https://doi.org/10.15446/recolma.v56n1.105613

Revista Colombiana de Matemáticas
Volumen 56(2022)1, páginas 35-61

Induced character in equivariant K-theory, wreath products and pullback of groups

Carácter inducido en K-teoría equivariante, productos wreath y pullbacks de grupos

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Abstract. Let G be a finite group and let X be a compact G-space. In this note we study the $(\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z})$-graded algebra

$$\mathcal{F}_G^q(X) = \bigoplus_{n\geq 0} q^n \cdot K_{G\wr\mathfrak{S}_n}(X^n) \otimes \mathbb{C},$$

defined in terms of equivariant K-theory with respect to wreath products as a symmetric algebra, we review some properties of $\mathcal{F}_G^q(X)$ proved by Segal and Wang. We prove a Kunneth type formula for this graded algebras, more specifically, let H be another finite group and let Y be a compact H-space, we give a decomposition of $\mathcal{F}_{G\times H}^q(X\times Y)$ in terms of $\mathcal{F}_G^q(X)$ and $\mathcal{F}_H^q(Y)$. For this, we need to study the representation theory of pullbacks of groups. We discuss also some applications of the above result to equivariant connective K-homology.

Key words and phrases. equivariant K-theory, wreath products, Fock space.

2020 Mathematics Subject Classification. 19L47, 19L41.

Resumen. Sea G un grupo finito y X un G-espacio compacto. En esta nota estudiamos el álgebra $(\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z})$-graduada

$$\mathcal{F}_G^q(X) = \bigoplus_{n\geq 0} q^n \cdot K_{G\wr\mathfrak{S}_n}(X^n) \otimes \mathbb{C},$$

definida en términos de K-teoría equivariante con respecto a productos guirnalda, como un álgebra simétrica, revisamos algunas de las propiedades de $\mathcal{F}^q_G(X)$ probadas por Segal y Wang. Probamos una formula tipo Kunneth para estas álgebras graduadas, más específicamente, sea H otro grupo finito y Y un H-espacio compacto, nosotros damos una descomposición de $\mathcal{F}^q_{G\times H}(X\times Y)$ en términos de $\mathcal{F}^q_G(X)$ y $\mathcal{F}^q_H(Y)$, para esto, debemos estudiar la teoría de representaciones de pullbacks de grupos. Discutimos tambien algunas aplicaciones de los resultados anteriores a K-homología equivariante conectiva.

Palabras y frases clave. K-teoría equivariante, productos wreath, espacio de Fock.

Notation

In this note we denote by $\mathfrak{S}_n$ the symmetric group in n letters. Let G be a finite group, let $g, g' \in G$, we say that g and g' are conjugated in G (denoted by $g \sim_G g'$) if there is $s \in G$ such that $g = sg's^{-1}$. We denote by

$$[g]_G = \{g' \in G \mid g \sim_G g'\}$$

the conjugacy class of g in G (or simply by $[g]$ when G is clear from the context). We denote by G_* the set of conjugacy classes of G. We denote by $C_G(g)$ the centralizer of g in G. Also $R(G)$ will be the complex representation ring of G, with operations given by direct sum and tensor product, and generated as an abelian group by the isomorphism classes of irreducible representations of G. The *class function ring of* G is the set

$$\mathrm{Class}(G) = \{f : G \to \mathbb{C} \mid f \text{ is constant in conjugacy classes}\}$$

with the usual operations.

1. Introduction

Let X be a finite CW-complex. In [14] Segal studied the vector spaces

$$\mathcal{F}(X) = \bigoplus_{n\geq 0} K_{\mathfrak{S}_n}(X^n) \otimes \mathbb{C},$$

these spaces carry several interesting structures, for example they admit a Hopf algebra structure with the product defined using induction on vector bundles and the coproduct defined using restriction.

Later in [18], Wang generalizes Segal's work to an equivariant context. Let G be a finite group and X be a finite G-CW-complex, Wang defines the vector space

$$\mathcal{F}_G(X) = \bigoplus_{n\geq 0} K_{G\wr\mathfrak{S}_n}(X^n) \otimes \mathbb{C},$$

where $G \wr \mathfrak{S}_n$ denotes the wreath product acting naturally over X^n. Wang proves that $\mathcal{F}_G(X)$ admits similar structures as $\mathcal{F}(X)$. In particular $\mathcal{F}_G(X)$ has a description as a supersymmetric algebra in terms of $K_G(X) \otimes \mathbb{C}$.

Following ideas of [14], in [16] appears another reason to study $\mathcal{F}_G(X)$. When X is a G-spin^c-manifold of even dimension, $\mathcal{F}_G(X)$ is isomorphic to the homology with complex coefficients of the G-fixed point set of a based configuration space $\mathfrak{C}(X, x_0, G)$ whose G-equivariant homotopy groups corresponds to the reduced G-equivariant connective K-homology groups of X. This description allows to relate generators of $\mathcal{F}_G(X)$ with some homological versions of the Chern classes.

Let G and H be finite groups, X be a finite G-CW-complex and Y be a finite H-CW-complex, we also prove a Künneth formula for $\mathcal{F}_{G\times H}(X \times Y)$, obtaining an isomorphism

$$\mathcal{F}_{G\times H}(X \times Y) \cong \mathcal{F}_G(X) \otimes_{\mathcal{F}(\{\bullet\})} \mathcal{F}_H(Y)$$

that is compatible with the decomposition as a supersymmetric algebra. In order to do this, we need to study the representation theory of pullbacks of groups.

Let

$$\begin{array}{ccc} \Gamma & \xrightarrow{p_2} & G \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ H & \xrightarrow{\pi_1} & K \end{array}$$

be a pullback diagram of finite groups, with π_1 and π_2 surjective, in this case Γ can be realized as a subgroup of $G \times H$. We prove that when Γ is conjugacy-closed (see Definition 5.1) in $G \times H$ then we have a ring isomorphism

$$\mathrm{Class}(\Gamma) \cong \mathrm{Class}(H) \otimes_{\mathrm{Class}(K)} \mathrm{Class}(G).$$

This paper has two goals, the first one is to be a expository note about the main properties of $\mathcal{F}(X)$ and as second we present a proof of a Kunneth-type formula for $\mathcal{F}_G(X \times Y)$.

This paper is organized as follows:

In Section 2 we recall basic facts about equivariant K-theory, in particular we recall the construction for the character. Following ideas of [15] we give an explicit definition of the induced bundle and recall a formula (proved in [7, Thm. D]) for a character of the induced bundle. In Section 3 we recall basic facts about wreath products and its action over X^n. In Section 4 we recall the definition of $\mathcal{F}_G(X)$ and give another way to obtain the description as a supersymmetric algebra using the formula of the induced character. In Section 5 we study the representation theory of pullbacks. In Section 6 we recall some basic properties of semidrect products of direct products. In Section 7 we use results in Section

5 to give a Künneth formula for the Hopf algebra $\mathcal{F}_{G\times H}(X\times Y)$. In Section 8 we do some final remarks about the relation of $\mathcal{F}(X)$ and homological versions of Chern classes.

2. Induced character in equivariant K-theory

In this section we recall a decomposition theorem for equivariant K-theory with complex coefficients obtained by Atiyah and Segal in [2]. In the next section we use that result to give a simple description of $\mathcal{F}^q_G(X)$. In this paper all CW-complexes (and G-CW-complexes) that we consider are finite.

Definition 2.1. Let X be a G-space. A G-vector bundle over X is a map $p : E \to X$, where E is a G-space satisfying the following conditions:

(1) $p : E \to X$ is a vector bundle.

(2) p is a G-map.

(3) For every $g \in G$ the left translation $E \to E$ by g is bundle map.

If $p : E \to X$ is a G-vector bundle we define *the fiber over* $x \in X$ to the set

$$p^{-1}(x) = \{v \in E \mid p(v) = x\},$$

when p is clear from the context we also denote this set by E_x. Also if $H \subseteq G$ is a subgroup, we can consider E as a H-vector bundle over X, we denote it by $\mathrm{res}^G_H(E)$.

Definition 2.2. Let X and Y be G-spaces. If $p : E \to Y$ is a G-vector bundle and $f : X \to Y$ is a G-map, then the pullback $p^*E \to X$ is a G-vector bundle over X defined as

$$p^*E = \{(e, x) \in E \times X \mid p(e) = f(x)\}.$$

When $i : X \to Y$ is an inclusion we usually denote $i^*(E)$ by $E \mid Y$.

Details about G-vector bundles can be found in [1].

Definition 2.3. Let G be a group, let X be a finite G-CW-complex (see [5]), the *equivariant K-theory group of* X,denoted by $K_G(X)$ is defined as the Grothendieck group of the monoid of isomorphism classes of G-equivariant vector bundles over X with the operation of direct sum. The functor $K_G(-)$ could be extended to an equivariant cohomology theory $K^*(-)$, defining for $n > 0$:

$$K_G^{-n}(X) = \ker\left(K_G(X \times S^n) \xrightarrow{i^*} K_G(X)\right).$$

And for any G-CW-pair (X, A), set

$$K_G^{-n}(X, A) = \ker\left(K_G^{-n}(X \cup_A X) \xrightarrow{i_2^*} K_G^{-n}(X)\right).$$

Finally for $n < 0$

$$K_G^{-n}(X) = K_G^n(X) \text{ and } K_G^{-n}(X, A) = K_G^n(X, A).$$

For more details about equivariant K-theory the reader can consult [13].

Example 2.4. If the action of G over X is free, then there is a canonical isomorphism of abelian groups

$$K_G(X) \cong K(X/G).$$

Example 2.5. If the action of G over X is trivial, then there is a canonical isomorphism of abelian groups

$$K_G(X) \cong R(G) \otimes_{\mathbb{Z}} K(X),$$

when $R(G)$ denotes the (complex) representation ring of G. In particular when $X = \{\bullet\}$ we obtain

$$K_G(\{\bullet\}) \cong R(G).$$

If Y is a finite G-CW-complex, we can define a G-action on $K(Y)$. Let $g \in G$, the pullback

$$g^* : K(Y) \to K(Y),$$

defines a G-action over $K(Y)$. We will need the following lemma.

Lemma 2.6. *Let Y be a finite G-CW-complex, then*

$$K(Y/G) \otimes \mathbb{C} \cong K(Y)^G \otimes \mathbb{C}$$

Proof. It is a consequence of the Chern character and the analogous fact for singular cohomology. ☑

In [2] a character for equivariant K-theory is constructed, that generalizes the character of representations. We will recall this construction briefly. Let E be a G-vector bundle over X and $g \in G$. Note that X^g is a $C_G(g)$-space, then if E is a G-vector bundle, $E|X^g$ is canonically a $C_G(g)$-vector bundle over X^g. Considering the action given by pullback we have that the isomorphism class $[(E|X^g)] \in K(X^g)$ is a $C_G(g)$-fixed point. Then $[(E|X^g)] \in K(X^g)^{C_G(g)}$. Finally for every element $\lambda \in S^1$, we can form the vector bundle of λ-eigenvectors considering the action of the element g over $\pi(E|X^g)$ denoted by $\pi(E|X^g)_\lambda$. Then we can define a map

$$\begin{aligned} \mathrm{char}_G : K_G(X) \otimes \mathbb{C} \to & \bigoplus_{[g]} K(X^g)^{C_G(g))} \otimes \mathbb{C} \\ [E] \mapsto & \left(\bigoplus_{\lambda \in S^1} [\pi(E|_{X^g})_\lambda] \otimes \lambda \right)_{[g]} . \end{aligned}$$

Using the above Lemma we identify $K(X^g)^{C_G(g)}$ with $K(X^g/C_G(g))$.

Theorem 2.7. *The map* char_G *is an isomorphism of complex vector spaces.*

For a proof of the theorem see [2].

2.1. The induced bundle

Now we will give an explicit construction of the induced vector bundle. It is a direct generalization of the induced representation defined for example in Section 3.3 in [15].

Let $H \subseteq G$ be a subgroup of G and $E \xrightarrow{\pi} X$ an H-vector bundle over a G-space X. If we choose an element from each left coset of H, we obtain a subset R of G called a *system of representatives of* $H \diagdown G$; each $g \in G$ can be written uniquely as $g = sr$, with $r \in R = \{r_1, \ldots, r_n\}$ and $s \in H$, $G = \coprod_{i=1}^{n} Hr_i$, we suppose that $r_1 = e$ the identity of the group G. Consider the vector bundle $F = \bigoplus_{i=1}^{n} (r_i)^* E$, with projection $\pi_F : F \to X$ and consider the following G-action defined over F:

Let $f \in F$, then

$$f = f_{r_1} \oplus \cdots \oplus f_{r_n},$$

where $f_{r_i} \in (r_i)^* E$. If $\pi_F(f) = x$ then $f_{r_i} = (x, e)$, where $e \in E_{r_i x}$.

Let $g \in G$, note that $r_i g^{-1}$ is in the same left coset of some r_j, i.e. $r_i g^{-1} = s r_j$, for some $s \in H$. Define

$$g(f_{r_i}) = (gx, s^{-1} e) \in (r_j)^* E,$$

and define the action of g on f by linearly.

Now we will see that F does not depend on the set of representatives up to isomorphism. Let $\{r'_1, \ldots, r'_n\}$ be another set of representatives of $H \diagdown G$ and let $F' = \bigoplus_{i=1}^{n} (r'_i)^* E$. By reordering we can assume that r_i and r'_i are in the same left coclass, then $r'_i r_i^{-1} \in H$.

We have an isomorphism of vector bundles over X

$$\begin{aligned} r'_i r_i^{-1} : (r_i)^* E &\to (r'_i)^* E \\ (x, e) &\to (x, r'_i r_i^{-1} e) \end{aligned}$$

inducing an isomorphism of G-vector bundles

$$r'_1 r_1^{-1} \oplus \ldots \oplus r'_n r_n^{-1} : F \to F'.$$

We only need to verify that this map commutes with the action of G. To see this, let $g \in G$ and $f_{r_i} = (x, e) \in (r_i)^* E$, there exist $s, s' \in H$ such that

$$r_i g^{-1} = s r_j \text{ and } r'_i g^{-1} = s' r'_j. \tag{1}$$

Note that $gf_{r_i} \in (r_j)^*E$, then

$$(r_j'r_j^{-1})g(f_{r_i}) = (gx, r_j'r_j^{-1}s^{-1}e).$$

On the other hand

$$g(r_i'r_i^{-1}f_{r_i}) = (gx, (s')^{-1}(r_i'r_i^{-1}e)),$$

but we know from (1)

$$(s')^{-1}r_i'r_i^{-1} = r_j'r_j^{-1}s^{-1}.$$

Then the map $r_1'r_1^{-1} \oplus \ldots \oplus r_n'r_n^{-1}$ commutes with the G-action and then F and F' are isomorphic as G-vector bundles.

We will denote the G-vector bundle F defined above by $\mathrm{Ind}_H^G(E)$. Summarizing we have.

Theorem 2.8. *Let G be a finite group, let $H \subseteq G$ be a subgroup. Let X be a G-CW-complex, and let E be a H-vector bundle over X, there is a unique G-vector bundle $\mathrm{Ind}_H^G(E)$ over X, up to isomorphism of G-vector bundles such that for every G-vector bundle F over X we have a natural identification*

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G(E), F) \cong \mathrm{Hom}_H(E, \mathrm{res}_H^G(F)).$$

Proof. Only remains to prove the identification. Let $\xi \in \mathrm{Hom}_G(\mathrm{Ind}_H^G(E), F)$, recall that we have an inclusion of H-vector bundles

$$E \to \mathrm{Ind}_H^G(E)$$
$$v \in E_x \mapsto (x, v).$$

Define $r(\xi) \in \mathrm{Hom}_H(E, \mathrm{res}_H^G(F))$ as follows, if $v \in E_x$

$$r(\xi)(v) = \xi(x, v).$$

It is clear that $r(\xi) \in \mathrm{Hom}_H(E, \mathrm{res}_H^G(F))$. On the other hand if $\eta \in \mathrm{Hom}_H(E, \mathrm{res}_H^G(F))$, define $I(\eta) : \mathrm{Ind}_H^G(E) \to F$ as follows, if $v_i \in r_i^*E$, then $v_i = (x, v)$ with $x \in X$ and $v \in E_{r_ix}$, then we define

$$I(\eta)(v_i) = r_i^{-1}(\eta(v)).$$

Extending linearly $I(\eta)$ to $\mathrm{Ind}_H^G(E)$.

Now we will see that $I(\eta)$ is G-equivariant. Let $g \in G$, let $s \in H$ such that

$$r_ig^{-1} = sr_j,$$

then

$$g \cdot v_i = (gx, s^{-1}v).$$

Now,

$$\begin{aligned}
I(\eta)(g \cdot v_i) &= I(\eta)(gx, s^{-1}v) \\
&= r_j^{-1}(\eta(s^{-1}v)) \\
&= r_j^{-1}s^{-1}\eta(v) \\
&= gr_i^{-1}\eta(v) \\
&= gI(\eta)(v_i).
\end{aligned}$$

Then $I(\eta) \in \mathrm{Hom}_G(\mathrm{Ind}_H^G(E), F)$. Now we will see that r and I are inverse of each other. It is clear that $r(I(\eta)) = \eta$. On the other hand,

$$\begin{aligned}
I(r(\xi))(v_i) &= r_i^{-1}(r(\xi)(v)) \\
&= r_i^{-1}\xi(r_i x, v) \\
&= \xi(x, v) \\
&= \xi(v_i).
\end{aligned}$$

☑

We have a formula for the character of an induced H-vector bundle, it is a particular case of a formula for induced character of generalized cohomology theories in [7] and [8]. We include a proof for completeness.

Theorem 2.9 (Formula for the induced character)**.** *Let X be a G-CW-complex, let H be a subgroup of G, let h be the order of H and E be a H-vector bundle, consider the map*

$$\mathrm{char}_G \circ \mathrm{Ind}_H^G : K_H(X) \otimes \mathbb{C} \to \bigoplus_{[g]} K(X^g)^{C_G(g)} \otimes \mathbb{C},$$

let R be a system of representatives of $H\diagdown G$. For each $g \in G$, we have

$$\begin{aligned}
\mathrm{char}_G(g) \circ \mathrm{Ind}_H^G([E]) &= \bigoplus_{r \in R, r^{-1}gr \in H} r^* \left(\mathrm{char}^H(r^{-1}gr)([E])\right) \\
&= \frac{1}{h} \bigoplus_{r \in G, r^{-1}gr \in H} r^* \left(\mathrm{char}^H(r^{-1}gr)([E])\right).
\end{aligned}$$

Proof. Our explicit definition of the induced bundle allows us to proof this result just by adapting the proof for representations contained in [15]. The vector bundle $F = \mathrm{Ind}_H^G(E)$ is the direct sum $\bigoplus_{i=1}^n r_i^*E$, with $R = \{r_1, \ldots, r_n\}$.

$$H\diagdown G = \{Hr_1, \ldots, Hr_n\}.$$

We know from the definition of the induced bundle that if we write $r_i g^{-1}$ in the form sr_j with $r_j \in R$ and $s \in H$, then g sends r_i^*E to r_j^*E. Considering the action of g in $H\diagdown G$, we have that

$$\mathrm{char}_G(g)\left(\mathrm{Ind}_H^G(E)\right) = \mathrm{char}_G(g)\left(\bigoplus_{Hr_i=Hr_ig^{-1}} r_i^*E \oplus \bigoplus_{Hr_i\neq Hr_ig^{-1}} r_i^*E\right)$$

Note that g acts in each term of the direct sum on the right hand side, hence the right hand side of the above equation can be written as

$$\mathrm{char}_G(g)\left(\bigoplus_{Hr_i=Hr_ig^{-1}} r_i^*E\right) \oplus \mathrm{char}_G(g)\left(\bigoplus_{Hr_i\neq Hr_ig^{-1}} r_i^*E\right)$$

We will see that $\mathrm{char}_G(g)\left(\bigoplus_{Hr_i\neq Hr_ig^{-1}} r_i^*E\right) = 0$. Because each 0-dimensional bundle is trivial, it suffices to check that this condition holds on fibers, i.e.,

$$\mathrm{char}_G(g)\left(\bigoplus_{Hr_i\neq Hr_ig^{-1}} r_i^*E\right)_x = 0,$$

for all $x \in X^g$. If we fix a basis for $(r_i^*E)_x$, the trace of the matrix representing the action of g is zero because $Hr_i \neq Hr_ig^{-1}$.

Now, if $Hr_i = Hr_ig^{-1}$ we have that $r_igr_i^{-1} = s_i$ with $s_i \in H$. Thus as the character is invariant under conjugation

$$\mathrm{char}_G(g)(r_i^*E) = \mathrm{char}_H(s_i)(r_i^*E).$$

Finally as the character commutes with pullbacks we have that,

$$\begin{aligned}\mathrm{char}_G(g)(\mathrm{Ind}_H^G(E)) &= \bigoplus_{r\in R, rgr^{-1}\in H} r_i^*\left(\mathrm{char}_H(rgr^{-1})(E)\right)\\ &= \frac{1}{h}\bigoplus_{s\in G, sgs^{-1}\in H} r_i^*\left(\mathrm{char}_H(s^{-1}hs)(E)\right).\end{aligned}$$

☑

3. Wreath product and its action on X^n

Let C be a set. There is a natural action of $\mathfrak{S}_n$ on C^n defined as

$$\sigma \bullet (c_1, \ldots, c_n) = (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)})$$

if G is a group, we define the wreath product as the semidirect product

$$G_n = G \wr \mathfrak{S}_n = G^n \rtimes \mathfrak{S}_n.$$

This section is dedicated to describe the conjugacy classes and centralizers of elements in G_n, we follow [10, Chapter 1, App. B] and [18].

First, we must recall the conjugacy classes in $\mathfrak{S}_n$. Two elements $s_1, s_2 \in \mathfrak{S}_n$ are conjugate if their cycle factorization correspond to the same partition of n. For example the elements $(1,2)(3,4,5)$ and $(1,4)(2,3,5)$ are conjugated and correspond to the partition $5 = 2+3$. Note that every partition of n can be view as a function $m : \{1, 2, \cdots, n\} \to \mathbb{N}$ as follows. If $s \in \mathfrak{S}_n$ then $m_s(r)$ is the number of r-cyces in s. Now in the general case, if $x = (\bar{g}, s) \in G_n$, then s can be decomposed as a product of disjoint cycles, if $z = (i_1 i_2 \dots i_r)$ is one of these cycles, the element $g_{i_r} g_{i_{r-1}} \cdots g_{i_1}$ is called the *cycle product* of x corresponding to z.

Recall that G_* denotes the set of conjugacy classes of G. If $x = (\bar{g}, s) \in G_n$, let $\rho(x) = m_x(r, c)$ denote the number of r-cycles in s whose cycle product belongs to c, where $c \in G_*$ and $r \in \{1, 2, \cdots n\}$. In this way every element $x \in G_n$ determines a matrix $\rho(x) = m_x(r, c)$ of non-negative integers such that $\sum_{r,c} r m_x(r, c) = n$.

For example let G be the cyclic group, $\{g^0, g^1, g^2, g^3\}$, of 4 elements generated by g and $s = (1,2)(3,4,5) \in \mathfrak{S}_5$. If $x = (g, g, g, g, g, s)$ then $\rho(x) = m_x(r, c)$ looks like

$$
\begin{array}{c|cccc|}
 & g^0 & g^1 & g^2 & g^3 \\
r=1 & 0 & 0 & 0 & 0 \\
r=2 & 0 & 0 & 1 & 0 \\
r=3 & 0 & 0 & 0 & 1 \\
r=4 & 0 & 0 & 0 & 0 \\
r=5 & 0 & 0 & 0 & 0
\end{array}
$$

the map $\rho : G_n \to M_{n,v}(\mathbb{Z})$ is called the *type* of $x \in G_n$, where $v = |G|$.

Proposition 3.1. *Two elements in G_n are conjugate iff they have the same type.*

Proof. See [10, Appendix 1.B] ☑

By the above proposition we can assume that every element $x \in G_n$ is conjugated to a product of elements of the form

$$((g, 1, \dots, 1), (i_{u_1}, \dots, i_{u_r})).$$

Denote by $g_r(c) = ((g, 1, \dots, 1), (1, \dots, r))$.

Proposition 3.2. *The elements in the centralizer* $C_{G_n}(g_n(c))$ *are of the form* $((gz,\dots,\underbrace{z}_{k+1},\dots,gz),(1,\dots,n)^k)$, *with* $z\in C_G(g)$. *Moreover* $C_{G_n}(g_n(c))\cong C_G(g)\times\langle(1,\dots n)\rangle$.

Proof. It follows from a direct computation. ☑

We have described centralizers of elements in G_n and in the next section we will use this description to write char_{G_n} in terms of char_G.

Let X be a G-space, there is canonical G_n-action over X^n defined from the G-action over X

$$\begin{aligned} G_n\times X^n &\to X^n\\ ((\bar g,\sigma),\bar x) &\mapsto \bar g(\sigma\bullet\bar x)\end{aligned}$$

where $\bar g$ acts component-wise.

In order to relate $\mathrm{char}_{G_n}$with with char_G we need to describe the fixed point set of a representative of each conjugacy class of G_n. Let us start with the conjugacy classes of elements $(\bar g,\sigma)$ where σ is an n-cycle. To this end we will need the following result.

Proposition 3.3. *Let* $\zeta=((g,1,\dots,1),\sigma)$ *whith* $g\in G$ *and* σ *is a* n*-cycle. There is a canonical homeomorphism*

$$(X^n)^\zeta/C_{G_n}(\zeta)\cong X^g/C_G(g).$$

Proof. We can assume $\sigma=(1,\dots,n)$. Let $(x_1,\dots,x_n)\in(X^n)^\zeta$, then

$$\zeta(x_1,\dots,x_n)=(x_1,\dots,x_n)$$

it implies

$$(gx_n,x_1,\dots,x_{n-1})=(x_1,\dots,x_n).$$

Therefore

$$x_n=x_{n-1}=\dots=x_1,\quad gx_n=x_1,$$

and then $(x_1,\dots,x_n)=(y,\dots,y)$ lies in the diagonal and $y\in X^g$. This proves that $(X^n)^\zeta\cong X^g$. On the other hand, if $\bar b\in C_{G_n}(\zeta)$ then by Proposition 3.2

$$\bar b=(((gz,\dots,\underbrace{z}_{k+1},\dots,gz),\sigma^k)$$

where $z\in C_G(g)$. Then we obtain

$$\bar b(y,\dots,y)=(((gz,\dots,\underbrace{z}_{k+1},\dots,gz),\sigma^k)\cdot(y\dots,y)=(gzy,\dots,\underbrace{zy}_{k+1},\dots,gzy),$$

showing that the orbit of $(y,\dots,y)$ by $C_{G_n}(\zeta)$ is

$$\{(gzy,\dots,\underbrace{zy}_{k+1},\dots,gzy)\,:\,z\in C_G(g)\}.$$

This proves the result. ☑

4. Fock space

Let X be a G-space, by the equivariant Bott periodicity theorem we know that $K_G^*(X) = K_G^0(X) \oplus K_G^1(X)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded group. Denote by

$$\mathcal{F}_G(X) = \bigoplus_{n\geq 0} K^*_{G_n}(X^n) \otimes \mathbb{C}, \quad \mathcal{F}^q_G(X) = \bigoplus_{n\geq 0} q^n K^*_{G_n}(X^n) \otimes \mathbb{C}$$

where q is formal variable giving a $\mathbb{Z}_+$-grading in $\mathcal{F}^q_G(X)$. They both have a natural structure of abelian groups, we endow them with a product $\cdot$, defined as the composition of the induced bundle and the Künneth isomorphism $\boxtimes$ (see [12])

$$K^*_{G_n}(X^n) \times K^*_{G_m}(X^m) \xrightarrow{\boxtimes} K^*_{G_n\times G_m}(X^{n+m}) \xrightarrow{\text{Ind}} K^*_{G_{n+m}}(X^{n+m})$$

Proposition 4.1. *With the above operations $\mathcal{F}^q_G(X)$ is a commutative $(\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z})$-graded ring.*

Proof. The associativity follows from the following fact. Let $[E_1] \in K^*_{G_n}(X^n)$, $[E_2] \in K^*_{G_m}(X^m)$ and $[E_3] \in K^*_{G_k}(X^k)$, then

$$\begin{aligned}(E_1 \cdot E_2) \cdot E_3 &\cong \text{Ind}^{G_{n+m+k}}_{G_{n+m}\times G_k}\left(\text{Ind}^{G_{n+m}}_{G_n\times G_n}(E_1 \boxtimes E_2) \boxtimes E_3\right)\\ &\cong \text{Ind}^{G_{n+m+k}}_{G_n\times G_m\times G_k}(E_1 \boxtimes E_2 \boxtimes E_3)\\ &\cong \text{Ind}^{G_{n+m+k}}_{G_n\times G_{m+k}}\left(E_1 \boxtimes \text{Ind}^{G_{m+k}}_{G_m\times G_k}(E_2 \boxtimes E_3)\right).\end{aligned}$$

For the graded commutativity, let $[E_1] \in K^*_{G_n}(X^n)$ and $[E_2] \in K^*_{G_m}(X^m)$, we will prove that $E_1 \cdot E_2$ and $E_2 \cdot E_1$ has the same character as G_{n+m}-vector bundles over X^{n+m}.

Consider two inclusions of $\mathfrak{S}_n$ into $\mathfrak{S}_{n+m}$. The first one is the inclusion by the first n letters denoted by $S_n \xrightarrow{i_1^n} S_{n+m}$; the second one is the inclusion by the last n letters denoted by $S_n \xrightarrow{i_2^m} S_{n+m}$. Let $x = (\bar{g}, \sigma) \in G_{n+m}$ and let $r = (\bar{h}, \tau) \in G_{n+m}$, such that $r^{-1}xr \in G_n \times G_m$, then, there is $\eta_1 \in \mathfrak{S}_n$ and $\eta_2 \in \mathfrak{S}_m$ such that $\tau^{-1}\sigma\tau = i_1(\eta_1)i_2(\eta_2)$, but $i_1(\eta_1)i_2(\eta_2)$ is conjugated in $\mathfrak{S}_{n+m}$ to $i_1(\eta_2)i_2(\eta_1)$, then, there is $\gamma \in \mathfrak{S}_{n+m}$ such that $\gamma^{-1}(\tau^{-1}\sigma\tau)\gamma = i_1(\eta_2)i_2(\eta_1)$. Then

$$((e, \dots, e), \gamma^{-1})(r^{-1}xr)((e, \dots, e), \gamma) \in G_m \times G_n.$$

It implies that we have bijective correspondence between elements $r \in G_{n+m}$ such that $r^{-1}xr \in G_n \times G_m$ and elements $s \in G_{n+m}$ such that $s^{-1}xs \in G_m \times G_n$. Then, if we apply Theorem 2.9, the number of terms in each direct sum computing

$$\text{Ind}^{G_{n+m}}_{G_n\times G_m}(E_1 \boxtimes E_2) \text{ and } \text{Ind}^{G_{n+m}}_{G_m\times G_n}(E_2 \boxtimes E_2)$$

are the same. Moreover, for every $(\alpha,\beta)\in G_n\times G_m$,

$$\mathrm{char}_{G_n\times G_m}(\alpha,\beta)(E_1\boxtimes E_2)\cong \mathrm{char}_{G_m\times G_n}(\beta,\alpha)(E_2\boxtimes E_1),$$

then, we have $E_1\cdot E_2$ and $E_2\cdot E_1$ have the same character, then the product is commutative. The other properties follows directly. ☑

Definition 4.2. Let R be a commutative ring, the graded-symmetric algebra of a $\mathbb{Z}$-graded R-module M (denoted by $\mathcal{S}(M)$) is the quotient of the tensor algebra of M by the ideal I generated by elements of the form

(1) $x\otimes y-(-1)^{\deg(x)\deg(y)}(y\otimes x)$,

(2) $x\otimes x$, when $\deg(x)$ is even.

Now we will give another proof of the description of $\mathcal{F}^q_G(X)$ as a graded-symmetric algebra given in [14] or Theorem 3 in [18]. Our proof gives an explicit isomorphism and moreover gives explicit generators of $\mathcal{F}^q_G(X)$ as $\mathbb{Z}_+\times\mathbb{Z}/2\mathbb{Z}$)-graded algebras.

Theorem 4.3. *There is an isomorphism of* $(\mathbb{Z}_+\times\mathbb{Z}/2\mathbb{Z})$*-graded algebras*

$$\Phi:\mathcal{S}(\oplus_{n\geq 1}q^nK^*_G(X)\otimes\mathbb{C})\to\mathcal{F}^q_G(X).$$

Proof. First note that using char_{G_n} we can define an injective group homomorphism in the following way. Consider the following sequence of maps:

$$\begin{aligned}K^*_G(X)\otimes\mathbb{C}&\xrightarrow{\cong}\bigoplus_{g\in G_*}K^*(X^g)^{C_G(g)}\otimes\mathbb{C}\\&\xrightarrow{\lambda}\bigoplus_{x\in G_{n*}}K^*((X^n)^x)^{C_{G_n}(x)}\otimes\mathbb{C}\xrightarrow{\cong}K^*_{G_n}(X^n)\otimes\mathbb{C}\end{aligned}$$

where the map λ is given by the assigning $[((g,1,\dots,1),(1,\dots,n))]_{G_n}$ to the conjugacy class $[g]_G$ and using the identification in Proposition 3.3. This map is certainly injective. Define

$$\phi:K^*_G(X)\otimes\mathbb{C}\to K^*_{G_n}(X^n)\otimes\mathbb{C}$$

by the composition of the above sequence so that ϕ is injective and by the universal property of the graded-symmetric algebra we have a unique map

$$\Phi:\mathcal{S}\left(\bigoplus_{n\geq 1}K^*_G(X)\otimes\mathbb{C}\right)\to\mathcal{F}^q_G(X)$$

extending ϕ.

Suppose inductively that $\mathrm{im}(\Phi)$ contains $K^*_{G_k}(X^k)\otimes\mathbb{C}$ for $k<n$. Then by induction we know that the image of the following composition

$$\mathcal{S}\left(\oplus_{n\geq 1}q^n K^*_G(X)\otimes\mathbb{C}\right)\times\mathcal{S}\left(\oplus_{n\geq 1}q^n K^*_G(X)\otimes\mathbb{C}\right)$$
$$\xrightarrow{\Phi\times\Phi}\mathcal{F}^q_G(X)\times\mathcal{F}^q_G(X)\xrightarrow{\cdot}\mathcal{F}^q_G(X)$$

contains

$$K^*_{G_k}(X^k)\otimes\mathbb{C}\cdot K^*_{G_{n-k}}(X^{n-k})\otimes\mathbb{C}\subseteq K^*_{G_n}(X)\otimes\mathbb{C}.$$

Now we have that the image under char_{G_n} of

$$\bigoplus_{k=1}^{n-1}(K^*_{G_k}(X^k)\otimes\mathbb{C})\cdot(K^*_{G_{n-k}}(X^{n-k})\otimes\mathbb{C})$$

coincides with

$$\bigoplus_{x\in J}K^*(((X^n)^x)/C_{G_n}(x))\otimes\mathbb{C},$$

where J is the set of conjugacy classes in G_n such that for every c, $m_\bullet(n,c)=0$, in other words J is the set of conjugacy classes whose components in $\mathfrak{S}_n$ are not an n-cycle.

On the other hand if $x=((g,1,\ldots,1),(1,\ldots,n))$ for some $g\in G$, then Proposition 3.3 gives us that $\mathrm{im}(\mathrm{char}_{G_n}\circ\Phi)$ contains $K^*((X^n)^x)^{C_{G_n}(x)}$. Finally, since char_{G_n} is an isomorphism we can conclude that Φ is surjective.

To see that Φ is injective we can use the formula for the induced character, because this formula implies that if $A\in\mathcal{S}(\bigoplus_{k\geq 1}q^kK^*_G(X)\otimes\mathbb{C})$ is not zero then there exists n and $x\in G_n$ such that $\mathrm{char}_{G_n}(x)(\Phi(A))\neq 0$, then $\Phi(A)\neq 0$. ☑

5. Pullback of groups

Let Γ be a group fitting into the following pullback diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "p_2"] \arrow[d, "p_1"'] & G \arrow[d, "\pi_2"'] \\
H \arrow[r, "\pi_1"] & K
\end{tikzcd} \tag{2}$$

If the group Γ comes from a diagram 2 then it is isomorphic to a subgroup of $G\times H$, namely $\Gamma\cong\{(g,h)\in G\times H\mid\pi_1(g)=\pi_2(h)\}$. When maps π_1 and π_2 are clear from the context we denote Γ by $G\times_K H$. We suppose that π_1 and π_2 are surjective.

In this section we describe the class function ring of Γ in terms of the class function rings of G, H and K. In order to obtain this description we need that Γ satisfies the following condition.

Definition 5.1. Let G be a finite group and let $H \subseteq G$ be a subgroup, let $[h]_H \in H_*$, we say that $[h]_H$ is closed in G if,

$$[h]_H = [h]_G \cap H.$$

We say that H is conjugacy-closed in G if, for every $h \in H$, $[h]_H$ is closed in G.

Example 5.2. The following are examples of conjugacy-closed subgroups:

- The general linear groups over subfields are conjugacy-closed.
- The symmetric group is conjugacy-closed in the general linear group.
- The symmetric group on subsets are conjugacy-closed.
- The orthogonal group is conjugacy-closed in the general linear group over real numbers.
- The unitary group is conjugacy-closed in the general linear group.

***Remark* 5.3.** Let G and H be groups

- If $H \subseteq G$ is conjugacy-closed in G, then the pullback of the inclusion

  $$i^* : \mathrm{Class}(G) \to \mathrm{Class}(H)$$

  is surjective.
- If H is a retract in G, the pullback of the inclusion

  $$i^* : R(G) \to R(H)$$

  is surjective.

When Γ is conjugacy-closed in $G \times H$, we have a way to express the class function ring of Γ in terms of the class function rings of G, H and K. The same is true for the representations ring when Γ is a retract of $G \times H$.

5.1. The class function ring of a pullback

Consider a pullback diagram of finite groups such as (2). If we apply the representation ring functor we obtain the following diagram

$$\begin{tikzcd} R(\Gamma) & R(G) \arrow[l, "p_2^*"'] \\ R(H) \arrow[u, "p_1^*"] & R(K) \arrow[l, "\pi_1^*"] \arrow[u, "\pi_2^*"'] \end{tikzcd} \tag{3}$$

This diagram endows the rings $R(G)$ and $R(H)$ with a $R(K)$-module structure. A similar statement is true changing the representation ring by the class function ring. We will prove that if Γ is a retract of $G \times H$, then the diagram (3) is a pushout. In fact, we have the following theorem.

Theorem 5.4. *Let Γ, G, H and K be finite groups such as in the diagram (2). If Γ is conjugacy-closed in $G \times H$, there is an isomorphism*

$$m : \mathrm{Class}(G) \otimes_{\mathrm{Class}(K)} \mathrm{Class}(H) \to \mathrm{Class}(\Gamma)$$

of $\mathrm{Class}(K)$*-modules*

Moreover, if Γ is a retract of $G \times H$, we have an isomorphism

$$f : R(G) \otimes_{R(K)} R(H) \to R(\Gamma)$$

of $R(K)$-modules.

Proof. In order to avoid confusion, in this proof we denote the product on $\mathrm{Class}(\Gamma)$, $\mathrm{Class}(G)$ and $\mathrm{Class}(H)$ by $\cdot$ and the generators of the tensor product by $\rho \otimes \gamma$.

The map f is defined as

$$\begin{aligned} f : \mathrm{Class}(G) \otimes_{\mathrm{Class}(K)} \mathrm{Class}(H) &\to \mathrm{Class}(\Gamma) \\ \rho \otimes \gamma &\mapsto p_1^*(\rho) \cdot p_2^*(\gamma) \end{aligned}$$

First we prove that the map f is well defined. Let $\xi \in \mathrm{Class}(K)$, $\rho \in \mathrm{Class}(G)$ and $\gamma \in \mathrm{Class}(H)$. Let $(g, h) \in \Gamma$

$$\begin{aligned} f\left(\pi_1^*(\xi) \cdot \rho \otimes \gamma\right)(g,h) &= \left(p_1^*\left(\pi_1^*(\xi) \cdot \rho\right) \cdot p_2^*\left(\gamma\right)\right)(g,h) \\ &= \left(\pi_1^*(\xi) \cdot \rho\right)(g)\gamma(h) \\ &= \xi(\pi_1(g))\rho(g)\gamma(h) \\ &= \rho(g)\xi(\pi_2(h))\gamma(h) \\ &= f(\rho \otimes \pi_2^*(\xi) \cdot \gamma)(g,h). \end{aligned}$$

Now we will prove that f is an isomorphism. Consider the following diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \ker(\pi) \arrow[r] \arrow[d, "f_2"] & \mathrm{Class}(G) \otimes_{\mathbb{C}} \mathrm{Class}(H) \arrow[r, "\pi"] \arrow[d, "f_1"] & \mathrm{Class}(G) \otimes_{\mathrm{Class}(K)} \mathrm{Class}(H) \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & \ker(i^*) \arrow[r] & \mathrm{Class}(G \times H) \arrow[r, "i^*"] & \mathrm{Class}(\Gamma) \arrow[r] & 0. \end{tikzcd}$$

Where map π is the quotient by the relations defining tensor product over $\mathrm{Class}(K)$, map i^* is the pullback of the inclusion $i : \Gamma \to G \times H$, map f_1 is the natural isomorphism given by tensor product over $\mathbb{C}$ and the map f_2 is the restriction of f_1 to $\ker(\pi)$. Note that as Γ is closed conjugacy in $G\times H$ the map i^* is surjective. We will prove that above diagram is commutative and that f_2 is an isomorphism.

First we need to verify that $f_1(\ker(\pi)) \subseteq \ker(i^*)$. Let $(g,h)\in\Gamma$,

$$\begin{aligned} &i^*[f_1(\pi_1^*(\xi)\cdot\rho\otimes\gamma - \rho\otimes\pi_2^*(\xi)\cdot\gamma)](g,h) = \\ &\qquad [p_1^*(\pi_1^*(\xi))\cdot p_1^*(\rho)\cdot p_2^*(\gamma) - p_1^*(\rho)\cdot p_2^*(\pi_2^*(\xi))\cdot p_2^*(\gamma)](g,h) = 0 \end{aligned}$$

Now we prove that $\ker(i^*) = f_1(\ker(\pi))$. For this we will prove that if f is a class function in $G\times H$ such that $i^*(f)\equiv 0$ and f is orthogonal to every element in $f_1(\ker(\pi))$, then f has to be zero.

Suppose that for every $\xi\in \mathrm{Class}(K)$, $\rho\in\mathrm{Class}(G)$ and $\gamma\in\mathrm{Class}(H)$ we have

$$\sum_{(g,h)\in G\times H}\overline{f(g,h)}\rho(g)\gamma(h)[\xi(\pi_2(h)) - \xi(\pi_1(g))] = 0.$$

Let us fix $\rho\in\mathrm{Class}(G)$ and let

$$\eta(g) = \sum_{h\in H}\overline{f(g,h)}\gamma(h)[\xi(\pi_2(h)) - \xi(\pi_1(g))].$$

We observe that η is a class function on G that is orthogonal to every ρ in $\mathrm{Class}(G)$, then $\eta\equiv 0$.

By a similar argument we conclude that for every $(g,h)\in G\times H$ and $\xi\in\mathrm{Class}(K)$

$$\overline{f(g,h)}[\xi(\pi_2(h)) - \xi(\pi_1(g))] = 0. \tag{4}$$

We already know that $f(g,h)=0$ if $(g,h)\in\Gamma$, then let $(g,h)\notin\Gamma$, we have two cases. First suppose that $\pi_1(g)$ is conjugate to $\pi_2(h)$ in K, in this case there is $\bar h\in H$ such that $(g,\bar h h\bar h^{-1})\in\Gamma$ and then $f(g,h) = f(g,\bar h h \bar h^{-1}) = 0$.

Suppose now that $\pi_1(g)$ is not conjugate to $\pi_2(h)$ in K, in this case there is $\xi\in\mathrm{Class}(K)$ such that $\xi(\pi_1(g))\neq\xi(\pi_2(g))$ and equation (4) gives us that $f(g,h)=0$. Then we conclude that $\ker(i^*) = f_1(\ker(\pi))$. The map f_2 is an isomorphism because it is the restriction of f_1 and as the diagram is commutative we conclude that f is $\mathrm{Class}(K)$-module isomorphism.

When Γ is a retract in $G\times H$, the same argument works changing characters by representations, in particular the map $i^*: R(G\times H)\to R(\Gamma)$ is surjective. ☑

Observe that the pullback is not always conjugacy-closed in the product as the following example shows.

Example 5.5. Consider the pullback of the symmetric groups $\mathfrak{S}_3$ over the cyclic group C_2

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & \mathfrak{S}_3 \arrow[d, "sgn"] \\ \mathfrak{S}_3 \arrow[r, "sgn"] & C_2 \end{tikzcd}$$

In this case the pullback Γ has 6 conjugacy classes, $\{\gamma_1, \cdots, \gamma_6\}$. The product $\mathfrak{S}_3 \times \mathfrak{S}_3$ has 9 conjugacy classes, $\{\chi_1 \cdots, \chi_9\}$. Observe that the elements $((1,2,3),(1,2,3))$ and $((1,2,3),(1,3,2))$ are conjugate in the group $\mathfrak{S}_3 \times \mathfrak{S}_3$ by the element $(e,(1,2))$ but they are not conjugate in Γ.

The pullback of the inclusion can be described in the class function ring as follows:

$$\begin{aligned} \mathrm{Class}(\mathfrak{S}_3 \times \mathfrak{S}_3) &\to \mathrm{Class}(\Gamma) \\ \chi_1 &\mapsto \gamma_1 \\ \chi_2 &\mapsto \gamma_1 \\ \chi_3 &\mapsto \gamma_2 \\ \chi_4 &\mapsto \gamma_2 \\ \chi_5 &\mapsto \gamma_3 \\ \chi_6 &\mapsto \gamma_3 \\ \chi_7 &\mapsto \gamma_4 \\ \chi_8 &\mapsto \gamma_4 \\ \chi_9 &\mapsto \gamma_5 + \gamma_6. \end{aligned}$$

This map is not surjective.

Example 5.6. Consider the following pullback

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & D_{12} \arrow[d, "\psi_1"'] \\ C_3 \ltimes C_4 \arrow[r, "\psi_2"] & \mathfrak{S}_3 \end{tikzcd}$$

In this case the pullback Γ is isomorphic to the group $C_2 \times (C_3 \ltimes C_4)$ and it is conjugacy closed in the group $D_{12} \times (C_3 \ltimes C_4)$. According with GAP[6] the group D_{12} has group id (12,4) and generators d_1, d_2 and d_3 of orders $2, 2$ and 3 respectively. The homomorphism ψ_1 is given by

$$\begin{aligned} \psi_1 : D_{12} &\to \mathfrak{S}_3 \\ d_1 &\mapsto (2,3) \\ d_2 &\mapsto (1) \\ d_3 &\mapsto (1,2,3). \end{aligned}$$

The group $C_3 \ltimes C_4$ has id (12,1) and it is generated by three elements g_1, g_2, g_3 of orders $4, 2$ and 3 respectively. The homomorphism ψ_2 is given by

$$\begin{aligned} \psi_2 : C_3 \ltimes C_4 &\to \mathfrak{S}_3 \\ g_1 &\mapsto (2,3) \\ g_2 &\mapsto (1) \\ g_3 &\mapsto (1,2,3). \end{aligned}$$

For these groups the pullback Γ is isomorphic to the group $C_2 \times (C_3 \ltimes C_4)$ with group id (24,7) and four generators f_1, f_2, f_3, f_4 of orders $4, 6, 2$ and 2 respectively.

Applying Theorem 5.4 we obtain an isomorphism

$$\mathrm{Class}(\Gamma) \cong \mathrm{Class}(D_{12}) \otimes_{\mathrm{Class}(\mathfrak{S}_3)} \mathrm{Class}(C_3 \ltimes C_4).$$

For more examples please see [4].

6. Semidirect product of a direct product

Let A_1, A_2 be groups with an action of a group G by automorphisms noted by $a_i^g = g \cdot a_i$, for $a_i \in A_i$ and $g \in G$. Note that G acts also on the direct product $A_1 \times A_2$ by acting on each component, i.e. $(a_1, a_2)^g := (a_1^g, a_2^g)$. In this section we describe the semidirect product of a direct product as a pullback of two semidirect products and then, we apply this for the wreath product of a direct product which will allow us to compute the Fock ring of a product.

Consider the projections $\pi_i : A_i \rtimes G \to G$ and the pullback Γ associated

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & A_1 \rtimes G \arrow[d] \\ A_2 \rtimes G \arrow[r] & G \end{tikzcd}$$

Proposition 6.1. *The pullback Γ is isomorphic to the semidirect product $(A_1 \times A_2) \rtimes G$.*

Proof. Note that the pullback is the subgroup of $(A_1 \rtimes G) \times (A_2 \rtimes G)$ given by

$$\Gamma = \{(a_1, g_1, a_2, g_2) \in (A_1 \rtimes G) \times (A_2 \rtimes G) : \pi_1(a_1, g_1) = \pi_2(a_2, g_2), a_1 \in A_1, g_i \in G\}$$

that is, $g_1 = g_2$. Consider the bijective function $\phi : \Gamma \to (A_1 \times A_2) \rtimes G$ given by $\phi(a_1, g, a_2, g) = (a_1, a_2, g)$. On one hand

$$\phi[(a_1, g, a_2, g) \cdot (b_1, h, b_2, h)] = \phi(a_1 b_1^g, gh, a_2 b_2^g, gh) = (a_1 b_1^g, a_2 b_2^g, gh).$$

On the other hand $(a_1, a_2, g) \cdot (b_1, b_2, h) = (a_1 b_1^g, a_2 b_2^g, gh)$ which shows that ϕ is a homomorphism of groups. ☑

Corollary 6.2. *Let A, B be groups, there is an isomorphism*

$$(A \times B)_n \cong A_n \times_{\mathfrak{S}_n} B_n.$$

Now we proof that certain conjugacy classes in $(A \times B)_n$ are closed in $A_n \times B_n$.

Proposition 6.3. *Let $(\bar{g}, \bar{h}, \sigma) \in (A \times B)_n$, where σ is an n-cycle. Then its conjugacy class in $A_n \times B_n$ is closed.*

***Proof*.** Let $x = (\bar{g}_1, \bar{h}_1, \sigma_1)$ and $y = (\bar{g}_2, \bar{h}_2, \sigma_2)$ be elements in $(A \times B)_n$ that are conjugated in $A_n \times B_n$, where σ_1 and σ_2 are n-cycles. We can suppose that $\sigma_1 = \sigma_2 = (1 \cdots n)$.

Note that

$$(\bar{g}_1, \sigma_1) \sim_{A_n} (\bar{g}_2, \sigma_2) \text{ and } (\bar{h}_1, \sigma_1) \sim_{B_n} (\bar{h}_2, \sigma_2).$$

Since as σ_1 and σ_2 are n-cycles, $\prod_{i=1}^n g_{1,i} \sim_A \prod_{i=1}^n g_{2,i}$, and $\prod_{i=1}^n h_{1,i} \sim_B \prod_{i=1}^n h_{2,i}$. On the other hand the type of x is given by

$$m_x(r, c) = \begin{cases} 1 & \text{if } r = n \text{ and } (\prod_{i=1}^n g_{1,i}, \prod_{i=1}^n h_{1,i}) \in c \\ 0 & \text{in any other case} \end{cases}$$

and the type of y is given by

$$m_y(r, c) = \begin{cases} 1 & \text{if } r = n \text{ and } (\prod_{i=1}^n g_{2,i}, \prod_{i=1}^n h_{2,i}) \in c \\ 0 & \text{in any other case} \end{cases}$$

Then the types of x and y are equal, hence x and y are conjugated in $(A \times B)_n$. ☑

7. The Fock space of a product of spaces

In this section we apply results of Section 5 in order to obtain a decomposition of $\mathcal{F}_{G\times H}(X \times Y)$ in terms of $\mathcal{F}_G(X)$ and $\mathcal{F}_H(Y)$. Let X be a G-space, we can endow to $\mathcal{F}_G(X)$ with natural module structures as follows:

- Consider the trivial G_n-space $\{\bullet\}$, and the unique G_n-map $\pi : X^n \to \{\bullet\}$, then the pullback

  $$\pi^* : \text{Class}(G_n) \to K^*_{G_n}(X^n) \otimes \mathbb{C}$$

  induces a $\text{Class}(G_n)$-module structure over $K^*_{G_n}(X) \otimes \mathbb{C}$, hence we have a $\mathcal{F}_G(\{\bullet\})$-module structure over $\mathcal{F}_G(X)$ defined componentwise.

- Note that we have a quotient map $s : G_n \to \mathfrak{S}_n$, then the pullback
$$(\pi \circ s)^* : \mathrm{Class}(\mathfrak{S}_n) \to K_{G_n}(X^n) \otimes \mathbb{C}$$
induce a $\mathrm{Class}(\mathfrak{S}_n)$-module structure over $K_{G_n}(X) \otimes \mathbb{C}$, hence we have a $\mathcal{F}(\{\bullet\})$-module structure over $\mathcal{F}_G(X)$ defined componentwise.

As we observe in Section 5.1, $(G \times H)_n$ is not closed conjugacy in $G_n \times H_n$, then we cannot expect a decomposition of $\mathrm{Class}((G \times H)_n)$ in terms of $\mathrm{Class}(G_n)$, $\mathrm{Class}(H_n)$ and $\mathrm{Class}(\mathfrak{S}_n)$, but as the conjugacy classes with an n-cycle as component in $\mathfrak{S}_n$ are closed we have a decomposition of $\mathcal{F}_{G\times H}(X \times Y)$ in terms of $\mathcal{F}_G(X)$, $\mathcal{F}_H(Y)$ and $\mathcal{F}(\{\bullet\})$, with X a G-space and Y a H-space.

Theorem 7.1. *There is an isomorphism of $\mathcal{F}(\{\bullet\})$-modules*
$$\mathcal{F}_{G\times H}(X \times Y) \xrightarrow{\mathcal{K}} \mathcal{F}_G(X) \otimes_{\mathcal{F}(\{\bullet\})} \mathcal{F}_H(Y).$$
The map $\mathcal{K}$ is compatible with the symmetric algebra decomposition in Thm. 4.3. That means, we have a commutative diagram
$$\begin{tikzcd}
\mathcal{F}_{G\times H}(X \times Y) \arrow[r, "K"] \arrow[d, "d"] & \mathcal{F}_G(X) \otimes_{\mathcal{F}(\{\bullet\})} \mathcal{F}_H(Y) \arrow[d, "d\otimes d"] \\
\mathcal{S}(X \times Y) \arrow[r, "\pi_G\otimes\pi_H"] & \mathcal{S}(X) \otimes_{\mathcal{F}(\{\bullet\})} \mathcal{S}(Y)
\end{tikzcd}.$$
Where $\mathcal{S}(X)$ stands for $\mathcal{S}(\bigoplus_{n\geq 0} K_G(X) \otimes \mathbb{C})$, and similarly for $\mathcal{S}(Y)$ and $\mathcal{S}(X \times Y)$.

***Proof*.** Let $\{E_1, \dots, E_m\}$ be a basis of $K^*_G(X) \otimes \mathbb{C}$ as complex vector space and let $\{F_1, \dots, F_s\}$ be a basis of $K^*_H(Y) \otimes \mathbb{C}$ as complex vector space. From the proof of Theorem 4.3 we can conclude that
$$\{\Delta_{G,n,c,k} \in K_{G_n}(X^n) \otimes \mathbb{C} \mid n \geq 0, c \in G_*, 1 \leq k \leq m\}$$
is a basis of $\mathcal{F}_G(X)$ as $\mathbb{C}$-algebra, where
$$\mathrm{char}_{G_n}(\Delta_{G,n,c,k})((g_1, \dots, g_n), \sigma) = \begin{cases} E_k & \text{if } \prod_{i=1}^n g_{\sigma^i(1)} \in c \text{ and } \sigma \text{ is an } n\text{-cycle} \\ 0 & \text{in any other case.} \end{cases}$$
In a similar way we define
$$\{\Delta_{H,n,d,l} \in K_{H_n}(Y^n) \otimes \mathbb{C} \mid n \geq 0, d \in H_*, 1 \leq l \leq s\}$$
a basis of $\mathcal{F}_H(Y)$ as $\mathbb{C}$-algebra, where
$$\mathrm{char}_{H_n}(\Delta_{H,n,d,l})((h_1, \dots, h_n), \sigma) = \begin{cases} F_l & \text{if } \prod_{i=1}^n h_{\sigma^i(1)} \in d \text{ and } \sigma \text{ is an } n\text{-cycle} \\ 0 & \text{in any other case.} \end{cases}$$

Recall that we have an isomorphism

$$K_{G\times H}(X\times Y)\otimes\mathbb{C}\xrightarrow{\boxtimes}(K_G(X)\otimes K_H(Y))\otimes\mathbb{C},$$

given by the external tensor product. It is proved in [12] or can be obtained (for complex coefficients) directly from the character.

Using the above identification we have that

$$\{\Delta_{G\times H,n,c\times d,(k,l)}\mid n\geq 0, c\in G_*, d\in H_*, 1\leq k\leq m, 1\leq l\leq s\}$$

is a basis as $\mathbb{C}$-algebra of $\mathcal{F}_{G\times H}(X\times Y)$, where

$$\begin{aligned}&\operatorname{char}_{(G\times H)_n}(\Delta_{G\times H,n,c\times d,(k,l)})(\bar{g},\bar{h},\sigma)=\\&\begin{cases}E_k\boxtimes F_l & \text{if } \prod_{i=1}^n g_{\sigma^i(1)}\in c, \prod_{i=1}^n h_{\sigma^i(1)}\in d \text{ and } \sigma \text{ is an } n\text{-cycle}\\0 & \text{in any other case.}\end{cases}\end{aligned}$$

As the conjugacy classes when the character of the above elements is not zero is closed in $G_n\times H_n$ we have

$$\Delta_{G\times H,n,c\times d,(k,l)}=\Delta_{G,n,c,k}\cdot\Delta_{H,n,d,l},$$

hence the map defined on generators as

$$\Delta_{G\times H,n,c\times d,(k,l)}\mapsto\Delta_{G,n,c,k}\otimes\Delta_{H,n,d,l}$$

is an isomorphism of $\mathcal{F}(\{\bullet\})$-modules satisfying the required conditions. ☑

8. Final remarks

In [16] and [17] a configuration space representing equivariant connective K-homology for finite groups was constructed. We recall the construction briefly.

Definition 8.1. Let G be a finite group and (X,x_0) be a based G-connected, G-CW-complex. Let $\mathfrak{C}(X,x_0,G)$ be the *G-space of configurations of complex vector spaces over* (X,x_0), defined as the increasing union, with respect to the inclusions $M_n(\mathbb{C}[G])\to M_{n+1}(\mathbb{C}[G])$

$$\mathfrak{C}(X,x_0,G)=\bigcup_{n\geq 0}\operatorname{Hom}^*(C_0(X),M_n(\mathbb{C}[G])),$$

with the compact open topology. Notice that * refers to *-homomorphism, $C_0(X)$ denotes the C*-algebra of complex valued continuous maps vanishing at x_0 and $\mathbb{C}[G]$ denotes the complex group ring.

We endow $\mathfrak{C}(X,x_0,G)$ with a continuous G-action as follows. If $F\in\mathfrak{C}(X,x_0,G)$, we define

$$\begin{aligned}g\cdot F: C_0(X)&\longrightarrow M_n(\mathbb{C}[G])\\f&\longmapsto g\cdot F(g^{-1}\cdot f).\end{aligned}$$

The space $\mathfrak{C}(X, x_0, G)$ can be described as the configuration space whose elements are formal sums

$$\sum_{i=1}^{n}(x_i, V_i),$$

when $x_i \in X - \{x_0\}$ and $V_i \subseteq \mathbb{C}[G]^\infty$ such that if $x_i \neq x_j$ then $V_i \perp V_j$, subject to some relations, for details see [17, Sec. 2.1]. We call the elements x_i the *points* and to the vector spaces V_i the *labels*.

***Remark* 1.** When the based G-CW-complex (X, x_0) is not supposed to be G-connected, we define the configuration space

$$\mathfrak{C}(X, x_0, G) = \Omega_0\mathfrak{C}(\Sigma X, x_0, G),$$

Where Ω_0 denotes the based loop space and Σ denotes the reduced suspension.

That description allow us to define a Hopf space structure on $\mathfrak{C}(X, x_0, G)$ by *putting together* two configurations when labels in both of them are mutually orthogonal.

We have the following result:

Theorem 2 (Thm. 5.2 in [16])**.** *Let (X, x_0) be a based finite G-connected G-CW-complex. If we denote by $k_n^G(X, x_0)$ the n-th G-equivariant connective K-homology groups of the pair (X, x_0), then there is a natural isomorphism*

$$\pi_n(\mathfrak{C}(X, x_0, G)^G) \xrightarrow{\mathfrak{A}^n} k_n^G(X, x_0).$$

When a Hopf space $\mathfrak{Y}$ is path-connected, consider the Hurewicz morphism

$$\lambda : \pi_*(\mathfrak{Y}; \mathbb{C}) = \bigoplus_{n\geq 0}\pi_i(\mathfrak{Y}) \otimes \mathbb{C} \to H_*(\mathfrak{Y}; \mathbb{C}).$$

We have the following result:

Theorem 3 (Thm. of the Appendix in [11])**.** *If $\mathfrak{Y}$ is a pathwise connected homotopy associative Hopf space with unit, and $\lambda : \pi_*(\mathfrak{Y}; \mathbb{C}) \to H_*(\mathfrak{Y}; \mathbb{C})$ is the Hurewicz morphism viewed as a morphism of $\mathbb{Z}$-graded Lie algebras, then it induces an isomorphism of Hopf algebras*

$$\bar{\lambda} : \mathcal{S}(\pi(\mathfrak{Y}; \mathbb{C})) \to H_*(\mathfrak{Y}; \mathbb{C}).$$

Applying the above theorem to $\mathfrak{C}(X, x_0, G)$ we obtain.

Corollary 4. *Let X be a finite G-CW-complex, if X is G-connected we have an isomorphism*

$$\mathcal{S}(k_*^G(X, x_0) \otimes \mathbb{C}) \cong H_*(\mathfrak{C}(X, x_0, G)^G; \mathbb{C}).$$

In order to relate $H_*(\mathfrak{C}(X, x_0, G)^G; \mathbb{C})$ with $\mathcal{F}_G(X)$ we need to recall the following result proved in Theorem 6.13 in [16] using the equivariant Chern character obtained in [9].

Theorem 5. *Let X be a G-CW-complex. There is a natural isomorphism of $\mathbb{Z}$-graded complex vector spaces (here the graduation is given by q)*

$$\bigoplus_{q\geq 0} k_n^G(X) \otimes \mathbb{C} \cong \bigoplus_{n\geq 0} K_n^G(X) \otimes \mathbb{C}[q].$$

Finally we can relate $H_*(\mathfrak{C}(X, x_0, G)^G; \mathbb{C})$ with $\mathcal{F}_G^q(X)$ when X is an even dimensional G-connected, G-Spin^c-manifold. First we recall *Poincaré duality* for equivariant K-theory.

Theorem 8.2. [3] *Let M be a n-dimensional G-Spin^c-manifold. Then there exists an isomorphism*

$$D : K_G^*(M_+) \longrightarrow K_{n-*}^G(M).$$

Applying Theorem 8.2 and Theorem 3 we can obtain the main result of the section.

Theorem 8.3. *Let (M, m_0) be an even dimensional G-connected, G-Spin^c-manifold. We have an isomorphism of $\mathbb{Z}$-graded Hopf algebras*

$$H_*(\mathfrak{C}(M, m_0, G)^G; \mathbb{C}) \cong \mathcal{F}_G^q(M).$$

Proof. Since M is a G-Spin^c manifold we can use Theorem 8.2 and obtain the following isomorphism of $\mathbb{Z}_+ \times \mathbb{Z}/2\mathbb{Z}$-graded Hopf algebras

$$\begin{aligned}
\mathcal{S}\left(k_*^G(M, m_0) \otimes \mathbb{C}\right) &\cong \mathcal{S}\left(\bigoplus_{n\geq 1} q^n K_*^G(M, m_0) \otimes \mathbb{C}\right) \\
&\cong \mathcal{S}\left(\bigoplus_{n\geq 1} q^n K_G^*(M_+, +) \otimes \mathbb{C}\right) \\
&\cong \mathcal{S}\left(\bigoplus_{n\geq 1} q^n K_G^*(M) \otimes \mathbb{C}\right).
\end{aligned}$$

Combining Corollary 4, Theorem 3 and Theorem 4.3 we obtain

$$H_*(\mathfrak{C}(M, m_0, G)^G; \mathbb{C}) \cong \mathcal{F}_G^q(M).$$

☑

For the case when M is not necessarily G-connected, we can obtain also a similar result. For details consult [16, Proposition 6.11].

Proposition 8.4. *Let X be a finite G-CW-complex, we have an isomorphism*

$$H_*(\Omega\mathfrak{C}(\Sigma X, G)^G; \mathbb{C}) \cong \mathcal{S}(k_*^G(X, x_0) \otimes \mathbb{C}).$$

In particular we have.

Example 8.5. For $X = S^0$ we have

$$\Omega\left(\mathfrak{C}(\Sigma(S^0), G)\right) \simeq BU_G.$$

Where BU_G can be taken as the Grassmannian of finite dimensional vector subspaces of a complete G-universe. A *complete G-universe* is a countably infinite-dimensional representation of G with an inner product such that contains a copy of every irreducible representation of G, contains countably many copies of each finite-dimensional subrepresentation. Applying the above discussion to this Hopf space we conclude that

$$\begin{aligned} H_*\left((BU_G)^G; \mathbb{C}\right) \cong& R(G) \otimes \mathcal{S}\left(\pi_*((BU_G)^G) \otimes \mathbb{C}\right) \\ \cong& R(G) \otimes \mathcal{S}\left(\bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C}\right) \\ \cong& \mathcal{S}\left(\bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C}\right). \end{aligned}$$

Summarizing, we have an isomorphism

$$H_*((BU_G)^G; \mathbb{C}) \cong \mathcal{F}_G^q(\{\bullet\}) = \mathcal{S}\left(\bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C}\right).$$

We also have

$$H_*\left((BU_G)^G; \mathbb{C}\right) \cong \mathcal{S}\left(\bigoplus_{n \geq 0} R(G_n) \otimes \mathbb{C}\right) \cong \mathbb{C}[\sigma_1^1, \ldots, \sigma_1^{k_1}, \sigma_2^1, \ldots]$$

where $\{\sigma_i^1, \cdots, \sigma_i^{k_i}\}$ is a complete set of non isomorphic irreducible representations of G_i. We expect that the elements σ_i^k correspond in some sense with duals of G-equivariant Chern classes.

Now suppose that M is a G-connected G-Spin^c-manifold and N is a H-connected H-Spin^c-manifold, then we have an isomorphism of $\mathbb{Z}$-graded Hopf algebras

$$H_*\left(\mathfrak{C}(M \times N, (m_0, n_0), G \times H)^{G \times H}; \mathbb{C}\right) \cong \mathcal{F}_G^q(M) \otimes_{\mathcal{F}(\{\bullet\})} \mathcal{F}_H^q(N)$$

In the case that $M = N = S^0$ with trivial action we obtain

$$H_*((BU_{G\times H})^{G\times H};\mathbb{C}) \cong \mathcal{F}_G(\{\bullet\}) \otimes_{\mathcal{F}(\{\bullet\})} \mathcal{F}_H(\{\bullet\}).$$

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(Recibido en noviembre de 2021. Aceptado en mayo de 2022)

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