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Combinatorial description around any vertex of a cubical n-manifold

La descripción combinatoria alrededor de cualquier vértice de una n-variedad cubulada

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Abstract. We say that a topological space N is a cubical n-manifold if it is a topological manifold of dimension n contained in the n-skeleton of the canonical cubulation of $\mathbb{R}^{n+2}$. For instance, any smooth n-knot $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+2}$ can be deformed by an ambient isotopy into a cubical n-knot. An open question is the following: Is any closed, oriented, cubical n-manifold N in $\mathbb{R}^{n+2}$, $n > 2$, smoothable? If the response is positive, we could give a discrete description of any smooth n-manifold; in particular, if we can stablish that for smooth n-knots, that fact can be useful to define invariants. One of the main difficulties to answer the above question lies in the understanding of how N looks at each vertex of the canonical cubulation. In this paper, we analyze all possible combinatorial behaviors around any vertex of any cubical manifold of dimension n, via the study of the cycles on the complete graph K_{2n}.

Key words and phrases. Cubical manifolds, complete graph, closed paths..

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Resumen. Una n-variedad cubulada N es una variedad topológica de dimensión n que está encajada en el n-esqueleto de la cubulación canónica de $\mathbb{R}^{n+2}$. En particular, cualquier n-nudo suave $\mathbb{S}^n \hookrightarrow \mathbb{R}^{n+2}$ puede ser deformado por una isotopía ambiente en un n-nudo cubulado. Una pregunta abierta es la siguiente ¿cualquier n-variedad cubulada, cerrada y orientable N en $\mathbb{R}^{n+2}$, $n > 2$, es suavizable? Si la respuesta es afirmativa, entonces podremos dar una descripción discreta de cualquier n-variedad suave; en específico, podremos aplicarla para n-nudos suaves y utilizarla para definir invariantes. Una de las principales dificultades para responder la pregunta anterior radica en la comprensión de como es N en cada vértice de la cubulación canónica. En este artículo, analizamos todos los posibles comportamientos combinatorios alrededor de cualquier vértice de una variedad cubulada de dimensión n, a través del estudio de los ciclos de la gráfica completa K_{2n}.

Palabras y frases clave. variedades cubuladas, gráfica completa, trayectorias cerradas.

1. Introduction

The *canonical cubulation* $\mathcal{C}^{n+2}$ of $\mathbb{R}^{n+2}$ is the decomposition into hypercubes of $\mathbb{R}^{n+2}$, which are the images of the unit cube $I^{n+2} = \{(x_1, \ldots, x_{n+2}) \,|\, 0 \leq x_i \leq 1\}$ by translations by vectors with integer coefficients ([1], [2]).

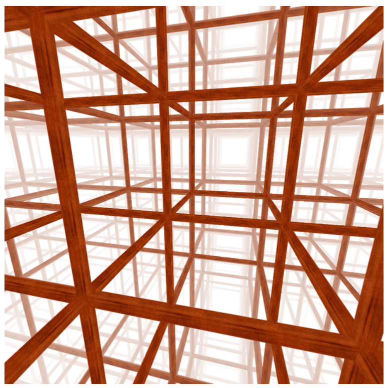

FIGURE 1. *The 3-dimensional cubic kaleidoscopic honeycomb* $\{4, 3, 4\}$. *This Figure is courtesy of Roice Nelson.*

Definition 1.1. Let N be a topological n-manifold embedded in $\mathbb{R}^{n+2}$. We say that N is a *cubical manifold* if it is contained in the n-skeleton of the canonical cubulation $\mathcal{C}^{n+2}$ of $\mathbb{R}^{n+2}$. In particular, for $n = 2$, N is called a *gridded surface.*

Observe that a cubical manifold can be subdivided into simplices to become a PL-manifold. M. Boege, G. Hinojosa and A. Verjovsky proved in [1] the following theorem.

Theorem 1.2. *Let N be a closed and smooth n-dimensional submanifold of $\mathbb{R}^{n+2}$ such that it has a trivial normal bundle. Then N can be deformed by an ambient isotopy into a cubical manifold.*

In particular, the above theorem implies that any smooth knot $K : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+2}$ can be deformed into a cubical knot.

A sort of reciprocal question is the following.

Question B. Is any closed, oriented, cubical n-manifold N in $\mathbb{R}^{n+2}$, $n \geq 2$, smoothable? More precisely, does N admit a global transverse field of 2-planes?

If it does then, by a theorem of J. H. C. Whitehead, there exists an arbitrarily small topological isotopy that moves N onto a smooth manifold in $\mathbb{R}^{n+2}$ (see [6], [8]).

If the answer is positive, we could give a discrete description of any smooth manifold, in particular we can get a combinatorial description of smooth n-knots that can be useful to define invariants. In fact, Matveev and Polyak [5] begin the exposition of finite type invariants from the "cubic" point of view and show how one can clearly describe invariants such as polynomial invariants. Cubic complexes may play a role in extending these invariants to higher dimensional knots. Recent work on the subject can be found in the work of Louis Funar [3] and Rade T. Živaljević [7].

We would like to point out that there exist PL-manifolds which are not smoothable. For instance, Kervaire ([4]) constructed an example of a PL triangulable closed manifold M of dimension 10 that does not admit any differentiable structure. Therefore the Kervaire manifold cannot be embedded as a codimension two cubical submanifold of $\mathbb{R}^{12}$.

For $n = 2$, the answer is positive as proved by J. P. Díaz, G. Hinojosa, R. Valdez and A. Verjovsky in [2]. In general, one of the main difficulties to answer the above question lies in the understanding of how N looks at any vertex of the canonical cubulation; in other words, we require to determine all the possible combinatorial configurations of N at any of its vertices.

Let $x \in N$ be a vertex. We can assume without loss of generality and for the sake of simplicity that the vertex x is the origin 0 and let $\mathcal{C}_0$ be the union of all n-faces $F \in \mathcal{C}^{n+2}$ such that $0 \in F$. The intersection $\mathcal{F}(N) = N \cap \mathcal{C}_0$ is called the *cubical-star* of N at 0. In particular, if N is a squared surface or gridded surface, the intersection $\mathcal{F}(N) = N \cap \mathcal{C}_0$ is called the *squared-star* of N at 0.

In order to analyze all possible combinatorial behaviors around any vertex of any cubical manifold of dimension 2, we need to study the space $\mathcal{F}(N)$. Notice that, if the face $F_{a,b}$ belongs to $\mathcal{F}(N)$, since N is a closed 2-manifold, there must exist two 2-faces $F_{a,c}$, $F_{b,d} \in \mathcal{F}(N)$. This allows us to describe $\mathcal{F}(N)$ as a finite path $\square \to \square \cdots \to \square$ of consecutive squares, *i.e.*, squares sharing an edge ($\square$ denotes one of the squares $F_{i,j} \in \mathcal{F}(N)$).

This problem is equivalent to consider the complete graph K_8 formed by 8 vertices labelled $1, -1, 2, -2, 3, -3, 4$ and -4, and edges $E_{i,j}$, $j \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ which correspond to the 2-faces of N.

In this paper, we count the number of closed paths (cycles) of lenght k, with $3 \leq k \leq 8$, with some restrictions (we do not use the edges $E_{-1,1}$, $E_{-2,2}$, $E_{-3,3}$, $E_{-4,4}$ and every vertex in the path has degree 2), which is a pure combinatorial problem but with implications in the theory exposed before since it gives us the number of all possible combinatorial behaviors around any vertex of a cubical manifold of dimension 2.

Using these ideas, we extend our study of the possible combinatorial behavior around any vertex of any cubical manifold of dimension n ($n > 2$), but in this case, we will consider the complete graph K_{2n}. More precisely, we have the following definition.

Definition 1.3. Let $\mathbb{F}_2$ be the set of all *squared-stars* of all possible gridded surfaces which have a vertex at 0.

We will prove the following theorem.

Theorem 3.1. *The cardinality of the set of all* squared-stars *of all possible gridded surfaces which have a vertex at* 0 *is equal to 36912. In other words,* $\|\mathbb{F}_2\| = 36912$.

In the general case, if $\tilde{\mathbb{F}}_n$ denotes the set of all *cubic-stars* of all possible cubical n-manifolds which have a vertex at 0, then we have the following result.

Theorem 3.2. *The cardinality of* $\tilde{\mathbb{F}}_n$ *is given by the formula*

$$\|\tilde{\mathbb{F}}_n\| = \sum_{k=3}^{2n+4} g(k)$$

where

$$\begin{aligned} g(k) = {} & \frac{(2n+4)!}{(2n+4-k)!} - (2n+4) \cdot P_2(k) \cdot \frac{(2n+4-2)!}{(2n+4-k)!} \\ & + (2n+4) \cdot (2n+4-2) \cdot P_{2,2}(k) \cdot \frac{(2n+4-4)!}{(2n+4-k)!} \\ & - (2n+4) \cdot (2n+4-2) \cdot (2n+4-4) \cdot P_{2,2,2}(k) \cdot \frac{(2n+4-6)!}{(2n+4-k)!} + \cdots \\ & + (-1)^{j+1}(2n+4) \cdots (2n+4-2j) \cdot P_{\underbrace{2,2,\ldots,2}_{j+1 \text{ numbers } 2}}(k) \cdot \frac{(2n+4-2(j+1))!}{(2n+4-k)!} \\ & + \cdots + (-1)^{n+2}(2n+4) \cdot (2n+4-2) \cdot \cdots \cdot 2 \cdot P_{\underbrace{2,2,\ldots,2}_{n+2 \text{ numbers } 2}}(k), \end{aligned}$$

such that

i) $P_{\underbrace{2,2,\ldots,2}_{j \text{ numbers } 2}}(k)$ *is the number of ways to choose* j *disjoint blocks of two consecutive points, from a set of* k *points lying in a circle;*

ii) $P_2(k) = k$, *for all* k*;*

iii) $P_{2,2}(k) = \frac{k(k-3)}{2}$, *for all* k*;*

iv) $P_{\underbrace{2,2,\ldots,2}_{n+2 \text{ numbers } 2}}(2n+4) = 2$*;*

v) $P_{\underbrace{2,2,\ldots,2}_{j \text{ numbers } 2}}(k) = 0$, *if* $k < 2j$.

2. Preliminaries

As we mentioned before, the *canonical cubulation* $\mathcal{C}^{n+2}$ of $\mathbb{R}^{n+2}$ is its decomposition into $(n+2)$-dimensional cubes which are the images of the unit $(n+2)$-cube $I^{n+2} = [0,1]^{n+2}$ by translations by vectors with integer coefficients. Then all vertices of $\mathcal{C}^{n+2}$ have integers in their coordinates. In other words, *the canonical cubulation* of $\mathbb{R}^{n+2}$ is its decomposition into a collection $\mathcal{C}^{n+2}$ of right-angled $(n+2)$-dimensional hypercubes called the *cells* such that any two are either disjoint or meet in one common $k-$face of some dimension k. This provides $\mathbb{R}^{n+2}$ with the structure of a cubic complex whose category is similar to the simplicial category PL ([1], [2]).

Any *cubulation* of $\mathbb{R}^n$ is obtained by applying a conformal transformation to the canonical cubulation. Remember that a *conformal transformation* is a transformation of the form $x \mapsto \lambda A(x) + a$, where $\lambda \neq 0$, $a \in \mathbb{R}^n$, $A \in SO(n)$.

Definition 2.1. The *k-skeleton* of the canonical cubulation $\mathcal{C}^n$ of $\mathbb{R}^n$, denoted by $\mathcal{C}^k$, consists of the union of the k-skeletons of the hypercubes in $\mathcal{C}^n$, *i.e.,* the union of all cubes of dimension k contained in the n-cubes in $\mathcal{C}^n$. We will call the 2-skeleton $\mathcal{C}^2$ of $\mathcal{C}^n$ the *canonical scaffolding* of $\mathbb{R}^n$.

We go back to our question. Let N be a cubical n manifold and let $x \in N$. We will discuss the case where x is a vertex of $\mathcal{C}$. We can assume without loss of generality and for the sake of simplicity that the vertex x is the origin $0 = (0, 0, \ldots, 0) \in \mathbb{R}^{n+2}$ and consider all the n-faces of N, $F_1, F_2, \ldots, F_j$, that contain x. Notice that each $(n-1)$-face of N must belong to only two n-faces and since there are only $2^{n-2}(n-2)n(n+1)(n+2)(\frac{1}{3})$ hyperfaces of dimension $(n-1)$ containing 0, it follows that $j \leq 2(m)$.

Consider the unitary canonical vectors on $\mathbb{R}^{n+2}$: $e_{\pm 1} = (\pm 1, 0, \ldots, 0)$, $e_{\pm 2} = (0, \pm 1, \ldots, 0)$, and $e_{\pm(n+2)} = (0, 0, 0, \pm 1)$. We will use throughout this paper, the following notation for this kind of n-faces:

$$F_{u_1, u_2, \ldots, u_n} = \{\sum_{i=1}^{n} a_i e_{u_i} \,:\, 0 \leq a_i \leq 1\}$$

where e_{u_i} ($u_i \in \{\pm 1, \pm 2, \ldots, \pm m\}$, $|u_i| \neq |u_j|$), denote the corresponding unitary canonical vectors.

Definition 2.2. Let $\mathcal{C}^0$ be the union of all n-faces $F \in \mathcal{C}$ such that $0 \in F$ and let N be a cubical n-manifold. The intersection $\mathcal{F}(N) = N \cap \mathcal{C}^0$ is called the *cubical-star* of N at 0.

In particular, if N is a cubical surface or gridded surface, the intersection $\mathcal{F}(N) = N \cap \mathcal{C}^0$ is called the *squared-star* of N at 0.

Definition 2.3. Let $\mathbb{F}_n$ be the set of all *cubical-stars* of all possible cubical n-manifolds which have a vertex at 0 and contain the face $F_{1,2,\ldots,n}$.

***Remark* 2.4.** Let N be a gridded surface. If the face $F_{a,b}$ belongs to $\mathcal{F}(N)$, then since N is a manifold there must exist two 2-faces $F_{a,c}$, $F_{b,c} \in \mathcal{F}(N)$. This allows us to describe $\mathcal{F}(N)$ as a finite path $\square \to \square \cdots \to \square$ of consecutive squares, *i.e.*, squares sharing an edge ($\square$ denotes one of the squares $F_{i,j} \in \mathcal{F}(N)$). For instance $F_{1,2} \to F_{1,3} \to F_{2,3}$ (see Figure 2). Moreover this remains true for any cubical manifold N replacing squares by n-faces.

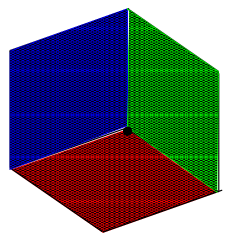

FIGURE 2. *squared-star consisting on* 3 *squares.*

Example 2.5. Next we will exhibit some square-stars consisting of any number, from 3 to 8, of squares.

(1) $F_{1,2} \to F_{1,3} \to F_{2,3}$ (see Figure 2).

(2) $F_{1,2} \to F_{1,-2} \to F_{2,3} \to F_{-2,3}$ (see Figure 3).

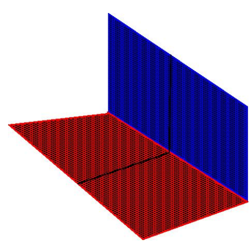

FIGURE 3. *square-star consisting of* 4 *squares.*

(3) $F_{1,2} \to F_{1,-2} \to F_{2,3} \to F_{-1,3} \to F_{-1,-2}$ (see Figure 4).

(4) $F_{1,-2} \to F_{-1,2} \to F_{1,-3} \to F_{2,-3} \to F_{-2,3} \to F_{-1,3}$ (see Figure 5).

(5) $F_{1,2} \to F_{1,-2} \to F_{2,3} \to F_{-1,3} \to F_{-2,-3} \to F_{-1,4} \to F_{-3,4}$.

(6) $F_{1,2} \to F_{1,-2} \to F_{-1,2} \to F_{-1,-3} \to F_{-2,-4} \to F_{-3,4} \to F_{3,4} \to F_{3,-4}$.

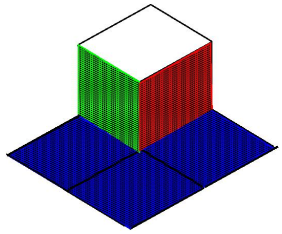

FIGURE 4. *square-star consisting of* 5 *squares.*

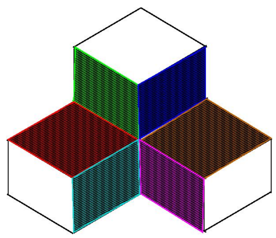

FIGURE 5. *square-star consisting of* 6 *squares.*

3. Main Theorem

We will start analyzing all the possible squared-starts of any gridded surface N, then we will give a formula to compute the cardinality of $\tilde{\mathbb{F}}_n$.

Theorem 3.1. *The cardinality of the set of all* squared-stars *of all possible gridded surfaces which have a vertex at* 0 *is equal to 36912. In other words,* $\|\mathbb{F}_2\| = 36912$.

***Proof*.** The problem to determine the number of all possible squared-stars of N is equivalent to the following problem.

Consider the complete graph K_8 formed with the eight vertices labelled 1, -1, 2, -2, 3, -3, 4, -4 and edges $E_{i,j}$, $i, j \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ which correspond to the faces $F_{i,j}$ of N. We would like to count the number of closed paths (cycles) of lenght k, with $3 \leq k \leq 8$, which do not use the edges $E_{-1,1}$, $E_{-2,2}$, $E_{-3,3}$ or $E_{-4,4}$ and such that every vertex in the path has degree 2.

In fact, we have that each squared-star can be described as a finite path $F_{a_1,a_2} \to F_{a_2,a_3} \to \cdots \to F_{a_k,a_1}$ of consecutive squares, such that $|a_i| \neq |a_{i+1}|$. This closed path corresponds to a cycle of lenght k in the graph K_8. Reciprocally, if we have a cycle $[a_1, a_2, \ldots, a_k]$ of lenght k in the graph K_8, we can construct a finite path $F_{a_1,a_2} \to F_{a_2,a_3} \to \cdots \to F_{a_k,a_1}$ which corresponds to a squared-star.

Our claim is that there are 36912 closed paths in total. For example, for $k = 3$, it is clear that there are only 4 paths that use the edge $E_{1,2}$: $E_{1,2} \to E_{2,3} \to E_{3,1}$, $E_{1,2} \to E_{2,-3} \to E_{-3,1}$, $E_{1,2} \to E_{2,4} \to E_{4,1}$ and $E_{1,2} \to E_{2,-4} \to E_{-4,1}$. However, for $k \geq 4$ the counting is not very simple, and we will have to give a general formula that counts the number of paths satisfying our conditions for every k.

The formula that describes the number of paths with the conditions mentioned above, for $3 \leq k \leq 8$, is

$$f(k) = \left(\frac{8!}{(8-k)!} - 8 \cdot P_2(k) \cdot \frac{6!}{(8-k)!} + 8 \cdot 6 \cdot P_{2,2}(k) \cdot \frac{4!}{(8-k)!} \right.$$
$$\left. - 8 \cdot 6 \cdot 4 \cdot P_{2,2,2}(k) \cdot \frac{2!}{(8-k)!} + 8 \cdot 6 \cdot 4 \cdot 2 \cdot P_{2,2,2,2}(k) \right),$$

where

i) $P_{\underbrace{2,2,\ldots,2}_{r \text{ numbers } 2}}(k)$ is the number of ways to choose r disjoint blocks of two consecutive points, from a set of k points lying in a circle;

ii) $P_2(k) = k$, for all k;

iii) $P_{2,2}(k) = \frac{k(k-3)}{2}$, for all k;

iv) $P_{2,2,2}(6) = 2$, $P_{2,2,2}(8) = 16$, $P_{2,2,2}(7) = 7$;

v) $P_{2,2,2,2}(8) = 2$;

vi) $P_{\underbrace{2,2,\ldots,2}_{r \text{ numbers } 2}}(k) = 0$, if $k < 2r$;

The proof of the formula will follow by the inclusion - exclusion principle. For example, for $k = 3$, there are $\frac{8!}{(8-5)!} = 8 \cdot 7 \cdot 6$ possible paths with three vertices. However some of them are not allowed under our conditions, that is, all the paths $E_{a,b} \to E_{b,c} \to E_{c,a}$, with $a + b = 0$, $a + c = 0$ or $b + c = 0$, are not allowed. There are three posible choices to get zero sum, and there are 8 ways to choose the pair, from the set $\{(1,-1), (2,-2), (3,-3), (4,-4), (-1,1), (-2,2), (-3,3), (-4,4)\}$; and finally the third vertex can be selected from the six remaining vertices. Hence, there are $3 \cdot 8 \cdot 6 = P_2(3) \cdot 8 \cdot \frac{6!}{(8-3)!}$ paths that are not allowed. Therefore for $k = 3$, we have that the total number of paths is

$$8 \cdot 7 \cdot 6 - 3 \cdot 8 \cdot 6 = 192.$$

Let us explain the meaning of $P_2(k)$, which is the number of ways to choose 2 consecutive points from a set of k points arranged in a circle. Each of the

k points belongs to two of such pairs, since there are k points, this generates $2k$ pairs of consecutive points, however we are counting each pair twice, that is, $P_2(k) = k$. In our case, the points considered are $-1, 1, -2, 2, \dots, -4, 4$, arranged in a circular way, and k can be $3, 4, \dots, 8$.

Now, in order to see that $P_{2,2}(k) = \frac{k(k-3)}{2}$, observe that if we take one pair of consecutive points, then the point just before the pair can not be taken, then the first point on the next pair has $k-3$ possible choices. Notice that we can choose one block of two points and then the next block, or viceversa, hence we need to divide $k(k-3)$ by 2.

To calculate $P_{2,2,2}(k)$ and $P_{2,2,2,2}(k)$, first note that if $P_{2,2,2}(k)$ and $P_{2,2,2,2}(k)$ are greater than zero then it is necessary that $k \geq 6$ and $k \geq 8$, respectively. It is clear that $P_{2,2,2,2}(8) = 2$, since choosing one point, we can form only two different blocks with this point, and the other blocks are determined. Similarly, $P_{2,2,2}(6) = 2$.

Let us calculate $P_{2,2,2}(7)$. Observe that if we take three disjoint blocks, there is one point left alone, but this isolated point can be any of the seven points in the circle, then $P_{2,2,2}(7) = 7$.

Finally, for $P_{2,2,2}(8)$, first note that after selecting the three blocks of pair of points, there are two isolated points, which can be selected with an even difference, otherwise we can not fit the three blocks of points. Now we choose that even difference between the isolated points, which can be 0, 2, 4 or 6. That is, if we fix one point as an isolated point, the other one can be chosen in 4 different ways. Since there are 8 possible points, we get $8 \times 4 = 32$ different ways to select the isolated point, but since we are considering every point twice in the counting, we must divide by 2. Therefore $P_{2,2,2}(8) = 16$.

Now, let fix $3 \leq k \leq 8$. Let us calculate the total number of paths with the given conditions. First of all, the number of total paths that used k vertices are the permutations of k elements taken from a set of 8 elements (the vertices of our graph) which are $\frac{8!}{(8-k)!}$. This is the total number of paths. Now we need to substract to this total, the number of paths that do not satisfy the condition. One path is not allowed if it has an edge of the form $E_{-i,i}$, then in the set of k vertices of our path, we need to see where the edge $E_{-i,i}$ is located, and there are $P_2(k)$ choices. The edge $E_{-i,i}$ has 8 possibilities and then we need to select the rest of the vertices of the path, which we can choose in $\frac{6!}{(8-k)}$ ways, therefore we need to substract $8 \cdot P_2(k) \cdot \frac{6!}{(8-k)!}$ from the total.

If $k \geq 4$, some paths were subtracted twice, for example, the path $E_{2,-2} \to E_{-2,3} \to E_{3,-3} \to E_{-3,2}$, then we need to add again this path. That is, we need to find the total number of two blocks of consecutive points among the k points, which is $P_{2,2}(k)$, for the first block we have 8 choices, and then for the second block we have only six choices (because if we choose the pair $-j, j$, then for the next block we can not use $j, -j$). Finally, we need to choose the other vertex of the path, for which we have $\frac{4!}{(8-k)!}$ choices. Hence we need to add $8 \cdot 6 \cdot P_{2,2}(k) \cdot \frac{4!}{(8-k)!}$ paths.

If $k \geq 6$, now we need to remove additional paths that do not satisfy the given conditions. With a similar analysis, we get that we need to remove $8 \cdot 6 \cdot 4 \cdot P_{2,2,2}(k) \cdot \frac{2!}{(8-k)!}$ paths. If $k = 8$, we are overcounting again, so we need to add this number $8 \cdot 6 \cdot 4 \cdot 2 \cdot P_{2,2,2,2}(k)$ of paths. Therefore the given formula follows.

Now we calculate de number of paths for $3 \leq k \leq 8$ using the formula. We already know that for $k = 3$, there are 4 allowed paths. Substituting the values of k in the formula, we get $f(3) = 192$, $f(4) = 816$, $f(5) = 2880$, $f(6) = 7680$, $f(7) = 13440$ and $f(8) = 11904$.

Therefore

$$\|\mathbb{F}_2\| = \sum_{k=3}^{8} f(k) = 36912.$$

☑

Next, we go back to our Question B. Let N be a cubical manifold of dimension n in $\mathbb{R}^{n+2}$. As we mention in the previous section, each n-face F around the origin can be denoted as follows:

$$F := F_{u_1,u_2,\ldots,u_n} = \left\{ \sum_{i=1}^{n} a_i e_{u_i} \,:\, 0 \leq a_i \leq 1 \right\},$$

where e_{u_i} ($u_i \in \{\pm 1, \pm 2, \ldots, \pm n+2\}$, $|u_i| \neq |u_j|$), are the corresponding unitary canonical vectors. Notice that in the previous description we take the sum over n canonical vectors; however we can take the sum over all the $n+2$ canonical vectors, as follows

$$F := F_{ij} = \left\{ \sum_{i=1}^{n+2} a_i e_i \in \mathbb{R}^{n+2} \mid 0 \leq a_k \leq 1 \text{ for } k \neq i,\, j, \text{ and } a_i = a_j = 0 \right\},$$

where e_i ($i \in \{\pm 1, \pm 2, \ldots, \pm n+2\}$, $|i| \neq |j|$), are again the corresponding unitary canonical vectors.

Keeping this in mind, we will restrict ourselves to a subset of cubical-starts $\tilde{\mathbb{F}}_n$ of $\mathbb{F}_n$, consisting of the following n-faces $F_{i,j}$ which are completely described by two indexes i, j ($|i| \neq |j|$), which correspond to the canonical vectors whose coefficients are equal zero. By simplicity, we will assume that $|i| < |j|$, then

$$F_{i,j} = \begin{cases} (\operatorname{sign}(i)a_1, \operatorname{sign}(j)a_2, a_3, \dots, \hat{a_i}, \dots, \hat{a_j}, \dots, a_{n+2}) & |j| > |i| \geq 3, \\ (\hat{a_1}, \operatorname{sign}(i)a_2, \operatorname{sign}(j)a_3, \hat{a_4}, a_5 \dots, a_{n+2}) & |i| = 1,\ |j| \geq 4, \\ (\hat{a_1}, \operatorname{sign}(i)a_2, \hat{a_3}, \operatorname{sign}(j)a_4, \dots, a_{n+2}) & |i| = 1,\ |j| = 3, \\ (\hat{a_1}, \hat{a_2}, \operatorname{sign}(i)a_3, \operatorname{sign}(j)a_4, \dots, a_{n+2}) & |i| = 1,\ |j| = 2, \\ (\operatorname{sign}(i)a_1, \hat{a_2}, \operatorname{sign}(j)a_3, \dots, \hat{a_j}, \dots, a_{n+2}) & |i| = 2,\ |j| \geq 4, \\ (\operatorname{sign}(i)a_1, \hat{a_2}, \hat{a_3}, \operatorname{sign}(j)a_4, \dots, a_{n+2}) & |i| = 2,\ |j| = 3; \end{cases}$$

where $0 \leq a_k \leq 1$ and the notation $\hat{a_i}$ means that the coefficient a_i is equal to zero.

In order to understand the set $\mathbb{F}_n$, we are interested in computing the cardinality of the subset $\tilde{\mathbb{F}}_n$ of $\mathbb{F}_n$.

Observe that this problem, by the same argument used in Theorem 3.1, is equivalent to following: Consider now the complete graph K_{2n+4} formed with the $2n+4$ vertices labelled $1,\ -1,\ 2,\ -2,\ 3,\ -3,\ \dots,\ n+2,\ -(n+2)$ and edges $E_{i,j}$, $j \in \{\pm 1, \pm 2, \pm 3, \dots, \pm(n+2)\}$ which would correspond to the n-hyperfaces of N at the vertex 0. We would like to find the number of closed paths (cycles) of lenght k, with $3 \leq k \leq 2n+4$, which do not use the edges $E_{-1,1}$, $E_{-2,2}$, $E_{-3,3}$, $\dots$, $E_{-(n+2),n+2}$ and such that every vertex in the path has degree 2.

Using the inclusion - exclusion principle as in the above proof, we have the following.

Theorem 3.2. *The cardinality of $\tilde{\mathbb{F}}_n$ is given by the formula*

$$\|\tilde{\mathbb{F}}_n\| = \sum_{k=3}^{2n+4} g(k),$$

where

$$\begin{aligned} g(k) = {} & \frac{(2n+4)!}{(2n+4-k)!} - (2n+4) \cdot P_2(k) \cdot \frac{(2n+4-2)!}{(2n+4-k)!} \\ & + (2n+4) \cdot (2n+4-2) \cdot P_{2,2}(k) \cdot \frac{(2n+4-4)!}{(2n+4-k)!} \\ & - (2n+4) \cdot (2n+4-2) \cdot (2n+4-4) \cdot P_{2,2,2}(k) \cdot \frac{(2n+4-6)!}{(2n+4-k)!} + \cdots \\ & + (-1)^{j+1}(2n+4) \cdots (2n+4-2j) \cdot P_{\underbrace{2,2,\dots,2}_{j+1 \text{ numbers } 2}}(k) \cdot \frac{(2n+4-2(j+1))!}{(2n+4-k)!} \\ & + \cdots + (-1)^{n+2}(2n+4) \cdot (2n+4-2) \cdot \cdots \cdot 2 \cdot P_{\underbrace{2,2,\dots,2}_{n+2 \text{ numbers } 2}}(k), \end{aligned}$$

such that

i) $P_{\underbrace{2,2,\ldots,2}_{j \text{ numbers } 2}}(k)$ *is the number of ways to choose* j *disjoint blocks of two consecutive points, from a set of* k *points lying in a circle;*

ii) $P_2(k) = k$, *for all* k;

iii) $P_{2,2}(k) = \frac{k(k-3)}{2}$, *for all* k;

iv) $P_{\underbrace{2,2,\ldots,2}_{n+2 \text{ numbers } 2}}(2n+4) = 2$;

v) $P_{\underbrace{2,2,\ldots,2}_{j \text{ numbers } 2}}(k) = 0$, *if* $k < 2j$.

***Remark* 3.3.** By the above result, we have that the set $\mathbb{F}_n$ of all *cubical-stars* of all possible cubical n-manifolds which have a vertex at 0, becomes extremely complicated as n grows, hence the difficulty to answer the Question B. Nevertheless, we think that any cubical manifold N of dimension n, contained in the scaffolding of the canonical cubulation of $\mathbb{R}^{n+2}$ is smoothable.

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