y|^2/p(y)} dy \le 1.$$

We set $g(y) = f(y)e^{-|y|^2/p(y)} = g_1(y) + g_2(y)$, where $g_1 = g\chi_{\{g\geq 1\}}$ and $g_2 = g\chi_{\{g<1\}}$. Applying (20), we have

$$
\begin{aligned}
II &\lesssim \int_{\mathbb{R}^d} \left(\int_{B_h^c(x)\cap E_x} P(x,y) f(y) e^{-|y|^2/p(y)} dy \right)^{p(x)} dx \\
&\lesssim \int_{\mathbb{R}^d} (J_1)^{p(x)}\, dx + \int_{\mathbb{R}^d} (J_2)^{p(x)}\, dx \\
&\lesssim \rho_{p(\cdot)} \left(M_{H-L}(f e^{-|\cdot|^2/p(\cdot)}) \right) + \int_{\mathbb{R}^d} \left(\frac{1}{C} \int_{C_x\cap E_x} P(x,y) g_1(y) dy \right)^{p(x)} dx \\
&+ \int_{\mathbb{R}^d} \left(\frac{1}{C} \int_{C_x\cap E_x} P(x,y) g_2(y) dy \right)^{p(x)} dx \\
&\lesssim 1 + II_1 + II_2.
\end{aligned}
$$

Now, we study the integrals II_1 y II_2.

$$
II_1 = \int_{\mathbb{R}^d} \left(\frac{1}{C} \int_{C_x\cap E_x} P(x,y) g_1(y) dy \right)^{p(x)} dx \leq \int_{\mathbb{R}^d} \left(\frac{1}{C} \int_{C_x\cap E_x} P(x,y) g_1(y) dy \right)^{p_-} dx
$$

On the other hand, using Lemma 1.7 with

$$
G(x) = \frac{1}{C} \int_{C_x\cap E_x} P(x,y) g_2(y) dy \leq 1
$$

and applying the inequality (7), we obtain that

$$
\begin{aligned}
II_2 &= \int_{\mathbb{R}^d} \left(\int_{C_x\cap E_x} \frac{1}{C} P(x,y) g_2(y) dy \right)^{p(x)} dx = \int_{\mathbb{R}^d} (G(x))^{p(x)} dx \\
&\lesssim \int_{\mathbb{R}^d} (G(x))^{p_\infty} dx + \int_{\mathbb{R}^d} \frac{dx}{(e+|x|)^{-dp_-}} \\
&= \int_{\mathbb{R}^d} \left(\frac{1}{C} \int_{C_x\cap E_x} P(x,y) g_2(y) dy \right)^{p_\infty} + C_{d,p},
\end{aligned}
$$

therefore

$$
II \lesssim \int_{\mathbb{R}^d} \left(\int_{C_x\cap E_x} P(x,y) g_1(y) dy \right)^{p_-} dx + \int_{\mathbb{R}^d} \left(\int_{C_x\cap E_x} P(x,y) g_2(y) dy \right)^{p_\infty} dx + C_{d,p}.
$$

Now, in order to estimate the last two integrals, we apply Hölder's inequality.

$$\int_{\mathbb{R}^d}\left(\int_{C_x\cap E_x}P(x,y)g_1(y)dy\right)^{p_-}dx \leq \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}P(x,y)^{\frac{1}{p'_-}}P(x,y)^{\frac{1}{p_-}}g_1(y)dy\right)^{p_-}dx$$
$$\leq \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(P(x,y))^{p'_-/p'_-}dy\right)^{p_-/p'_-}\left(\int_{\mathbb{R}^d}(P(x,y))^{p_-/p_-}g_1^{p_-}(y)dy\right)^{p_-/p_-}dx$$
$$= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}P(x,y)dy\right)^{p_-/p'_-}\left(\int_{\mathbb{R}^d}P(x,y)g_1^{p_-}(y)dy\right)dx$$
$$\lesssim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}P(x,y)g_1^{p_-}(y)dydx.$$

Then, by Fubbini's theorem we get,

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\int_{C_x\cap E_x}P(x,y)g_1(y)dy\right)^{p_-}dx &\lesssim \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}P(x,y)g_1^{p_-}(y)dydx\\ &= \int_{\mathbb{R}^d}g_1^{p_-}(y)\left(\int_{\mathbb{R}^d}P(x,y)dx\right)dy\\ &\lesssim \int_{\mathbb{R}^d}(g_1(y))^{p(y)}dy\\ &\lesssim \int_{\mathbb{R}^d}f(y)^{p(y)}e^{-|y|^2}dy \lesssim \rho_{p(\cdot),\gamma_d}(f)\end{aligned}$$

Now, we need to estimate the integral

$$\int_{\mathbb{R}^d}\left(\int_{C_x\cap E_x}P(x,y)g_2(y)dy\right)^{p_\infty}dx.$$

We proceed in an analogous way, but applying the Hölder's inequality to the exponent p_∞, and applying the inequality (8) in Lemma 1.7. Thus, it follows

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\int_{C_x\cap E_x}P(x,y)g_2(y)dy\right)^{p_\infty}dx &\lesssim \int_{\mathbb{R}^d}g_2^{p(y)}(y)dy + C_{d,p}\\ &\lesssim \rho_{p(\cdot),\gamma_d}(f)+C,\end{aligned}$$

therefore,

$$II \lesssim \int_{\mathbb{R}^d}\left(\int_{C_x\cap E_x}P(x,y)g_1(y)dy\right)^{p_-}dx+\int_{\mathbb{R}^d}\left(\int_{C_x\cap E_x}P(x,y)g_2(y)dy\right)^{p_\infty}dx + C_{d,p}$$
$$\leq 2\rho_{p(\cdot),\gamma_d}(f)+C_{d,p}$$

With this we obtain that $\|T_1^*(f\chi_{E_{(\cdot)}})\|_{p(\cdot),\gamma_d} \leq C_{d,p}$, then by homogenity of the norm the result holds for all function $f \in L^{p(\cdot)}(\mathbb{R}^d,\gamma_d)$.

Hence

$$\|T_1^* f\|_{p(\cdot),\gamma_d} \lesssim \|T_1^*(f\chi_{E_{(\cdot)}})\|_{p(\cdot),\gamma_d} + \|T_1^*(f\chi_{E^c_{(\cdot)}})\|_{p(\cdot),\gamma_d} \lesssim \|f\|_{p(\cdot),\gamma_d}.$$

☑

Now, the proof of the $L^{p(\cdot)}(\gamma_d)$-boundedness of the maximal function T^* of the Ornstein-Uhlenbeck semigroup, Theorem 1.1, is a immediate consequence of Theorems 2.7 and 2.8, since we have

$$||T^* f||_{p(\cdot),\gamma_d} \leq \|T_0^* f\|_{p(\cdot),\gamma_d} + \|T_1^* f\|_{p(\cdot),\gamma_d} \leq C\|f\|_{p(\cdot),\gamma_d}. \quad ☑$$

On the other hand, the $L^{p(\cdot)}(\gamma_d)$-boundedness of the Ornstein-Uhlenbeck semigroup, Corollary 1.2 is immediate from Theorem 1.1. Moreover, one can get a direct proof of it repeating the proof of Theorems 2.7 and 2.8 by simply using (16), (18) and (19).

Additionally, from the $L^{p(\cdot)}(\mathbb{R}^d,\gamma_d)$-boundedness of T^* we obtain

Theorem 2.9. *Let $p(\cdot) \in \mathcal{P}^\infty_{\gamma_d}(\mathbb{R}^d) \cap LH_0(\mathbb{R}^d)$ with $1 < p_- \leq p_+ < \infty$, and $f \in L^{p(\cdot)}(\mathbb{R}^d,\gamma_d)$. The application $t \to T_t f$ is continuous from $[0,\infty)$ to $L^{p(\cdot)}(\mathbb{R}^d,\gamma_d)$.*

Proof. We have to prove that $T_t f \to T_{t_0} f$ on $L^{p(\cdot)}(\mathbb{R}^d,\gamma_d)$ if $t \to t_0$. By the property of semigroup, it is enough to prove that $T_t f \to f$ in $L^{p(\cdot)}(\mathbb{R}^d,\gamma_d)$ if $t \to 0^+$.

As $f \in L^{p(\cdot)}(\mathbb{R}^d,\gamma_d)$, then $f(x) < \infty$ *a.e.* $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d,\gamma_d)$. Let $f_t(x) = |T_t f(x) - f(x)|^{p(x)}$, from the pointwise convergence of the Ornstein-Uhlenbeck semigroup (see [9]), we have,

$$\lim_{t\to 0^+} f_t(x) = \lim_{t\to 0^+} |T_t f(x) - f(x)|^{p(x)} = 0, \ a.e. \ x \in \mathbb{R}^d$$

On the other hand,

$$\begin{aligned} |T_t f(x) - f(x)|^{p(x)} &\leq 2^{p_+}\left(|T_t f(x)|^{p(x)} + |f(x)|^{p(x)}\right) \\ &\leq 2^{p_+}\left(|T^* f(x)|^{p(x)} + |f(x)|^{p(x)}\right). \end{aligned}$$

Set $g(x) = 2^{p_+}\left(|T^* f(x)|^{p(x)} + |f(x)|^{p(x)}\right)$ for all $x \in \mathbb{R}^d$. Then g is integrable, in fact

$$\begin{aligned} \int_{\mathbb{R}^d} g(x)\gamma_d(dx) &= \int_{\mathbb{R}^d} 2^{p_+}\left(|T^* f(x)|^{p(x)} + |f(x)|^{p(x)}\right)\gamma_d(dx) \\ &= 2^{p_+}\left(\int_{\mathbb{R}^d} |T^* f(x)|^{p(x)}\gamma_d(dx) + \int_{\mathbb{R}^d} |f(x)|^{p(x)}\gamma_d(dx)\right) \\ &= 2^{p_+}\left(\rho_{p(\cdot),\gamma_d}(T^* f) + \rho_{p(\cdot),\gamma_d}(f)\right) < \infty, \end{aligned}$$

since f, $T^*f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$.

Applying Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} f_t(x)\gamma_d(dx) = \int_{\mathbb{R}^d} \lim_{t\to 0^+} f_t(x)\gamma_d(dx) = 0.$$

Thus,

$$0=\lim_{t\to 0^+}\int_{\mathbb{R}^d} f_t(x)\gamma_d(dx)=\lim_{t\to 0^+}\int_{\mathbb{R}^d} |T_tf(x) - f(x)|^{p(x)}\, \gamma_d(dx)=\lim_{t\to 0^+} \rho_{p(\cdot),\gamma_d}(T_tf-f)$$

Then, $\rho_{p(\cdot),\gamma_d}(T_tf - f) \to 0,\;\; t \to 0^+$ and hence $\|T_tf - f\|_{p(\cdot),\gamma_d} \to 0,\;\; t \to 0^+$. Therefore, $T_tf \to f$ on $L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ as $t \to 0^+$. ☑

2.2. Consequences of the Boundedness of the Ornstein-Uhlenbeck semigroup

Another consequence of Theorem 1.1 is the boundedness of Poisson-Hermite semigroup in $L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$, Theorem 1.3:

Proof*. *of Theorem 1.3. Let $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ with $\|f\|_{p(\cdot),\gamma_d} \leq 1$, then by Corollary 1.2, we have for every $s > 0$

$$||T_sf||_{p(\cdot),\gamma_d} \leq C||f||_{p(\cdot),\gamma_d} \leq C,$$

and therefore

$$\left\|\frac{T_sf}{C}\right\|_{p(\cdot),\gamma_d} \leq 1.$$

Thus

$$\rho_{p(\cdot),\gamma_d}\left(\frac{T_sf}{C}\right) \leq 1.$$

For fixed $t > 0$, since the measure $\mu_t^{1/2}(ds)$ is a probability measure, using Jensen's inequality, and Fubini's theorem we get that the modular is less or equal to 1. In fact,

$$\begin{aligned}
\rho_{p(\cdot),\gamma_d}\left(\frac{P_tf}{C}\right) &= \int_{\mathbb{R}^d} \left(\frac{P_tf(x)}{C}\right)^{p(x)} \gamma_d(dx) \\
&\leq \int_{\mathbb{R}^d} \int_0^{+\infty} \left|\frac{T_sf(x)}{C}\right|^{p(x)} \mu_t^{1/2}(ds)\gamma_d(dx) \\
&= \int_0^{+\infty} \int_{\mathbb{R}^d} \left|\frac{T_sf(x)}{C}\right|^{p(x)} \gamma_d(dx)\mu_t^{1/2}(ds) \\
&= \int_0^{+\infty} \rho_{p(\cdot),\gamma_d}\left(\frac{T_sf}{C}\right)\mu_t^{1/2}(ds) \leq 1.
\end{aligned}$$

Thus, $P_t f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ and $\|P_t f\|_{p(\cdot),\gamma_d} \leq C, \quad$ for all $t > 0$.

Now, by homogeneity of the norm and the linearity of P_t we obtain the general result.

$$\|P_t f\|_{p(\cdot),\gamma_d} \leq C\|f\|_{p(\cdot),\gamma_d}$$

for any function $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ and $t > 0$. ☑

Additionally, as a consequence of Theorem 1.3 we obtain the boundedness of Gaussian Bessel potentials, Theorem 1.4:

Proof. of Theorem 1.4. Let $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ with $\|f\|_{p(\cdot),\gamma_d} \leq 1$, we already know, from the proof of Theorem 1.3 that, for every $s > 0$, $||P_s f||_{p(\cdot),\gamma_d} \leq C||f||_{p(\cdot),\gamma_d} \leq C$ and therefore $\rho_{p(\cdot),\gamma_d}\left(\frac{P_s f}{C}\right) \leq 1$.

Now, for fixed $\beta > 0$, using the Jensen's inequality and Fubini's theorem, we get,

$$\begin{aligned}
\rho_{p(\cdot),\gamma_d}\left(\frac{\mathscr{I}_\beta f}{C}\right) &= \int_{\mathbb{R}^d} \left|\frac{\mathscr{I}_\beta f(x)}{C}\right|^{p(x)} \gamma_d(dx) \\
&\leq \int_{\mathbb{R}^d} \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} \left|\frac{P_s f(x)}{C}\right|^{p(x)} ds\, \gamma_d(dx) \\
&= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} \int_{\mathbb{R}^d} \left|\frac{P_s f(x)}{C}\right|^{p(x)} \gamma_d(dx)\, ds \\
&= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} \rho_{p(\cdot),\gamma_d}\left(\frac{P_s f}{C}\right) ds \leq 1.
\end{aligned}$$

Thus $\mathscr{I}_\beta f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$ and

$$\|\mathscr{I}_\beta f\|_{p(\cdot),\gamma_d} \leq C,$$

for any $\beta > 0$. Now, again by homogeneity of the norm and linearity of $\mathscr{I}_\beta$ we get the general result,

$$\|\mathscr{I}_\beta f\|_{p(\cdot),\gamma_d} \leq C\|f\|_{p(\cdot),\gamma_d}$$

for any function $f \in L^{p(\cdot)}(\mathbb{R}^d, \gamma_d)$. ☑

We want to thank the reviewer for his/her commentaries and corrections which improved greatly the presentation as well as the clarity of the results of the paper.

References

[1] D. Bakry, *Functional inequalities for Markov semigroups.*, Probability measures on groups: recent directions and trends, Tata Inst. Fund. Res., Mumbai, 2006.

[2] D. Bakry and O. Mazet, *Characterization of Markov semigroups on $\mathbb{R}$ associated to some families of orthogonal polynomials*, Sem. Prob. XXXVII. Lec. Notes in Math 1832 Springer (2003), 60–80.

[3] E. Berezhnoǐ, *Two-weighted estimations for the Hardy-Littlewood maximal function in ideal banach spaces*, Proc Amer Math Soc **127** (1999), no. 1, 79–87.

[4] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces*, Foundations and Harmonic Analysis, Applied and Numerical Harmonic Analysis, Birkhäuser-Springer, 2013.

[5] E. Dalmasso and R. Scotto, *Riesz transforms on variable Lebesgue spaces with Gaussian measure*, Integral Transforms and Special Functions **28** (2017), no. 5, 403–420.

[6] L. Diening, P. Harjulehto, P. Hästö, and M. Rů̆žička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Springer, 2011.

[7] E. Fabes, C. Gutiérrez, and R. Scotto, *Weak-type estimates for the Riesz transforms associated with the gaussian measure*, Rev Mat Iber **10** (1994), no. 2, 229–281.

[8] S. Pérez, *Estimaciones puntuales y en normas para operadores relacionados con el semigrupo de Ornstein-Uhlenbeck*, Tesis doctoral, Universidad Autónoma de Madrid, 1996.

[9] E. Pineda and W. Urbina, *Non tangential convergence for the Ornstein-Uhlenbeck semigroup*, Divulgaciones Matemáticas **16** (2007), no. 1, 107–124.

[10] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press., 1970.

[11] W. Urbina, *Semigroups of operators for classical orthogonal polynomials and functional inequalities*, Séminaires et Congrès (25), CIMPA Workshop Mérida, Venezuela, French Mathematical Society (SMF), 2012.

[12] ———, *Gaussian harmonic analysis*, Springer Monographs in Mathematics, Springer-Nature, 2019.

(Recibido en junio de 2020. Aceptado en febrero de 2021)

Departamento de Matemática
Decanato de Ciencia y Tecnologia, UCLA
Apartado 400 Barquisimeto 3001 Venezuela
e-mail: jorge.moreno@ucla.edu.ve

Departamento de Matemática
Facultad de Ciencias Naturales y Matemáticas,
ESPOL Guayaquil 09-01-5863, Ecuador.
e-mail: epineda@espol.edu.ec

Department of Mathematics and Actuarial Science
Roosevelt University, 430 S. Michigan Ave.,
Chicago, IL, 60605, USA
e-mail: wurbinaromero@roosevelt.edu