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E-infinity coalgebra structure on chain complexes with integer coefficients

E-infinito coalgebra estructura en complejos de cadenas con coeficientes enteros

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Abstract. The aim of this paper is to construct an E_∞-operad inducing an E_∞-coalgebra structure on chain complexes with integer coefficients, which is an alternative description to the E_∞-coalgebra by the Barrat-Eccles operad.

Key words and phrases. Operad theory, Chain complexes, E_∞-coalgebras, Barrat-Eccles operad.

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Resumen. El objetivo de este artículo es construir un E_∞-operad que induce una estructura de E_∞-coalgebra en los complejos de cadenas con coeficientes enteros. Esta construcción produce una descripción alternativa a la E_∞-coalgebra del operad de Barrat-Eccles.

Palabras y frases clave. Teoría de operads, complejos de cadenas, E_∞-coalgebras, operad de Barrat-Eccles.

1. Introduction

An E_∞-coalgebra structure on chain complexes of simplicial sets with coefficients in $\mathbb{Z}$ is introduced by Smith in [10] (in his work operads are called symmetric formal coalgebras and E_∞-operads, f-resolutions). He used an E_∞-operad, denoted by $\mathfrak{S}$, with each component $\mathrm{R}\Sigma_n$ a Σ_n-free bar resolution of $\mathbb{Z}$. The morphisms $f_n : \mathrm{R}\Sigma_n \otimes C_*(X) \to C_*(X)^{\otimes n}$ determined by the E_∞-coalgebra structure contain a family of higher diagonals on $C_*(X)$, starting with an homotopic version of the iterated Alexander-Whitney diagonal.

The operad $\mathfrak{S}$ is also defined by Berger and Fresse in [2] as a differential graded operad $\mathcal{E}$ by taking chains on the simplicial Barratt-Eccles operad $\mathcal{W}$(see

[1]). With $\mathcal{E}$ (also called Barratt-Eccles operad), they construct an explicit coaction on normalized chain complexes extending the structure given by the Alexander-Whitney diagonal.

In this paper we present a new operad E_∞-operad $\mathcal{R}$ inducing an E_∞-coalgebra structure on chain complexes, which is defined following the ideas by Smith in his construction of $\mathfrak{S}$. We show in section 3 that $\mathfrak{S}$ can be obtained from $\mathcal{R}$ by an operadic quotient (see corollary 3.4), which is a direct consequence of the operadic composition definition of $\mathfrak{S}$. The associated operad morphism between $\mathcal{R}$ and $\mathfrak{S}$ is a quasi-isomorphism because both of them are E_∞-operads.

It is worth pointing out that $\mathcal{R}$ presents similarities with the bar-cobar resolution of Ginzburg-Kapranov (see [6]). Berger and Moerdijk in [3] identified this resolution with the W-construction of Boardman and Vogt (see [4]). As a consequence, the W-construction of the Barratt-Eccles operad gives a cofibrant resolution of it. Then, our operad $\mathcal{R}$ may be seen as a middle point between the Barratt-Eccles operad $\mathfrak{S}$(or $\mathcal{E}$) and its W-construction.

The results in this paper are based on the author's PhD thesis [9], where E_∞-coalgebras are identified over structures associated to chain complexes. They generalize uniqueness properties described by Prouté in [7] and [8] of the Eilenberg-Mac Lane transformation.

2. Preliminaries

2.1. Differential graded modules

A $\mathbb{Z}$-module M is graded if there is a collection $\{M_i\}_{i\in\mathbb{Z}}$ of submodules of M such that $M = \bigoplus_{i\in\mathbb{Z}} M_i$. A differential graded module with augmentation and coefficients in $\mathbb{Z}$, or DGA-module for short, is a graded $\mathbb{Z}$-module M together with a morphism $\partial : M \to M$ of degree -1 such that $\partial^2 = 0$, and morphisms $\epsilon : M \to \mathbb{Z}$, $\eta : \mathbb{Z} \to M$ of degree 0, called augmentation and coaugmentation of M, respectively, such that $\epsilon \circ \eta = \mathrm{id}$. The category of DGA-modules is denoted DGA-Mod.

2.2. Operads

An operad P in the monoidal category DGA-Mod is a collection of DGA-modules $\{P(n)\}_{n\geq 1}$ together with a right action of the symmetric group Σ_n on each component $P(n)$, and morphisms of the form $\gamma : P(r) \otimes P(i_1) \otimes P(i_r) \to P(i_1 + \cdots + i_r)$, which satisfy the usual conditions of existence of an unit, associativity and equivariance. The morphisms γ will be called composition morphisms of the operad. A morphism between operads $f : P \to Q$, is a collection of DGA-morphisms $f_n : P(n) \to Q(n)$ of degree 0, respecting units, composition and equivariance. The category of operads is denoted $\mathcal{OP}$.

If we forget the composition of morphisms of an operad P, the collection of DGA-modules with right actions that remains is called an $\mathbb{S}$-module. They form a category denoted $\mathbb{S}$-Mod. The forgetful functor $U : \mathcal{OP} \to \mathbb{S}$-Mod has a right adjoint denoted $F : \mathbb{S}\text{-Mod} \to \mathcal{OP}$, called the free operad functor.

Definition 2.1. Let $\mathcal{P}$ be an operad on the category DGA-Mod, with composition γ. A sub $\mathbb{S}$-module $\mathcal{I}$ of $U(\mathcal{P})$ is called an operadic ideal of $\mathcal{P}$ if it satisfies $\gamma(x \otimes y_1 \otimes \cdots \otimes y_k) \in \mathcal{I}$, whenever some elements $x, y_1, \dots, y_k$ belong to $\mathcal{I}$.

Definition 2.2. Let $\mathcal{P}$ be an operad and $\mathcal{I}$ an operadic ideal of $\mathcal{P}$. We define the quotient operad $\mathcal{P}/\mathcal{I}$ as the operad given by $(\mathcal{P}/\mathcal{I})(n) = P(n)/I(n)$ for every $n \geq 1$, and composition induced by the composition of $\mathcal{P}$.

Remark **2.3.** Clearly, the operad structure of $\mathcal{P}/\mathcal{I}$ is well defined by the properties of the operadic ideal $\mathcal{P}$, which allow to induce the composition $\mathcal{P}$ on the quotient (see [6] §2.1).

2.3. The Bar Resolution

The chain complex with coefficients in $\mathbb{Z}$ given by the Σ_n-free bar resolution of $\mathbb{Z}$ is and it denoted $R\Sigma_n$. Recall that the degree m elements of $R\Sigma_n$ are $\mathbb{Z}$-linear combinations of elements of the form $\sigma[\sigma_1/\cdots/\sigma_m]$, where $\sigma, \sigma_1, \dots, \sigma_m \in \Sigma_n$ and the boundary map is $\partial = \sum_{i=0}^{m}(-1)^i\partial_i$, where $\partial_0[\sigma_1/\cdots/\sigma_m] = \sigma_1[\sigma_2/\cdots/\sigma_m]$, for $0 < i < m$, $\partial_i[\sigma_1/\cdots/\sigma_m] = [\sigma_1/\cdots/\sigma_i\sigma_{i+1}/\cdots/\sigma_m]$, and $\partial_m[\sigma_1/\cdots/\sigma_m] = [\sigma_1/\cdots/\sigma_{m-1}]$. In degree zero, the $\mathbb{Z}[\Sigma_n]$-module is generated by one element, written $[\,]$. $R\Sigma_n$ is acyclic with contracting chain homotopy the map $\psi_n : R\Sigma_n \to R\Sigma_n$ of degree 1 defined by the relations $\psi_n[\sigma_1/\cdots/\sigma_m] = 0$ and $\psi_n\sigma[\sigma_1/\cdots/\sigma_m] = [\sigma/\sigma_1/\cdots/\sigma_m]$.

2.4. E_∞-Operads

Definition 2.4. An operad $\mathcal{P}$ in the category DGA-Mod is called E_∞-operad if each component $P(n)$ is a Σ_n-free resolution of $\mathbb{Z}$.

Definition 2.5. We call E_∞-coalgebra(algebra) any $\mathcal{P}$-coalgebra(algebra) with $\mathcal{P}$ an E_∞-operad.

We introduce a notion of morphism between E_∞-coalgebras which is well suited for our purpose.

Definition 2.6. Let $\mathcal{P}$ be an E_∞-operad in the category DGA-Mod, and let A, B be $\mathcal{P}$-coalgebras. A morphism $f : A \to B$ of $\mathcal{P}$-coalgebras is a morphism of DGA-Mod which preserves the $\mathcal{P}$-coalgebra structure up to homotopy, that is, the following diagram:

$$\begin{array}{ccc} \mathcal{P}(n) \otimes A & \xrightarrow{\varphi_n^A} & A^{\otimes n} \\ {\scriptstyle 1\otimes f}\downarrow & & \downarrow{\scriptstyle f^{\otimes n}} \\ \mathcal{P}(n) \otimes B & \xrightarrow[\varphi_n^B]{} & B^{\otimes n} \end{array} \qquad (1)$$

is commutative up to homotopy for every $n > 0$, where φ_n^A and φ_n^B are the associated morphisms of the $\mathcal{P}$-coalgebra structure of A and B, respectively. The category of $\mathcal{P}$-coalgebras is denoted $\mathcal{P}$-CoAlg.

3. The Operad $\mathcal{R}$

In this section we construct an E_∞-operad $\mathcal{R}$ which is used to describe the complex $C_*(X)$ as an E_∞-coalgebra.

Definition 3.1. Let S be the $\mathbb{S}$-module in the category DGA-Mod, with components $S(n) = \mathrm{R}\Sigma_n$, the $\mathbb{Z}[\Sigma_n]$-free bar resolution of $\mathbb{Z}$. Define the operad $\mathcal{R}$ as the quotient operad $F(S)/\mathcal{J}$, where $\mathcal{J}$ is the operadic ideal of the free operad $F(S)$ generated by the elements of zero degree of $F(S)$ of the form $x - y$, where x and y are not null.

Theorem 3.2. *The operad $\mathcal{R}$ is an E_∞-operad and induces an E_∞-coalgebra structure on $C_*(X)$.*

Proof. It suffices to exhibit in each arity a contracting chain homotopy. In arity n, the contracting chain homotopy $\Phi_n : R(n) \to R(n)$ is obtained by extending on $R(n)$ the contracting chain homotopy ψ_n from the component $\mathrm{R}\Sigma_n$ of S as follows.

$R(2)$ is isomorphic to $S(2)$, so the contracting chain homotopy remains the same. When $n > 2$, $R(n)$ has two types of elements: the elements from the injection $S(n) \to R(n)$ and the elements of the form $\gamma(x; y_1, \dots, y_r)$, where $x \in S(r)$ and $y_j \in R(i_j)$. In the first case Φ_n will behave as the contracting chain homotopy in $S(n)$, and for the second case, we define $\Phi_n\gamma(x; y_1, \dots, y_r) = \gamma(\Phi_n(x); y_1, \dots, y_r)$.

To check that $\partial\Phi_n + \Phi_n\partial = 1$, let x of the form $[\sigma_1|\cdots|\sigma_l]$, with $\sigma_j \in \Sigma_r$. Now $\partial\Phi_n\gamma(x; y_1, \dots, y_r) = \partial\gamma(\Phi_n(x); y_1, \dots, y_r) = 0$. On the other hand,

$$\Phi_n\partial\gamma(x; y_1, \dots, y_r) = \Phi_n\gamma(\partial x; y_1, \dots, y_r) + (\mathrm{sign}) \sum \Phi_n\gamma(x; y_1, \dots, \partial y_j, \dots, y_r) \tag{2}$$

$$= \gamma(\Phi_n\partial x; y_1, \dots, y_r) + (\mathrm{sign}) \sum \gamma(\Phi_n x; y_1, \dots, \partial y_j, \dots, y_r) \tag{3}$$

$$= \gamma(x - \partial\Phi_n x; y_1, \dots, y_r) \tag{4}$$

$$= \gamma(x; y_1, \dots, y_r) \tag{5}$$

When x has the form $\sigma[\sigma_1|\cdots|\sigma_l]$ the verification is similar, since the composition is γ equivariant:
$\gamma(\sigma[\sigma_1|\cdots|\sigma_l]; y_1, \dots, y_r) = \gamma([\sigma_1|\cdots|\sigma_l]; y_{\sigma^{-1}(1)}, \dots, y_{\sigma^{-1}(l)})$.

Now, the universal property of the coaugmentation ι of the adjunction $F \vdash U$, gives the commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F(S) \\ & {\scriptstyle i}\searrow & \downarrow{\scriptstyle p} \\ & & \mathfrak{S} \end{array} \tag{6}$$

Here the morphism i is the identity of $\mathbb{S}$-modules. It is easy to see that p respects the ideal $\mathcal{J}$ because, when the free operad construction is interpreted by rooted trees, p is essentially the contraction of vertices of trees. Thus p passes to the quotient and we obtain a morphism of operads $\overline{p} : \mathcal{R} \to \mathfrak{S}$, which implies that every $\mathfrak{S}$-coalgebra is an $\mathcal{R}$-coalgebra. ☑

Corollary 3.3. *The construction in theorem 3.2 is functorial.*

Proof. The functoriality of the $\mathcal{R}$-coalgebra structure is inherited by the $\mathfrak{S}$-coalgebra structure by the operad morphism $\overline{p} : \mathcal{R} \to \mathfrak{S}$ in the proof of theorem 3.2. Indeed, for every morphism $f : X \to Y$ the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{R} & \xrightarrow{\overline{p}} & \mathfrak{S} & \longrightarrow & \mathrm{CoEnd}(C_*(X)) \\ & & & \searrow & \downarrow{\scriptstyle f_*} \\ & & & & \mathrm{CoEnd}(C_*(Y)) \end{array} \tag{7}$$

☑

We can understand the relation between the operad $\mathcal{R}$ and the operad $\mathfrak{S}$ with the following proposition.

Corollary 3.4. *There is an operad ideal $\mathcal{I}$ such that $\mathfrak{S} \cong \mathcal{R}/\mathcal{I}$.*

Proof. This is because the underlying $\mathbb{S}$-module of $\mathfrak{S}$ is S, and a direct consequence of the definition of compositions γ of $\mathfrak{S}$ (see [10] or [2]). In other words, the operadic ideal $\mathcal{I}$ is defined by the identification needed for γ. ☑

In [5] Vallette and Dehling describe an operad similar to $\mathcal{R}$ and state (by the use relations) a definition of E_∞-algebras. In this sense, $\mathcal{R}$-coalgebras can be described as follows.

Corollary 3.5. *Let A be a DGA-module together with:*

(1) For every integer $m \geq 1$, $n \geq 1$ and $\sigma, \sigma_1, \ldots, \sigma_n \in \Sigma_m$, morphisms of degree n:

$$\mu_{\sigma[\sigma_1/\cdots/\sigma_n]_m} : A \to A^{\otimes n}.$$

(2) For every integer $m \geq 1$ and $\sigma \in \Sigma_m$, maps of degree 0:

$$\mu_{\sigma[\ ]_m} : A \to A^{\otimes n}.$$

Suppose these morphisms satisfy the following relations:

(1) $\mu_{\sigma x} = \mu_x \sigma$, where σ is the right action on n factors.

(2) $\mu_{x+y} = \mu_x + \mu y$ and $\partial \mu_x = \mu_{\partial x}$.

(3) $(\mu_{[\ ]_{m_1}} \otimes \cdots \otimes \mu_{[\ ]_{m_n}})\mu_{[\ ]_n} = \mu_{[\ ]_{m_1+\cdots+m_n}}$.

Then, A is an $\mathcal{R}$-coalgebra. The converse is also true.

Proof. This is directly implied by the operad morphism $\mathcal{R} \to \mathrm{Coend}(A)$. ☑

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