Induced character in equivariant K-theory, wreath products and pullback of groups

Carácter inducido en K-teoría equivariante, productos wreath y pullbacks de grupos

German Combariza\textsuperscript{1}, Juan Rodríguez\textsuperscript{2}, Mario Velasquez\textsuperscript{3,35}

\textsuperscript{1}Fundación Universitaria Konrad Lorenz, Bogotá, Colombia

\textsuperscript{2}École normale supérieure de Lyon, Lyon, France

\textsuperscript{3}Universidad Nacional de Colombia, Bogotá, Colombia

Abstract. Let $G$ be a finite group and let $X$ be a compact $G$-space. In this note we study the $(\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z})$-graded algebra

$$\mathcal{F}_G^q(X) = \bigoplus_{n \geq 0} q^n \cdot K_{G/\mathcal{G}_n}(X^n) \otimes \mathbb{C},$$

defined in terms of equivariant K-theory with respect to wreath products as a symmetric algebra, we review some properties of $\mathcal{F}_G^q(X)$ proved by Segal and Wang. We prove a Kunneth type formula for this graded algebras, more specifically, let $H$ be another finite group and let $Y$ be a compact $H$-space, we give a decomposition of $\mathcal{F}_{G \times H}^q(X \times Y)$ in terms of $\mathcal{F}_G^q(X)$ and $\mathcal{F}_H^q(Y)$. For this, we need to study the representation theory of pullbacks of groups. We discuss also some applications of the above result to equivariant connective K-homology.

Key words and phrases. equivariant K-theory, wreath products, Fock space.

2020 Mathematics Subject Classification. 19L47, 19L41.

Resumen. Sea $G$ un grupo finito y $X$ un $G$-espacio compacto. En esta nota estudiamos el álgebra $(\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z})$-graduada

$$\mathcal{F}_G^q(X) = \bigoplus_{n \geq 0} q^n \cdot K_{G/\mathcal{G}_n}(X^n) \otimes \mathbb{C},$$

defined in terms of equivariant K-theory with respect to wreath products as a symmetric algebra, we review some properties of $\mathcal{F}_G^q(X)$ proved by Segal and Wang. We prove a Kunneth type formula for this graded algebras, more specifically, let $H$ be another finite group and let $Y$ be a compact $H$-space, we give a decomposition of $\mathcal{F}_{G \times H}^q(X \times Y)$ in terms of $\mathcal{F}_G^q(X)$ and $\mathcal{F}_H^q(Y)$. For this, we need to study the representation theory of pullbacks of groups. We discuss also some applications of the above result to equivariant connective K-homology.

Key words and phrases. equivariant K-theory, wreath products, Fock space.

2020 Mathematics Subject Classification. 19L47, 19L41.
definida en términos de K-teoría equivariante con respecto a productos guiernalda, como un álgebra simétrica, revisamos algunas de las propiedades de $\mathcal{F}_G^q(X)$ probadas por Segal y Wang. Probamos una fórmula tipo Kunneth para estas álgebras graduadas, más específicamente, sea $H$ otro grupo finito y $Y$ un $H$-espacio compacto, nosotros damos una descomposición de $\mathcal{F}_G^q(X \times Y)$ en términos de $\mathcal{F}_G^q(X)$ y $\mathcal{F}_H^q(Y)$, para esto, debemos estudiar la teoría de representaciones de pullbacks de grupos. Discutimos también algunas aplicaciones de los resultados anteriores a K-homología equivariante conectiva.

Palabras y frases clave. K-teoría equivariante, productos wreath, espacio de Fock.

Notation

In this note we denote by $\mathfrak{S}_n$ the symmetric group in $n$ letters. Let $G$ be a finite group, let $g, g' \in G$, we say that $g$ and $g'$ are conjugated in $G$ (denoted by $g \sim_G g'$) if there is $s \in G$ such that $g = sg's^{-1}$. We denote by

$$[g]_G = \{g' \in G \mid g \sim_G g'\}$$

the conjugacy class of $g$ in $G$ (or simply by $[g]$ when $G$ is clear from the context). We denote by $G_*$ the set of conjugacy classes of $G$. We denote by $C_G(g)$ the centralizer of $g$ in $G$. Also $R(G)$ will be the complex representation ring of $G$, with operations given by direct sum and tensor product, and generated as an abelian group by the isomorphism classes of irreducible representations of $G$. The class function ring of $G$ is the set

$$\text{Class}(G) = \{f : G \to \mathbb{C} \mid f \text{ is constant in conjugacy classes}\}$$

with the usual operations.

1. Introduction

Let $X$ be a finite CW-complex. In [14] Segal studied the vector spaces

$$\mathcal{F}(X) = \bigoplus_{n \geq 0} K_{\mathfrak{S}_n}(X^n) \otimes \mathbb{C},$$

these spaces carry several interesting structures, for example they admit a Hopf algebra structure with the product defined using induction on vector bundles and the coproduct defined using restriction.

Later in [18], Wang generalizes Segal’s work to an equivariant context. Let $G$ be a finite group and $X$ be a finite $G$-CW-complex, Wang defines the vector space

$$\mathcal{F}_G(X) = \bigoplus_{n \geq 0} K_{G \mathfrak{S}_n}(X^n) \otimes \mathbb{C},$$
where $G \wr \mathfrak{S}_n$ denotes the wreath product acting naturally over $X^n$. Wang proves that $\mathcal{F}_G(X)$ admits similar structures as $\mathcal{F}(X)$. In particular $\mathcal{F}_G(X)$ has a description as a supersymmetric algebra in terms of $K_G(X) \otimes \mathbb{C}$.

Following ideas of [14], in [16] appears another reason to study $\mathcal{F}_G(X)$. When $X$ is a $G$-spin$^c$-manifold of even dimension, $\mathcal{F}_G(X)$ is isomorphic to the homology with complex coefficients of the $G$-fixed point set of a based configuration space $\mathfrak{C}(X, x_0, G)$ whose $G$-equivariant homotopy groups corresponds to the reduced $G$-equivariant connective K-homology groups of $X$. This description allows to relate generators of $\mathcal{F}_G(X)$ with some homological versions of the Chern classes.

Let $G$ and $H$ be finite groups, $X$ be a finite $G$-CW-complex and $Y$ be a finite $H$-CW-complex, we also prove a K"unneth formula for $\mathcal{F}_{G \times H}(X \times Y)$, obtaining an isomorphism

$$\mathcal{F}_{G \times H}(X \times Y) \cong \mathcal{F}_G(X) \otimes_{\mathcal{F}(\bullet)} \mathcal{F}_H(Y)$$

that is compatible with the decomposition as a supersymmetric algebra. In order to do this, we need to study the representation theory of pullbacks of groups.

Let

$$
\begin{array}{ccc}
\Gamma & \longrightarrow & G \\
p_2 \downarrow & & \downarrow \pi_2 \\
H & \longrightarrow & K \\
p_1 \downarrow & & \downarrow \pi_1
\end{array}
$$

be a pullback diagram of finite groups, with $\pi_1$ and $\pi_2$ surjective, in this case $\Gamma$ can be realized as a subgroup of $G \times H$. We prove that when $\Gamma$ is conjugacy-closed (see Definition 5.1) in $G \times H$ then we have a ring isomorphism

$$\text{Class}(\Gamma) \cong \text{Class}(H) \otimes_{\text{Class}(K)} \text{Class}(G).$$

This paper has two goals, the first one is to be a expository note about the main properties of $\mathcal{F}(X)$ and as second we present a proof of a Kunneth-type formula for $\mathcal{F}_G(X \times Y)$.

This paper is organized as follows:

In Section 2 we recall basic facts about equivariant K-theory, in particular we recall the construction for the character. Following ideas of [15] we give an explicit definition of the induced bundle and recall a formula (proved in [7, Thm. D]) for a character of the induced bundle. In Section 3 we recall basic facts about wreath products and its action over $X^n$. In Section 4 we recall the definition of $\mathcal{F}_G(X)$ and give another way to obtain the description as a supersymmetric algebra using the formula of the induced character. In Section 5 we study the representation theory of pullbacks. In Section 6 we recall some basic properties of semidirect products of direct products. In Section 7 we use results in Section

Revista Colombiana de Matemáticas
5 to give a Künneth formula for the Hopf algebra $F_{G \times H}(X \times Y)$. In Section 8 we do some final remarks about the relation of $\mathcal{F}(X)$ and homological versions of Chern classes.

2. Induced character in equivariant K-theory

In this section we recall a decomposition theorem for equivariant K-theory with complex coefficients obtained by Atiyah and Segal in [2]. In the next section we use that result to give a simple description of $F_G^q(X)$. In this paper all CW-complexes (and $G$-CW-complexes) that we consider are finite.

Definition 2.1. Let $X$ be a $G$-space. A $G$-vector bundle over $X$ is a map $p : E \to X$, where $E$ is a $G$-space satisfying the following conditions:

1. $p : E \to X$ is a vector bundle.
2. $p$ is a $G$-map.
3. For every $g \in G$ the left translation $E \to E$ by $g$ is bundle map.

If $p : E \to X$ is a $G$-vector bundle we define the fiber over $x \in X$ to the set $p^{-1}(x) = \{ v \in E \mid p(v) = x \}$, when $p$ is clear from the context we also denote this set by $E_x$. Also if $H \subseteq G$ is a subgroup, we can consider $E$ as a $H$-vector bundle over $X$, we denote it by $\text{res}_H^G(E)$.

Definition 2.2. Let $X$ and $Y$ be $G$-spaces. If $p : E \to Y$ is a $G$-vector bundle and $f : X \to Y$ is a $G$-map, then the pullback $p^*E \to X$ is a $G$-vector bundle over $X$ defined as $p^*E = \{(e, x) \in E \times X \mid p(e) = f(x)\}$. When $i : X \to Y$ is an inclusion we usually denote $i^*(E)$ by $E|_Y$.

Details about $G$-vector bundles can be found in [1].

Definition 2.3. Let $G$ be a group, let $X$ be a finite $G$-CW-complex (see [5]), the equivariant K-theory group of $X$, denoted by $K_G^n(X)$ is defined as the Grothendieck group of the monoid of isomorphism classes of $G$-equivariant vector bundles over $X$ with the operation of direct sum. The functor $K_G^*(-)$ could be extended to an equivariant cohomology theory $K^*(-)$, defining for $n > 0$: $K_G^{-n}(X) = \ker \left( K_G(X \times S^n) \xrightarrow{i_*} K_G(X) \right)$. And for any $G$-CW-pair $(X, A)$, set $K_G^{-n}(X, A) = \ker \left( K_G^{-n}(X \cup_A X) \xrightarrow{i_*} K_G^{-n}(X) \right)$.
Finally for $n < 0$

$$K^{-n}_G(X) = K^0_G(X) \text{ and } K^{-n}_G(X, A) = K^n_G(X, A).$$

For more details about equivariant K-theory the reader can consult [13].

**Example 2.4.** If the action of $G$ over $X$ is free, then there is a canonical isomorphism of abelian groups

$$K_G(X) \cong K(X/G).$$

**Example 2.5.** If the action of $G$ over $X$ is trivial, then there is a canonical isomorphism of abelian groups

$$K_G(X) \cong R(G) \otimes \mathbb{Z} K(X),$$

when $R(G)$ denotes the (complex) representation ring of $G$. In particular when $X = \{\bullet\}$ we obtain

$$K_G(\{\bullet\}) \cong R(G).$$

If $Y$ is a finite $G$-CW-complex, we can define a $G$-action on $K(Y)$. Let $g \in G$, the pullback $g^* : K(Y) \to K(Y)$, defines a $G$-action over $K(Y)$. We will need the following lemma.

**Lemma 2.6.** Let $Y$ be a finite $G$-CW-complex, then

$$K(Y/G) \otimes \mathbb{C} \cong K(Y)^G \otimes \mathbb{C}$$

**Proof.** It is a consequence of the Chern character and the analogous fact for singular cohomology. $\square$

In [2] a character for equivariant K-theory is constructed, that generalizes the character of representations. We will recall this construction briefly. Let $E$ be a $G$-vector bundle over $X$ and $g \in G$. Note that $X^g$ is a $C_G(g)$-space, then if $E$ is a $G$-vector bundle, $E|X^g$ is canonically a $C_G(g)$-vector bundle over $X^g$. Considering the action given by pullback we have that the isomorphism class $[(E|X^g)] \in K(X^g)$ is a $C_G(g)$-fixed point. Then $[(E|X^g)] \in K(X^g)^{C_G(g)}$. Finally for every element $\lambda \in S^1$, we can form the vector bundle of $\lambda$-eigenvectors considering the action of the element $g$ over $\pi(E|X^g)$ denoted by $\pi(E|X^g)_\lambda$. Then we can define a map

$$\text{char}_G : K_G(X) \otimes \mathbb{C} \to \bigoplus_{[g]} K((X^g)^{C_G(g)}) \otimes \mathbb{C}$$

$$[E] \mapsto \left( \bigoplus_{\lambda \in S^1} \pi(E|X^g)_\lambda \otimes \lambda \right).$$

Using the above Lemma we identify $K(X^g)^{C_G(g)}$ with $K(X^g/C_G(g))$. 

Revista Colombiana de Matemáticas
Theorem 2.7. The map $\text{char}_G$ is an isomorphism of complex vector spaces.

For a proof of the theorem see [2].

2.1. The induced bundle

Now we will give an explicit construction of the induced vector bundle. It is a direct generalization of the induced representation defined for example in Section 3.3 in [15].

Let $H \subseteq G$ be a subgroup of $G$ and $E \overset{\pi}{\to} X$ an $H$-vector bundle over a $G$-space $X$. If we choose an element from each left coset of $H$, we obtain a subset $R$ of $G$ called a system of representatives of $H \backslash G$; each $g \in G$ can be written uniquely as $g = sr$, with $r \in R = \{r_1, \ldots, r_n\}$ and $s \in H$, $G = \bigsqcup_{i=1}^n Hr_i$, we suppose that $r_1 = e$ the identity of the group $G$. Consider the vector bundle $F = \bigoplus_{i=1}^n (r_i)^*E$, with projection $\pi_F : F \to X$ and consider the following $G$-action defined over $F$:

Let $f \in F$, then
$$f = f_{r_1} \oplus \cdots \oplus f_{r_n},$$
where $f_{r_i} \in (r_i)^*E$. If $\pi_F(f) = x$ then $f_{r_i} = (x, e)$, where $e \in E_{r_i,x}$.

Let $g \in G$, note that $r_ig^{-1}$ is in the same left coset of some $r_j$, i.e. $r_ig^{-1} = sr_j$, for some $s \in H$. Define
$$g(f_{r_i}) = (gx, s^{-1}e) \in (r_j)^*E,$$
and define the action of $g$ on $f$ by linearly.

Now we will see that $F$ does not depend on the set of representatives up to isomorphism. Let $\{r'_1, \ldots, r'_n\}$ be another set of representatives of $H \backslash G$ and let $F' = \bigoplus_{i=1}^n (r'_i)^*E$. By reordering we can assume that $r_i$ and $r'_i$ are in the same left coset, then $r'_ir_i^{-1} \in H$.

We have an isomorphism of vector bundles over $X$
$$r'_ir_i^{-1} : (r_i)^*E \to (r'_i)^*E$$
$$(x, e) \to (x, r'_ir_i^{-1}e)$$
inducing an isomorphism of $G$-vector bundles
$$r'_1r_1^{-1} \oplus \cdots \oplus r'_nr_n^{-1} : F \to F'.$$

We only need to verify that this map commutes with the action of $G$. To see this, let $g \in G$ and $f_{r_i} = (x, e) \in (r_i)^*E$, there exist $s, s' \in H$ such that
$$r_ig^{-1} = sr_j \text{ and } r'_ig^{-1} = s' r'_j.$$ (1)
Note that \( gf_r, \in (r_j)^*E \), then
\[
(r_j^{-1})g(f_r) = (gx, r_j^{-1}r_j^{-1}e).
\]
On the other hand
\[
g(r_j^{-1}f_r) = (gx, (s')^{-1}(r_j^{-1}e)),
\]
but we know from (1)
\[
(s')^{-1}r_j^{-1} = r_j^{-1}r_j^{-1}e.
\]
Then the map \( r_j^{-1} + \ldots + r_j^{-1} \) commutes with the \( G \)-action and then \( F \) and \( F' \) are isomorphic as \( G \)-vector bundles.

We will denote the \( G \)-vector bundle \( F \) defined above by \( \text{Ind}_H^G(E) \). Summarizing we have.

**Theorem 2.8.** Let \( G \) be a finite group, let \( H \subseteq G \) be a subgroup. Let \( X \) be a \( G \)-CW-complex, and let \( E \) be a \( H \)-vector bundle over \( X \), there is a unique \( G \)-vector bundle \( \text{Ind}_H^G(E) \) over \( X \), up to isomorphism of \( G \)-vector bundles such that for every \( G \)-vector bundle \( F \) over \( X \) we have a natural identification
\[
\text{Hom}_G(\text{Ind}_H^G(E), F) \cong \text{Hom}_H(E, \text{res}_H^G(F)).
\]

**Proof.** Only remains to prove the identification. Let \( \xi \in \text{Hom}_G(\text{Ind}_H^G(E), F) \), recall that we have an inclusion of \( H \)-vector bundles
\[
E \to \text{Ind}_H^G(E)
\]
\[
v \in E_x \mapsto (x, v).
\]
Define \( r(\xi) \in \text{Hom}_H(E, \text{res}_H^G(F)) \) as follows, if \( v \in E_x \)
\[
r(\xi)(v) = \xi(x, v).
\]
It is clear that \( r(\xi) \in \text{Hom}_H(E, \text{res}_H^G(F)) \). On the other hand if \( \eta \in \text{Hom}_H(E, \text{res}_H^G(F)) \), define \( I(\eta) : \text{Ind}_H^G(E) \to F \) as follows, if \( v_i \in r_i^{-1}E \), then \( v_i = (x, v) \) with \( x \in X \) and \( v \in E_{r_i,x} \), then we define
\[
I(\eta)(v_i) = r_i^{-1}(\eta(v)).
\]
Extending linearly \( I(\eta) \) to \( \text{Ind}_H^G(E) \).

Now we will see that \( I(\eta) \) is \( G \)-equivariant. Let \( g \in G \), let \( s \in H \) such that
\[
rg^{-1} = sr_j,
\]
then
\[
g \cdot v_i = (gx, s^{-1}v).
\]
Now,
\[
I(\eta)(g \cdot v_i) = I(\eta)(gx, s^{-1}v)
= r_j^{-1}(\eta(s^{-1}v))
= r_j^{-1}s^{-1}\eta(v)
= gr_j^{-1}\eta(v)
= gI(\eta)(v_i).
\]
Then \(I(\eta) \in \text{Hom}_G(\text{Ind}_H^G(E), F)\). Now we will see that \(r\) and \(I\) are inverse of each other. It is clear that \(r(I(\eta)) = \eta\). On the other hand,
\[
I(r(\xi))(v_i) = r_i^{-1}(r(\xi)(v))
= r_i^{-1}\xi(r_ix, v)
= \xi(x, v)
= \xi(v_i).
\]
\[\square\]

We have a formula for the character of an induced \(H\)-vector bundle, it is a particular case of a formula for induced character of generalized cohomology theories in [7] and [8]. We include a proof for completeness.

**Theorem 2.9 (Formula for the induced character).** Let \(X\) be a \(G\)-CW-complex, let \(H\) be a subgroup of \(G\), let \(h\) be the order of \(H\) and \(E\) be a \(H\)-vector bundle, consider the map
\[
\text{char}_G \circ \text{Ind}_H^G : K_H(X) \otimes \mathbb{C} \to \bigoplus_{[g]} K(X^g)^{CG(g)} \otimes \mathbb{C},
\]
let \(R\) be a system of representatives of \(H \setminus G\). For each \(g \in G\), we have
\[
\text{char}_G(g) \circ \text{Ind}_H^G([E]) = \bigoplus_{r \in R, r^{-1}gr \in H} r^* \left( \text{char}^H(r^{-1}gr)([E]) \right)
= \frac{1}{h} \bigoplus_{r \in G, r^{-1}gr \in H} r^* \left( \text{char}^H(r^{-1}gr)([E]) \right).
\]

**Proof.** Our explicit definition of the induced bundle allows us to proof this result just by adapting the proof for representations contained in [15]. The vector bundle \(F = \text{Ind}_H^G(E)\) is the direct sum \(\bigoplus_{i=1}^n r_i^*E\), with \(R = \{r_1, \ldots, r_n\}\).
\[
H \setminus G = \{Hr_1, \ldots, Hr_n\}.
\]
We know from the definition of the induced bundle that if we write \(r_i g^{-1}\) in the form \(sr_j\) with \(r_j \in R\) and \(s \in H\), then \(g\) sends \(r_i^*E\) to \(r_j^*E\). Considering
the action of $g$ in $H \setminus G$, we have that
\[
\text{char}_G(g) \left( \text{Ind}_H^G(E) \right) = \text{char}_G(g) \left( \bigoplus_{Hr_i = Hr_i g} r_i^* E \oplus \bigoplus_{Hr_i \neq Hr_i g} r_i^* E \right)
\]
Note that $g$ acts in each term of the direct sum on the right hand side, hence the right hand side of the above equation can be written as
\[
\text{char}_G(g) \left( \bigoplus_{Hr_i = Hr_i g} r_i^* E \right) \oplus \text{char}_G(g) \left( \bigoplus_{Hr_i \neq Hr_i g} r_i^* E \right)
\]
We will see that $\text{char}_G(g) \left( \bigoplus_{Hr_i \neq Hr_i g} r_i^* E \right) = 0$. Because each 0-dimensional bundle is trivial, it suffices to check that this condition holds on fibers, i.e.,
\[
\text{char}_G(g) \left( \bigoplus_{Hr_i \neq Hr_i g} r_i^* E \right)_x = 0,
\]
for all $x \in X^g$. If we fix a basis for $(r_i^* E)_x$, the trace of the matrix representing the action of $g$ is zero because $Hr_i \neq Hr_i g^{-1}$.

Now, if $Hr_i = Hr_i g^{-1}$ we have that $r_i g r_i^{-1} = s_i$ with $s_i \in H$. Thus as the character is invariant under conjugation
\[
\text{char}_G(g)(r_i^* E) = \text{char}_H(s_i)(r_i^* E).
\]
Finally as the character commutes with pullbacks we have that,
\[
\text{char}_G(g)(\text{Ind}_H^G(E)) = \bigoplus_{r \in R, r g r^{-1} \in H} r_i^* \left( \text{char}_H(r g r^{-1})(E) \right)
\]
\[
= \frac{1}{h} \bigoplus_{s \in G, s g s^{-1} \in H} r_i^* \left( \text{char}_H(s^{-1} h s)(E) \right).
\]

3. Wreath product and its action on $X^n$

Let $C$ be a set. There is a natural action of $\mathfrak{S}_n$ on $C^n$ defined as
\[
\sigma \bullet (c_1, \ldots, c_n) = (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)})
\]
if $G$ is a group, we define the wreath product as the semidirect product
\[
G_n = G \wr \mathfrak{S}_n = G^n \rtimes \mathfrak{S}_n.
\]
This section is dedicated to describe the conjugacy classes and centralizers of elements in $G_n$, we follow [10, Chapter 1, App. B] and [18].

First, we must recall the conjugacy classes in $S_n$. Two elements $s_1, s_2 \in S_n$ are conjugate if their cycle factorization correspond to the same partition of $n$. For example, the elements $(1, 2)(3, 4, 5)$ and $(1, 4)(2, 3, 5)$ are conjugated and correspond to the partition $5 = 2 + 3$. Note that every partition of $n$ can be view as a function $m : \{1, 2, \ldots, n\} \rightarrow \mathbb{N}$ as follows. If $s \in S_n$ then $m_s(r)$ is the number of $r$-cycles in $s$. Now in the general case, if $x = (g, s) \in G_n$, then $s$ can be decomposed as a product of disjoint cycles, if $z = (i_1, i_2, \ldots, i_r)$ is one of these cycles, the element $g_{i_1}g_{i_{r-1}} \cdots g_{i_1}$ is called the cycle product of $x$ corresponding to $z$.

Recall that $G_*$ denotes the set of conjugacy classes of $G$. If $x = (g, s) \in G_n$, let $\rho(x) = m_x(r, c)$ denote the number of $r$-cycles in $s$ whose cycle product belongs to $c$, where $c \in G_*$ and $r \in \{1, 2, \ldots, n\}$. In this way every element $x \in G_n$ determines a matrix $\rho(x) = m_x(r, c)$ of non-negative integers such that $\sum_{r, c} r m_x(r, c) = n$.

For example let $G$ be the cyclic group, $\{g^0, g^1, g^2, g^3\}$, of 4 elements generated by $g$ and $s = (1, 2)(3, 4, 5) \in S_5$. If $x = (g, g, g, g, g, s)$ then $\rho(x) = m_x(r, c)$ looks like

\[
\begin{array}{cccc}
gr^0 & g^1 & g^2 & g^3 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 \\
5 & 0 & 0 & 0 \\
\end{array}
\]

the map $\rho : G_n \rightarrow M_{n,v}(\mathbb{Z})$ is called the type of $x \in G_n$, where $v = |G|$.

**Proposition 3.1.** Two elements in $G_n$ are conjugate iff they have the same type.

**Proof.** See [10, Appendix 1.B] \(\Box\)

By the above proposition we can assume that every element $x \in G_n$ is conjugated to a product of elements of the form

$$((g, 1, \ldots, 1), (i_v, 1, \ldots, i_u)).$$

Denote by $g_v(c) = ((g, 1, \ldots, 1), (1, \ldots, v))$. 

Volumen 56, Número 1, Año 2022
**Proposition 3.2.** The elements in the centralizer $C_{G_n}(g_n(c))$ are of the form $(gz, z, \ldots, z_{k+1}, \ldots, gz)$, with $z \in C_G(g)$. Moreover $C_{G_n}(g_n(c)) \cong C_G(g) \times \langle (1, \ldots, n) \rangle$.

**Proof.** It follows from a direct computation. ✓

We have described centralizers of elements in $G_n$ and in the next section we will use this description to write $\text{char}_{G_n}$ in terms of $\text{char}_G$.

Let $X$ be a $G$-space, there is canonical $G_n$-action over $X^n$ defined from the $G$-action over $X$

$$G_n \times X^n \to X^n$$

$$(g, \sigma, \bar{x}) \mapsto g(\sigma \cdot \bar{x})$$

where $\bar{g}$ acts component-wise.

In order to relate $\text{char}_{G_n}$ with $\text{char}_G$ we need to describe the fixed point set of a representative of each conjugacy class of $G_n$. Let us start with the conjugacy classes of elements $(\bar{g}, \sigma)$ where $\sigma$ is an $n$-cycle. To this end we will need the following result.

**Proposition 3.3.** Let $\zeta = ((g, 1, \ldots, 1), \sigma)$ with $g \in G$ and $\sigma$ is a $n$-cycle. There is a canonical homeomorphism

$$(X^n)^\zeta / C_{G_n}(\zeta) \cong X^g / C_G(g).$$

**Proof.** We can assume $\sigma = (1, \ldots, n)$. Let $(x_1, \ldots, x_n) \in (X^n)^\zeta$, then

$$\zeta(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$$

it implies

$$(gx_n, x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_n).$$

Therefore

$$x_n = x_{n-1} = \cdots = x_1, \quad gx_n = x_1,$$

and then $(x_1, \ldots, x_n) = (y, \ldots, y)$ lies in the diagonal and $y \in X^g$. This proves that $(X^n)^\zeta \cong X^g$. On the other hand, if $\bar{b} \in C_{G_n}(\zeta)$ then by Proposition 3.2

$$\bar{b} = \left(\left((gz, \ldots, z_{k+1}, \ldots, gz), \sigma^k\right) \right)$$

where $z \in C_G(g)$. Then we obtain

$$\bar{b}(y, \ldots, y) = \left(\left((gz, \ldots, z_{k+1}, \ldots, gz), \sigma^k\right) \cdot (y, \ldots, y) = (gzy, \ldots, zy_{k+1}, \ldots, gzy), \right)$$

showing that the orbit of $(y, \ldots, y)$ by $C_{G_n}(\zeta)$ is

$$\{ (gzy, \ldots, zy_{k+1}, \ldots, gzy) : z \in C_G(g) \}.$$

This proves the result. ✓
4. Fock space

Let $X$ be a $G$-space, by the equivariant Bott periodicity theorem we know that $K^*_G(X) = K^0_G(X) \oplus K^1_G(X)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded group. Denote by

$$F^*_G(X) = \bigoplus_{n \geq 0} K^*_G(n) \otimes \mathbb{C}, \quad F^n_G(X) = \bigoplus_{n \geq 0} q^n K^*_G(n) \otimes \mathbb{C}$$

where $q$ is formal variable giving a $\mathbb{Z}_+$-grading in $F^*_G(X)$. They both have a natural structure of abelian groups, we endow them with a product $\cdot$, defined as the composition of the induced bundle and the Künneth isomorphism $\boxtimes$ (see [12])

$$K^*_G(X^n) \times K^*_G(X^m) \boxrightarrow{\boxtimes} K^*_G(X^{n+m}) \xrightarrow{\text{Ind}} K^*_G(X^{n+m})$$

**Proposition 4.1.** With the above operations $F^*_G(X)$ is a commutative $(\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z})$-graded ring.

**Proof.** The associativity follows from the following fact. Let $[E_1] \in K^*_G(X^n)$, $[E_2] \in K^*_G(X^m)$ and $[E_3] \in K^*_G(X^k)$, then

$$(E_1 \cdot E_2) \cdot E_3 \cong \text{Ind}_{G_n \times G_m \times G_k}^{G_{n+m+k}} \left( \text{Ind}_{G_n \times G_m}^{G_{n+m}} (E_1 \boxtimes E_2) \boxtimes E_3 \right)$$

$$\cong \text{Ind}_{G_n \times G_m \times G_k}^{G_{n+m+k}} (E_1 \boxtimes E_2 \boxtimes E_3)$$

$$\cong \text{Ind}_{G_n \times G_m \times G_k}^{G_{n+m+k}} \left( E_1 \boxtimes \text{Ind}_{G_m \times G_k}^{G_{m+k}} (E_2 \boxtimes E_3) \right).$$

For the graded commutativity, let $[E_1] \in K^*_G(X^n)$ and $[E_2] \in K^*_G(X^m)$, we will prove that $E_1 \cdot E_2$ and $E_2 \cdot E_1$ has the same character as $G_{n+m}$-vector bundles over $X^{n+m}$.

Consider two inclusions of $\mathfrak{S}_n$ into $\mathfrak{S}_{n+m}$. The first one is the inclusion by the first $n$ letters denoted by $S_n \overset{\iota_1}{\rightarrow} S_{n+m}$; the second one is the inclusion by the last $n$ letters denoted by $S_n \overset{\iota_2}{\rightarrow} S_{n+m}$. Let $x = (\tilde{g}, \sigma) \in G_{n+m}$ and let $r = (\tilde{h}, \tau) \in G_{n+m}$, such that $r^{-1}xr \in G_n \times G_m$, then, there is $\eta_1 \in \mathfrak{S}_n$ and $\eta_2 \in \mathfrak{S}_m$ such that $\tau^{-1} \sigma \tau = i_1(\eta_1) i_2(\eta_2)$, but $i_1(\eta_1) i_2(\eta_2)$ is conjugated in $\mathfrak{S}_{n+m}$ to $i_1(\eta_2) i_2(\eta_1)$, then, there is $\gamma \in \mathfrak{S}_{n+m}$ such that $\gamma^{-1}(\tau^{-1} \sigma \tau) \gamma = i_1(\eta_1) i_2(\eta_1)$. Then

$$((e, \ldots, e), \gamma^{-1}(\tau^{-1} \sigma \tau)((e, \ldots, e), \gamma) \in G_m \times G_n.$$ 

It implies that we have bijective correspondence between elements $r \in G_{n+m}$ such that $r^{-1}xr \in G_n \times G_m$ and elements $s \in G_{n+m}$ such that $s^{-1}xs \in G_m \times G_n$. Then, if we apply Theorem 2.9, the number of terms in each direct sum computing

$$\text{Ind}_{G_n \times G_m}^{G_{n+m}} (E_1 \boxtimes E_2) \quad \text{and} \quad \text{Ind}_{G_m \times G_n}^{G_{n+m}} (E_2 \boxtimes E_2)$$
are the same. Moreover, for every \((\alpha, \beta) \in G_n \times G_m\),
\[
\text{char}_{G_n \times G_m}(\alpha, \beta)(E_1 \boxtimes E_2) \cong \text{char}_{G_m \times G_n}(\beta, \alpha)(E_2 \boxtimes E_1),
\]
then, we have \(E_1 \cdot E_2 \) and \(E_2 \cdot E_1\) have the same character, then the product is commutative. The other properties follows directly. □✓

Definition 4.2. Let \(R\) be a commutative ring, the graded-symmetric algebra of a \(\mathbb{Z}\)-graded \(R\)-module \(M\) (denoted by \(S(M)\)) is the quotient of the tensor algebra of \(M\) by the ideal \(I\) generated by elements of the form

\[
\begin{align*}
(1) \quad & x \otimes y - (-1)^{\deg(x) \deg(y)} (y \otimes x), \\
(2) \quad & x \otimes x, \text{ when } \deg(x) \text{ is even}.
\end{align*}
\]

Now we will give another proof of the description of \(\mathcal{F}_G^q(X)\) as a graded-symmetric algebra given in [14] or Theorem 3 in [18]. Our proof gives an explicit isomorphism and moreover gives explicit generators of \(\mathcal{F}_G^q(X)\) as \(\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z}\)-graded algebras.

Theorem 4.3. There is an isomorphism of \((\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z})\)-graded algebras
\[
\Phi : S(\bigoplus_{n \geq 1} q^n K^*_G(X) \otimes \mathbb{C}) \to \mathcal{F}_G^q(X).
\]

Proof. First note that using \(\text{char}_{G_n}\) we can define an injective group homomorphism in the following way. Consider the following sequence of maps:

\[
K^*_G(X) \otimes \mathbb{C} \xrightarrow{\lambda} \bigoplus_{x \in G_n} K^*((X^n)^x)_{C_G(x)} \otimes \mathbb{C} \xrightarrow{\text{map}} K^*_G(X^n) \otimes \mathbb{C}
\]

where the map \(\lambda\) is given by the assigning \([((g, 1, \ldots, 1), (1, \ldots, n))]_{G_n}\) to the conjugacy class \([g]_G\) and using the identification in Proposition 3.3. This map is certainly injective. Define

\[
\phi : K^*_G(X) \otimes \mathbb{C} \to K^*_G(X^n) \otimes \mathbb{C}
\]

by the composition of the above sequence so that \(\phi\) is injective and by the universal property of the graded-symmetric algebra we have a unique map

\[
\Phi : S \left( \bigoplus_{n \geq 1} K^*_G(X) \otimes \mathbb{C} \right) \to \mathcal{F}_G^q(X)
\]

extending \(\phi\).
Suppose inductively that $\text{im}(\Phi)$ contains $K^*_{G_k}(X^k) \otimes \mathbb{C}$ for $k < n$. Then by induction we know that the image of the following composition

$$S(\oplus_{n \geq 1} q^n K^*_G(X) \otimes \mathbb{C}) \times S(\oplus_{n \geq 1} q^n K^*_G(X) \otimes \mathbb{C})$$

contains

$$K^*_G(X^k) \otimes \mathbb{C} \cdot K^*_{G_{n-k}}(X^{n-k}) \otimes \mathbb{C} \subseteq K^*_G(X) \otimes \mathbb{C}.$$ 

Now we have that the image under $\text{char}_{G_n}$ of

$$\bigoplus_{k=1}^{n-1} (K^*_G(X^k) \otimes \mathbb{C}) \cdot (K^*_{G_{n-k}}(X^{n-k}) \otimes \mathbb{C})$$

coincides with

$$\bigoplus_{x \in J} K^*((X^n)^{x}/C_{G_n}(x)) \otimes \mathbb{C},$$

where $J$ is the set of conjugacy classes in $G_n$ such that for every $c, m_{\bullet}(n, c) = 0$, in other words $J$ is the set of conjugacy classes whose components in $S_n$ are not an n-cycle.

On the other hand if $x = ((g, 1, \ldots, 1), (1, \ldots, n))$ for some $g \in G$, then Proposition 3.3 gives us that $\text{im}(\text{char}_{G_n} \circ \Phi)$ contains $K^*((X^n)^{x}/C_{G_n}(x))$. Finally, since $\text{char}_{G_n}$ is an isomorphism we can conclude that $\Phi$ is surjective.

To see that $\Phi$ is injective we can use the formula for the induced character, because this formula implies that if $A \in S(\bigoplus_{k \geq 1} q^k K^*_G(X) \otimes \mathbb{C})$ is not zero then there exists $n$ and $x \in G_n$ such that $\text{char}_{G_n}(x)(\Phi(A)) \neq 0$, then $\Phi(A) \neq 0$. □✓

5. Pullback of groups

Let $\Gamma$ be a group fitting into the following pullback diagram

$$\begin{array}{ccc}
\Gamma & \xrightarrow{p_2} & G \\
p_1 \downarrow & & \downarrow \pi_2 \\
H & \xrightarrow{\pi_1} & K
\end{array}$$

(2)

If the group $\Gamma$ comes from a diagram 2 then it is isomorphic to a subgroup of $G \times H$, namely $\Gamma \cong \{(g, h) \in G \times H \mid \pi_1(g) = \pi_2(h)\}$. When maps $\pi_1$ and $\pi_2$ are clear from the context we denote $\Gamma$ by $G \times_K H$. We suppose that $\pi_1$ and $\pi_2$ are surjective.

In this section we describe the class function ring of $\Gamma$ in terms of the class function rings of $G$, $H$ and $K$. In order to obtain this description we need that $\Gamma$ satisfies the following condition.
Definition 5.1. Let $G$ be a finite group and let $H \subseteq G$ be a subgroup, let $[h]_H \in H_*$, we say that $[h]_H$ is closed in $G$ if,

$$[h]_H = [h]_G \cap H.$$ 

We say that $H$ is conjugacy-closed in $G$ if, for every $h \in H$, $[h]_H$ is closed in $G$.

Example 5.2. The following are examples of conjugacy-closed subgroups:

- The general linear groups over subfields are conjugacy-closed.
- The symmetric group is conjugacy-closed in the general linear group.
- The symmetric group on subsets are conjugacy-closed.
- The orthogonal group is conjugacy-closed in the general linear group over real numbers.
- The unitary group is conjugacy-closed in the general linear group.

Remark 5.3. Let $G$ and $H$ be groups

- If $H \subseteq G$ is conjugacy-closed in $G$, then the pullback of the inclusion

  $$i^* : \text{Class}(G) \to \text{Class}(H)$$

  is surjective.

- If $H$ is a retract in $G$, the pullback of the inclusion

  $$i^* : R(G) \to R(H)$$

  is surjective.

When $\Gamma$ is conjugacy-closed in $G \times H$, we have a way to express the class function ring of $\Gamma$ in terms of the class function rings of $G$, $H$ and $K$. The same is true for the representations ring when $\Gamma$ is a retract of $G \times H$.

5.1. The class function ring of a pullback

Consider a pullback diagram of finite groups such as (2). If we apply the representation ring functor we obtain the following diagram

$$
\begin{array}{c}
R(\Gamma) \\
\downarrow p_1^* \\
R(H) \\
\downarrow p_1^*
\end{array}
\xleftarrow{p_2^*}
\begin{array}{c}
R(G) \\
\downarrow p_2^* \\
R(K) \\
\downarrow p_2^*
\end{array}
\tag{3}
$$
This diagram endows the rings $R(G)$ and $R(H)$ with a $R(K)$-module structure. A similar statement is true changing the representation ring by the class function ring. We will prove that if $\Gamma$ is a retract of $G \times H$, then the diagram (3) is a pushout. In fact, we have the following theorem.

**Theorem 5.4.** Let $\Gamma$, $G$, $H$ and $K$ be finite groups such as in the diagram (2). If $\Gamma$ is conjugacy-closed in $G \times H$, there is an isomorphism

$$m : \text{Class}(G) \otimes_{\text{Class}(K)} \text{Class}(H) \to \text{Class}(\Gamma)$$

of $\text{Class}(K)$-modules.

Moreover, if $\Gamma$ is a retract of $G \times H$, we have an isomorphism

$$f : R(G) \otimes_{R(K)} R(H) \to R(\Gamma)$$

of $R(K)$-modules.

**Proof.** In order to avoid confusion, in this proof we denote the product on $\text{Class}(\Gamma)$, $\text{Class}(G)$ and $\text{Class}(H)$ by $\cdot$ and the generators of the tensor product by $\rho \otimes \gamma$.

The map $f$ is defined as

$$f : \text{Class}(G) \otimes_{\text{Class}(K)} \text{Class}(H) \to \text{Class}(\Gamma)$$

$$\rho \otimes \gamma \mapsto p_1^*(\rho) \cdot p_2^*(\gamma)$$

First we prove that the map $f$ is well defined. Let $\xi \in \text{Class}(K)$, $\rho \in \text{Class}(G)$ and $\gamma \in \text{Class}(H)$. Let $(g, h) \in \Gamma$

$$f(\pi_1^*(\xi) \cdot \rho \otimes \gamma)(g, h) = (p_1^*(\pi_1^*(\xi)) \cdot p_2^*(\gamma))(g, h)$$

$$= (\pi_1^*(\xi) \cdot \rho)(g) \gamma(h)$$

$$= \xi(\pi_1(g)) \rho(g) \gamma(h)$$

$$= \rho(g) \xi(\pi_2(h)) \gamma(h)$$

$$= f(\rho \otimes \pi_2^*(\xi) \cdot \gamma)(g, h).$$

Now we will prove that $f$ is an isomorphism. Consider the following diagram with exact rows

\[
\begin{array}{ccc}
0 & \to & \ker(\pi) \\
\downarrow{f_2} & & \downarrow{f_1} \\
\text{Class}(G) \otimes_{\mathbb{C}} \text{Class}(H) & \to & \text{Class}(G) \otimes_{\text{Class}(K)} \text{Class}(H) & \to & 0 \\
\downarrow{f} & & \downarrow{f_1} \\
\text{Class}(G \times H) & \to & \text{Class}(\Gamma) & \to & 0.
\end{array}
\]

Where map $\pi$ is the quotient by the relations defining tensor product over $\text{Class}(K)$, map $i^*$ is the pullback of the inclusion $i : \Gamma \to G \times H$, map $f_1$ is the natural isomorphism given by tensor product over $\mathbb{C}$ and the map $f_2$ is the
restriction of $f_1$ to $\ker(\pi)$. Note that as $\Gamma$ is closed conjugacy in $G \times H$ the map $i^*$ is surjective. We will prove that above diagram is commutative and that $f_2$ is an isomorphism.

First we need to verify that $f_1(\ker(\pi)) \subseteq \ker(i^*)$. Let $(g,h) \in \Gamma$,

\[ i^*[f_1(\pi_1^*(\xi) \cdot \rho \otimes \gamma - \rho \otimes \pi_2^*(\xi) \cdot \gamma)](g,h) = \]

\[ [p_1^*(\pi_1^*(\xi)) \cdot p_1^*(\rho) \cdot p_2^*(\gamma) - p_1^*(\rho) \cdot p_2^*(\pi_2^*(\xi)) \cdot p_2^*(\gamma)](g,h) = 0 \]

Now we prove that $\ker(i^*) = f_1(\ker(\pi))$. For this we will prove that if $f$ is a class function in $G \times H$ such that $i^*(f) \equiv 0$ and $f$ is orthogonal to every element in $f_1(\ker(\pi))$, then $f$ has to be zero.

Suppose that for every $\xi \in \text{Class}(K)$, $\rho \in \text{Class}(G)$ and $\gamma \in \text{Class}(H)$ we have

\[ \sum_{(g,h) \in G \times H} f(g,h) \rho(g) \gamma(h)[\xi(\pi_2(h)) - \xi(\pi_1(g))] = 0. \]

Let us fix $\rho \in \text{Class}(G)$ and let

\[ \eta(g) = \sum_{h \in H} f(g,h) \gamma(h)[\xi(\pi_2(h)) - \xi(\pi_1(g))]. \]

We observe that $\eta$ is a class function on $G$ that is orthogonal to every $\rho$ in $\text{Class}(G)$, then $\eta \equiv 0$.

By a similar argument we conclude that for every $(g,h) \in G \times H$ and $\xi \in \text{Class}(K)$

\[ f(g,h)[\xi(\pi_2(h)) - \xi(\pi_1(g))] = 0. \] (4)

We already know that $f(g,h) = 0$ if $(g,h) \in \Gamma$, then let $(g,h) \notin \Gamma$, we have two cases. First suppose that $\pi_1(g)$ is conjugate to $\pi_2(h)$ in $K$, in this case there is $h \in H$ such that $(g, hh^{-1}) \in \Gamma$ and then $f(g,h) = f(g, hh^{-1}) = 0$.

Suppose now that $\pi_1(g)$ is not conjugate to $\pi_2(h)$ in $K$, in this case there is $\xi \in \text{Class}(K)$ such that $\xi(\pi_1(g)) \neq \xi(\pi_2(g))$ and equation (4) gives us that $f(g,h) = 0$. Then we conclude that $\ker(i^*) = f_1(\ker(\pi))$. The map $f_2$ is an isomorphism because it is the restriction of $f_1$ and as the diagram is commutative we conclude that $f$ is $\text{Class}(K)$-module isomorphism.

When $\Gamma$ is a retract in $G \times H$, the same argument works changing characters by representations, in particular the map $i^*: R(G \times H) \to R(\Gamma)$ is surjective.

Observe that the pullback is not always conjugacy-closed in the product as the following example shows.
Example 5.5. Consider the pullback of the symmetric groups $\mathfrak{S}_3$ over the cyclic group $C_2$

\[
\begin{array}{ccc}
\Gamma & \rightarrow & \mathfrak{S}_3 \\
\downarrow & & \downarrow \text{sgn} \\
\mathfrak{S}_3 & \rightarrow & C_2
\end{array}
\]

In this case the pullback $\Gamma$ has 6 conjugacy classes, $\{\gamma_1, \ldots, \gamma_6\}$. The product $\mathfrak{S}_3 \times \mathfrak{S}_3$ has 9 conjugacy classes, $\{\chi_1, \ldots, \chi_9\}$. Observe that the elements $((1, 2, 3), (1, 2, 3))$ and $((1, 2, 3), (1, 3, 2))$ are conjugate in the group $\mathfrak{S}_3 \times \mathfrak{S}_3$ by the element $(e, (1, 2))$ but they are not conjugate in $\Gamma$.

The pullback of the inclusion can be described in the class function ring as follows:

<table>
<thead>
<tr>
<th>Class($\mathfrak{S}_3 \times \mathfrak{S}_3$)</th>
<th>$\rightarrow$</th>
<th>Class($\Gamma$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>$\mapsto$</td>
<td>$\gamma_1$</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>$\mapsto$</td>
<td>$\gamma_1$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>$\mapsto$</td>
<td>$\gamma_2$</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>$\mapsto$</td>
<td>$\gamma_2$</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>$\mapsto$</td>
<td>$\gamma_3$</td>
</tr>
<tr>
<td>$\chi_6$</td>
<td>$\mapsto$</td>
<td>$\gamma_3$</td>
</tr>
<tr>
<td>$\chi_7$</td>
<td>$\mapsto$</td>
<td>$\gamma_4$</td>
</tr>
<tr>
<td>$\chi_8$</td>
<td>$\mapsto$</td>
<td>$\gamma_4$</td>
</tr>
<tr>
<td>$\chi_9$</td>
<td>$\mapsto$</td>
<td>$\gamma_5 + \gamma_6$</td>
</tr>
</tbody>
</table>

This map is not surjective.

Example 5.6. Consider the following pullback

\[
\begin{array}{ccc}
\Gamma & \rightarrow & D_{12} \\
\downarrow & & \downarrow \psi_1 \\
C_3 \times C_4 & \rightarrow & \mathfrak{S}_3
\end{array}
\]

In this case the pullback $\Gamma$ is isomorphic to the group $C_2 \times (C_3 \times C_4)$ and it is conjugacy closed in the group $D_{12} \times (C_3 \times C_4)$. According with GAP[6] the group $D_{12}$ has group id (12,4) and generators $d_1, d_2$ and $d_3$ of orders 2, 2 and 3 respectively. The homomorphism $\psi_1$ is given by

$\psi_1 : D_{12} \rightarrow \mathfrak{S}_3$

- $d_1 \mapsto (2, 3)$
- $d_2 \mapsto (1)$
- $d_3 \mapsto (1, 2, 3)$. 
The group $C_3 \rtimes C_4$ has id (12,1) and it is generated by three elements $g_1, g_2, g_3$ of orders 4, 2 and 3 respectively. The homomorphism $\psi_2$ is given by

$$
\psi_2 : C_3 \rtimes C_4 \to S_3
$$

$$
g_1 \mapsto (2, 3) \\
g_2 \mapsto (1) \\
g_3 \mapsto (1, 2, 3).
$$

For these groups the pullback $\Gamma$ is isomorphic to the group $C_2 \times (C_3 \rtimes C_4)$ with group id (24,7) and four generators $f_1, f_2, f_3, f_4$ of orders 4, 6, 2 and 2 respectively.

Applying Theorem 5.4 we obtain an isomorphism

$$
\text{Class}(\Gamma) \cong \text{Class}(D_{12}) \otimes \text{Class}(S_3) \otimes \text{Class}(C_3 \rtimes C_4).
$$

For more examples please see [4].

6. Semidirect product of a direct product

Let $A_1, A_2$ be groups with an action of a group $G$ by automorphisms noted by $a_i^g = g \cdot a_i$, for $a_i \in A_i$ and $g \in G$. Note that $G$ acts also on the direct product $A_1 \times A_2$ by acting on each component, i.e. $(a_1, a_2)^g := (a_1^g, a_2^g)$. In this section we describe the semidirect product of a direct product as a pullback of two semidirect products and then, we apply this for the wreath product of a direct product which will allow us to compute the Fock ring of a product.

Consider the projections $\pi_i : A_i \rtimes G \to G$ and the pullback $\Gamma$ associated

$$
\begin{array}{ccc}
\Gamma & \longrightarrow & A_1 \rtimes G \\
\downarrow & & \downarrow \\
A_2 \rtimes G & \longrightarrow & G
\end{array}
$$

Proposition 6.1. The pullback $\Gamma$ is isomorphic to the semidirect product $(A_1 \times A_2) \rtimes G$.

Proof. Note that the pullback is the subgroup of $(A_1 \times G) \times (A_2 \times G)$ given by

$$
\Gamma = \{ (a_1, g_1, a_2, g_2) \in (A_1 \times G) \times (A_2 \times G) : \pi_1(a_1, g_1) = \pi_2(a_2, g_2), a_1 \in A_1, g_i \in G \}
$$

that is, $g_1 = g_2$. Consider the bijective function $\phi : \Gamma \to (A_1 \times A_2) \rtimes G$ given by $\phi(a_1, g, a_2, g) = (a_1, a_2, g)$. On one hand

$$
\phi((a_1, g, a_2, g) \cdot (b_1, h, b_2, h)) = \phi(a_1 b_1^g, gh, a_2 b_2^g, gh) = (a_1 b_1^g, a_2 b_2^g, gh).
$$

On the other hand $(a_1, a_2, g) \cdot (b_1, b_2, h) = (a_1 b_1^g, a_2 b_2^g, gh)$ which shows that $\phi$ is a homomorphism of groups.

Revista Colombiana de Matemáticas
Corollary 6.2. Let $A, B$ be groups, there is an isomorphism

$$(A \times B)_n \cong A_n \times B_n.$$ 

Now we proof that certain conjugacy classes in $(A \times B)_n$ are closed in $A_n \times B_n$.

Proposition 6.3. Let $(\bar{g}, \bar{h}, \sigma) \in (A \times B)_n$, where $\sigma$ is an $n$-cycle. Then its conjugacy class in $A_n \times B_n$ is closed.

Proof. Let $x = (\bar{g}_1, \bar{h}_1, \sigma_1)$ and $y = (\bar{g}_2, \bar{h}_2, \sigma_2)$ be elements in $(A \times B)_n$ that are conjugated in $A_n \times B_n$, where $\sigma_1$ and $\sigma_2$ are $n$-cycles. We can suppose that $\sigma_1 = \sigma_2 = (1 \cdots n)$.

Note that

$$(\bar{g}_1, \sigma_1) \sim_{A_n} (g_2, \sigma_2) \text{ and } (\bar{h}_1, \sigma_1) \sim_{B_n} (\bar{h}_2, \sigma_2).$$

Since as $\sigma_1$ and $\sigma_2$ are $n$-cycles, $\prod_{i=1}^n g_{1,i} \sim_A \prod_{i=1}^n g_{2,i}$, and $\prod_{i=1}^n h_{1,i} \sim_B \prod_{i=1}^n h_{2,i}$. On the other hand the type of $x$ is given by

$$m_x(r, c) = \begin{cases} 1 & \text{if } r = n \text{ and } (\prod_{i=1}^n g_{1,i}, \prod_{i=1}^n h_{1,i}) \in c \\ 0 & \text{in any other case} \end{cases}$$

and the type of $y$ is given by

$$m_y(r, c) = \begin{cases} 1 & \text{if } r = n \text{ and } (\prod_{i=1}^n g_{2,i}, \prod_{i=1}^n h_{2,i}) \in c \\ 0 & \text{in any other case} \end{cases}$$

Then the types of $x$ and $y$ are equal, hence $x$ and $y$ are conjugated in $(A \times B)_n$. □✓

7. The Fock space of a product of spaces

In this section we apply results of Section 5 in order to obtain a decomposition of $\mathcal{F}_G \times \mathcal{H}(X \times Y)$ in terms of $\mathcal{F}_G(X)$ and $\mathcal{F}_H(Y)$. Let $X$ be a $G$-space, we can endow to $\mathcal{F}_G(X)$ with natural module structures as follows:

- Consider the trivial $G_n$-space $\{\bullet\}$, and the unique $G_n$-map $\pi : X^n \to \{\bullet\}$, then the pullback

$$\pi^* : \text{Class}(G_n) \to K_{G_n}^*(X^n) \otimes \mathbb{C}$$

induces a $\text{Class}(G_n)$-module structure over $K_{G_n}^*(X) \otimes \mathbb{C}$, hence we have a $\mathcal{F}_G(\{\bullet\})$-module structure over $\mathcal{F}_G(X)$ defined componentwise.
• Note that we have a quotient map \(s : G_n \to \mathfrak{S}_n\), then the pullback \((\pi \circ s)^* : \text{Class}(\mathfrak{S}_n) \to K_{G_n}(X^n) \otimes \mathbb{C}\) induce a \(\text{Class}(\mathfrak{S}_n)\)-module structure over \(K_{G_n}(X) \otimes \mathbb{C}\), hence we have a \(\mathcal{F}((\bullet))\)-module structure over \(F_{G}((X))\) defined componentwise.

As we observe in Section 5.1, \((G \times H)_n\) is not closed conjugacy in \(G_n \times H_n\), then we cannot expect a decomposition of \(\text{Class}((G \times H)_n)\) in terms of \(\text{Class}(G_n)\), \(\text{Class}(H_n)\) and \(\text{Class}(\mathfrak{S}_n)\), but as the conjugacy classes with an \(n\)-cycle as component in \(S_n\) are closed we have a decomposition of \(F_{G \times H}(X \times Y)\) in terms of \(F_{G}(X)\), \(F_{H}(Y)\) and \(\mathcal{F}((\bullet))\), with \(X\) a \(G\)-space and \(Y\) a \(H\)-space.

**Theorem 7.1.** There is an isomorphism of \(\mathcal{F}((\bullet))\)-modules

\[
F_{G \times H}(X \times Y) \xrightarrow{\mathcal{K}} F_{G}(X) \otimes_{\mathcal{F}((\bullet))} F_{H}(Y).
\]

The map \(\mathcal{K}\) is compatible with the symmetric algebra decomposition in Thm. 4.3. That means, we have a commutative diagram

\[
\begin{array}{ccc}
F_{G \times H}(X \times Y) & \xrightarrow{\mathcal{K}} & F_{G}(X) \otimes_{\mathcal{F}((\bullet))} F_{H}(Y) \\
\downarrow d & & \downarrow d \otimes d \\
S(X \times Y) & \xrightarrow{\pi_{G} \otimes \pi_{H}} & S(X) \otimes_{\mathcal{F}((\bullet))} S(Y)
\end{array}
\]

Where \(S(X)\) stands for \(S(\bigoplus_{n \geq 0} K_{G}(X) \otimes \mathbb{C})\), and similarly for \(S(Y)\) and \(S(X \times Y)\).

**Proof.** Let \(\{E_1, \ldots, E_m\}\) be a basis of \(K_{G}^*(X) \otimes \mathbb{C}\) as complex vector space and let \(\{F_1, \ldots, F_s\}\) be a basis of \(K_{H}^*(Y) \otimes \mathbb{C}\) as complex vector space. From the proof of Theorem 4.3 we can conclude that

\[
\{\Delta_{G,n,c,k} \in K_{G_n}(X^n) \otimes \mathbb{C} \mid n \geq 0, c \in G_*, 1 \leq k \leq m\}
\]

is a basis of \(F_{G}(X)\) as \(\mathbb{C}\)-algebra, where

\[
\text{char}_{G_*}(\Delta_{G,n,c,k})(\{g_1, \ldots, g_n\}, \sigma) = \begin{cases} E_k & \text{if } \prod_{i=1}^{n} g_\sigma(1) \in c \text{ and } \sigma \text{ is an } n\text{-cycle} \\ 0 & \text{in any other case} \end{cases}
\]

In a similar way we define

\[
\{\Delta_{H,n,d,l} \in K_{H_n}(Y^n) \otimes \mathbb{C} \mid n \geq 0, d \in H_*, 1 \leq l \leq s\}
\]

a basis of \(F_{H}(Y)\) as \(\mathbb{C}\)-algebra, where

\[
\text{char}_{H_*}(\Delta_{H,n,d,l})(\{h_1, \ldots, h_n\}, \sigma) = \begin{cases} F_l & \text{if } \prod_{i=1}^{n} h_\sigma(1) \in d \text{ and } \sigma \text{ is an } n\text{-cycle} \\ 0 & \text{in any other case} \end{cases}
\]
Recall that we have an isomorphism
\[ K_{G \times H}(X \times Y) \otimes \mathbb{C} \xrightarrow{\cong} (K_G(X) \otimes K_H(Y)) \otimes \mathbb{C}, \]
given by the external tensor product. It is proved in [12] or can be obtained (for complex coefficients) directly from the character.

Using the above identification we have that
\[ \{ \Delta_{G \times H, n, c \times d, (k, l)} \mid n \geq 0, c \in G_*, d \in H_*, 1 \leq k \leq m, 1 \leq l \leq s \} \]
is a basis as \( \mathbb{C} \)-algebra of \( F_{G \times H}(X \times Y) \), where
\[
\text{char}_{(G \times H)_n}(\Delta_{G \times H, n, c \times d, (k, l)})(\tilde{g}, \tilde{h}, \sigma) = \begin{cases} E_k \otimes F_l & \text{if } \prod_{i=1}^n g_{\sigma^i(1)} \in c, \prod_{i=1}^n h_{\sigma^i(1)} \in d \text{ and } \sigma \text{ is an } n\text{-cycle} \\ 0 & \text{in any other case.} \end{cases}
\]
As the conjugacy classes when the character of the above elements is not zero is closed in \( G_n \times H_n \) we have
\[ \Delta_{G \times H, n, c \times d, (k, l)} = \Delta_{G, n, c, k} \cdot \Delta_{H, n, d, l}, \]
hence the map defined on generators as
\[ \Delta_{G \times H, n, c \times d, (k, l)} \mapsto \Delta_{G, n, c, k} \otimes \Delta_{H, n, d, l} \]
is an isomorphism of \( \mathcal{F}(\{\bullet\}) \)-modules satisfying the required conditions. \( \square \)

8. Final remarks

In [16] and [17] a configuration space representing equivariant connective K-homology for finite groups was constructed. We recall the construction briefly.

**Definition 8.1.** Let \( G \) be a finite group and \((X, x_0)\) be a based \( G \)-connected, \( G \)-CW-complex. Let \( \mathcal{C}(X, x_0, G) \) be the \textit{G-space of configurations of complex vector spaces over} \((X, x_0)\), defined as the increasing union, with respect to the inclusions \( M_n(\mathbb{C}[G]) \to M_{n+1}(\mathbb{C}[G]) \)
\[ \mathcal{C}(X, x_0, G) = \bigcup_{n \geq 0} \text{Hom}^*(C_0(X), M_n(\mathbb{C}[G])), \]
with the compact open topology. Notice that \( * \) refers to \( * \)-homomorphism, \( C_0(X) \) denotes the \( * \)-algebra of complex valued continuous maps vanishing at \( x_0 \) and \( \mathbb{C}[G] \) denotes the complex group ring.
We endow \( \mathcal{C}(X, x_0, G) \) with a continuous \( G \)-action as follows. If \( F \in \mathcal{C}(X, x_0, G) \), we define
\[ g \cdot F : C_0(X) \to M_n(\mathbb{C}[G]) \]
\[ f \mapsto g \cdot F(g^{-1} \cdot f). \]
The space $\mathcal{C}(X, x_0, G)$ can be described as the configuration space whose elements are formal sums
$$
\sum_{i=1}^{n}(x_i, V_i),
$$
when $x_i \in X - \{x_0\}$ and $V_i \subseteq \mathbb{C}[G]^{\infty}$ such that if $x_i \neq x_j$ then $V_i \perp V_j$, subject to some relations, for details see [17, Sec. 2.1]. We call the elements $x_i$ the points and to the vector spaces $V_i$ the labels.

**Remark 1.** When the based $G$-CW-complex $(X, x_0)$ is not supposed to be $G$-connected, we define the configuration space $\mathcal{C}(X, x_0, G) = \Omega_0 \mathcal{C}(\Sigma X, x_0, G)$, where $\Omega_0$ denotes the based loop space and $\Sigma$ denotes the reduced suspension.

That description allow us to define a Hopf space structure on $\mathcal{C}(X, x_0, G)$ by putting together two configurations when labels in both of them are mutually orthogonal.

We have the following result:

**Theorem 2** (Thm. 5.2 in [16]). Let $(X, x_0)$ be a based finite $G$-connected $G$-CW-complex. If we denote by $k_n^G(X, x_0)$ the $n$-th $G$-equivariant connective $K$-homology groups of the pair $(X, x_0)$, then there is a natural isomorphism
$$
\pi_n(\mathcal{C}(X, x_0, G)^G) \overset{\cong}{\longrightarrow} k_n^G(X, x_0).
$$

When a Hopf space $\mathcal{Y}$ is path-connected, consider the Hurewicz morphism
$$
\lambda : \pi_*(\mathcal{Y}; \mathbb{C}) = \bigoplus_{n \geq 0} \pi_n(\mathcal{Y}) \otimes \mathbb{C} \to H_*(\mathcal{Y}; \mathbb{C}).
$$

We have the following result:

**Theorem 3** (Thm. of the Appendix in [11]). If $\mathcal{Y}$ is a pathwise connected homotopy associative Hopf space with unit, and $\lambda : \pi_*(\mathcal{Y}; \mathbb{C}) \to H_*(\mathcal{Y}; \mathbb{C})$ is the Hurewicz morphism viewed as a morphism of $\mathbb{Z}$-graded Lie algebras, then it induces an isomorphism of Hopf algebras
$$
\bar{\lambda} : S(\pi_*(\mathcal{Y}; \mathbb{C})) \to H_*(\mathcal{Y}; \mathbb{C}).
$$

Applying the above theorem to $\mathcal{C}(X, x_0, G)$ we obtain.

**Corollary 4.** Let $X$ be a finite $G$-CW-complex, if $X$ is $G$-connected we have an isomorphism
$$
S(k_*^G(X, x_0) \otimes \mathbb{C}) \cong H_*(\mathcal{C}(X, x_0, G)^G; \mathbb{C}).
$$
In order to relate $H_*(\mathcal{C}(X,x_0,G)\otimes; \mathbb{C})$ with $\mathcal{F}_G(X)$ we need to recall the following result proved in Theorem 6.13 in [16] using the equivariant Chern character obtained in [9].

**Theorem 5.** Let $X$ be a $G$-CW-complex. There is a natural isomorphism of $\mathbb{Z}$-graded complex vector spaces (here the graduation is given by $q$)

$$\bigoplus_{q \geq 0} k^G_n(X) \otimes \mathbb{C} \cong \bigoplus_{n \geq 0} K^G_n(X) \otimes \mathbb{C}[[q]].$$

Finally we can relate $H_*(\mathcal{C}(X,x_0,G)\otimes; \mathbb{C})$ with $\mathcal{F}_{qG}(X)$ when $X$ is an even dimensional $G$-connected, $G$-Spin$^c$-manifold. First we recall Poincaré duality for equivariant K-theory.

**Theorem 8.2.** [3] Let $M$ be a $n$-dimensional $G$-Spin$^c$-manifold. Then there exists an isomorphism

$$D : K_n^G(M) \rightarrow K_{n-\ast}^G(M).$$

Applying Theorem 8.2 and Theorem 3 we can obtain the main result of the section.

**Theorem 8.3.** Let $(M,m_0)$ be an even dimensional $G$-connected, $G$-Spin$^c$-manifold. We have an isomorphism of $\mathbb{Z}$-graded Hopf algebras

$$H_*(\mathcal{C}(M,m_0,G)\otimes; \mathbb{C}) \cong \mathcal{F}_G^q(M).$$

**Proof.** Since $M$ is a $G$-Spin$^c$ manifold we can use Theorem 8.2 and obtain the following isomorphism of $\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z}$-graded Hopf algebras

$$S\left(k^G_\ast(M,m_0) \otimes \mathbb{C}\right) \cong S\left(\bigoplus_{n \geq 1} q^n K^G_\ast(M,m_0) \otimes \mathbb{C}\right) \cong S\left(\bigoplus_{n \geq 1} q^n K^G_\ast(M,+ \otimes \mathbb{C}\right) \cong S\left(\bigoplus_{n \geq 1} q^n K^G_\ast(M) \otimes \mathbb{C}\right).$$

Combining Corollary 4, Theorem 3 and Theorem 4.3 we obtain

$$H_*(\mathcal{C}(M,m_0,G)\otimes; \mathbb{C}) \cong \mathcal{F}_G^q(M).$$

✓
For the case when $M$ is not necessarily $G$-connected, we can obtain also a similar result. For details consult [16, Proposition 6.11].

**Proposition 8.4.** Let $X$ be a finite $G$-CW-complex, we have an isomorphism

$$H_*((\Omega \mathcal{C}(\Sigma X, G); \mathbb{C}) \cong S(k^G_*(X, x_0) \otimes \mathbb{C}).$$

In particular we have.

**Example 8.5.** For $X = S^0$ we have

$$\Omega (\mathcal{C}(\Sigma(S^0), G)) \simeq BU_G.$$

Where $BU_G$ can be taken as the Grassmannian of finite dimensional vector subspaces of a complete $G$-universe. A complete $G$-universe is a countably infinite-dimensional representation of $G$ with an inner product such that contains a copy of every irreducible representation of $G$, contains countably many copies of each finite-dimensional subrepresentation. Applying the above discussion to this Hopf space we conclude that

$$H_*((BU_G)^G; \mathbb{C}) \cong R(G) \otimes S(\pi_*(\mathcal{C}(S^0))^G)$$

$$\cong R(G) \otimes S(\bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C})$$

$$\cong S(\bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C}).$$

Summarizing, we have an isomorphism

$$H_*((BU_G)^G; \mathbb{C}) \cong F^G_{BU_G}(\bullet) = S(\bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C}).$$

We also have

$$H_*((BU_G)^G; \mathbb{C}) \cong S(\bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C}) \cong \mathbb{C}[\sigma^1_1, \ldots, \sigma^k_1, \sigma^1_2, \ldots]$$

where $\{\sigma^1_1, \ldots, \sigma^k_1\}$ is a complete set of non isomorphic irreducible representations of $G$. We expect that the elements $\sigma^k_i$ correspond in some sense with duals of $G$-equivariant Chern classes.

Now suppose that $M$ is a $G$-connected $G$-Spin$^c$-manifold and $N$ is a $H$-connected $H$-Spin$^c$-manifold, then we have an isomorphism of $\mathbb{Z}$-graded Hopf algebras

$$H_*((C(M \times N, (m_0, n_0), G \times H)^{G \times H}; \mathbb{C}) \cong F^G_{BU_G}(M) \otimes F^H_{BU_H}(N)$$
In the case that $M = N = S^0$ with trivial action we obtain
\[ H_\ast((BU_{G \times H})^{G \times H}; \mathbb{C}) \cong \mathcal{F}_G(\{\bullet\}) \otimes_{\mathcal{F}(\{\bullet\})} \mathcal{F}_H(\{\bullet\}). \]

References


(Recibido en noviembre de 2021. Aceptado en mayo de 2022)

**Departamento de Matemáticas**
**Fundación Universitaria Konrad Lorenz**
**Bogotá D.C, Colombia**
e-mail: german.combarizag@konradlorenz.edu.co

**UMPA L’unité de Mathématiques Pures et Appliquées**
**ENS de Lyon site Monod UMPA UMR 5669 CNRS 46, allé d’Italie**
**69364 Lyon Cedex 07, France**
e-mail: juan-esteban.rodriguez-camargo@ens-lyon.fr

**Departamento de Matemáticas**
**Universidad Nacional de Colombia, sede Bogotá**
**Cra. 30 Calle 45, Ciudad Universitaria**
**Bogotá D.C, Colombia**
e-mail: mavelasquezme@unal.edu.co

Revista Colombiana de Matemáticas