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Sizes of flats of cycle matroids of complete graphs

Los tamaños de los cerrados de la matroide gráfica del grafo completo

Christo Kriel, Eunice Mphako-Banda✉

University of the Witwatersrand, Johannesburg, South Africa

Abstract. We show that the problem of counting the number of flats of size k for a cycle matroid of a complete graph is equivalent to the problem of counting the number of partitions of an integer k into triangular numbers. In addition, we give some values of k such that there is no flat of size k in a cycle matroid of a complete graph of order n. Finally, we give a minimum bound for the number of values, k, for which there are no flats of size k in the given cycle matroid.

Key words and phrases. Compositions, cycle matroid, flats, triangular number partitions, bad colouring.

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Resumen. Demostraremos que el problema de contar los conjuntos cerrados de tamaño k de la matroide gráfica de un grafo completo es equivalente al problema de contar las particiones de un entero k en números triangulares. Adicionalmente, daremos unos valores de k tales que no existe ningún cerrado de tamaño k en la matroide gráfica de un grafo completo de orden n. Finalmente, daremos una cota inferior para el número de valores k para los cuales no existe ningún cerrado de tamaño k en la matroide gráfica.

Palabras y frases clave. Composiciones, matroide, particiones de números triangulares.

1. Introduction

A matroid can be defined in several equivalent ways, using the set of independent sets, the set of circuits, the set of bases, the rank function and the closure operator. In this paper, we define a matroid using the rank function.

A matroid $M(E)$ is a set E with a rank function r, for which the following properties hold:

(R1) If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

(R3) If X and Y are subsets of E, then

$$r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y).$$

Let $\mathrm{cl} : 2^E \to 2^E$ for all $X \subseteq E$, given by $\mathrm{cl}(X) = X \cup \{x \in E : r(X \cup x) = r(X)\}$, where $r(X)$ is the rank of X, then cl(X) is called the closure of the set X. In addition, if $X = \mathrm{cl}(X)$, then X is called a *flat* of a matroid, see [5]. Furthermore, each graph G is a matroid, where E is the edge set, $X \subseteq E$, and the rank function, $r(X) = |V(X)| - k(X)$ where $|V(X)|$ is the number of vertices and $k(X)$ is the number of components of X. This matroid is called the cycle matroid of G and is denoted by $M(G)$. The contraction of X from a matroid M denoted by M/X is a matroid with a rank function $r_{M/X}(T) = r(X \cup T) - r(X)$ for all $T \subset E - X$. Furthermore, the characteristic polynomial of a matroid M is given by $\chi(M; \lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{r(M) - r(X)}$. For this introduction and further reading on matroids, we refer to [5].

This paper was motivated by the fact that projective geometries play a similar role in matroid theory to the role complete graphs play in graph theory. For example, just as every simple graph on n vertices can be obtained from K_n by deleting edges, so too can every simple F-representable matroid of rank r be obtained from $PG(r-1, q)$ by deleting elements. Furthermore, there is a simple explicit expression of the Tutte polynomial of a projective geometry which is computed using nice properties of this structure, stated in the following proposition, see [4].

Let q be a prime power and r a positive integer, we denote the rank r projective geometry over a finite field $GF(q)$ by $PG(r-1, q)$.

The *Gaussian coefficient* $\begin{bmatrix} r \\ j \end{bmatrix}_q$ for all integers r and j with $0 \leq j \leq r$ is defined by

$$\begin{bmatrix} r \\ j \end{bmatrix}_q = \frac{(q^r - 1)(q^r - q) \cdots (q^r - q^{j-1})}{(q^j - 1)(q^j - q) \cdots (q^j - q^{j-1})}.$$

Proposition 1.1. *Let q be a prime power and r a positive integer. Then the following hold for the projective geometry of rank r over a finite field q, $PG(r-1, q)$.*

(i) All flats of $PG(r-1, q)$ of rank j are isomorphic to $PG(j-1, q)$.

(ii) $PG(r-1, q)$ has $\begin{bmatrix} r \\ j \end{bmatrix}_q$ flats of rank j.

(iii) The simplification of $PG(r-1,q)/PG(j-1,q)$ *is isomorphic to* $PG(r-j-1,q)$.

(iv) $PG(r-1,q)$ *has* $\frac{q^r-1}{q-1}$ *elements.*

(v) The characteristic polynomial of $PG(r-1,q)$ *is*

$$\chi(PG(r-1,q);\lambda) = \prod_{i=0}^{r-1}(\lambda - q^i).$$

Unlike a projective geometry, a cycle matroid of a complete graph, $M(K_n)$, has flats of size k which are non-isomorphic and some having different ranks, making counting the number of flats of size k very complicated. On the other hand, one attractive property of $M(K_n)$ is that it is closed up to simplification under flat contraction just like projective geometries. For example in $M(K_6)$, flats of size 3 are of two types: first type is a flat isomorphic to $M(K_3)$ of rank 2 and the second type is a flat isomorphic to union $M(K_2) \cup M(K_2) \cup M(K_2)$ of rank 3. Furthermore, the minor $M(K_6)/M(K_3)$ is isomorphic up to simplification to $M(K_4)$. The following proposition states without proof some properties of flats of $M(K_n)$ which can be easily verified.

Proposition 1.2. *Let* $M(K_n)$ *be the cycle matroid of* K_n *and* X_k *a flat of* $M(K_n)$ *of size* k.

(i) All flats X_k *are of the form* $X_k = \cup_i M(K_i)$ $i \in \{1, 2, \cdots, n\}$.

(ii) There is no simple formula for the number of flats of $M(K_n)$ *of size* k.

(iii) The simplification of $M(K_n)/X_k$ *is isomorphic to* $M(K_{n-r(X_k)})$.

(iv) $M(K_n)$ *has* $\frac{n(n-1)}{2}$ *elements.*

(v) The characteristic polynomial of $M(K_n)$ *is* $\prod_{i=1}^{n-1}(\lambda - i)$.

The main problem of this paper is addressing part (i) and (ii) of Proposition 1.1 for projective geometries. We count the number of non-isomorphic flats of size k in $M(K_n)$. Further, we address the question of finding a number k such that $M(K_n)$ does not have a flat of size k.

2. Counting flats of a cycle matroid of a complete graph

In this section, we count the number of flats of size k for the matroid $M(K_n)$. We state some facts on triangular numbers and integer partitions, which we need, without proof.

In the following definition of a triangular number, 0 is the first triangular number, hence differs slightly from the usual definition, see [2].

Definition 2.1. The n-th triangular number is $\Delta_n = \binom{n}{2}$. In addition, Δ_{n+1} can be computed recursively as

$$\Delta_{n+1} = \Delta_n + n = \binom{n}{2} + n = \frac{n(n-1)+2n}{2} = \frac{(n+1)n}{2}.$$

The following theorem is well known in the literature. We refer the reader to [1, 3] for further discussion and proof.

Theorem 2.2 (Gauss Eureka)**.** *Every integer can be written as the sum of three triangular numbers.*

It is clear from this theorem that every integer can be partitioned into triangular numbers.

The relationship between triangular numbers and complete graphs is stated in the following lemma.

Lemma 2.3. *There is a one-to-one correspondence between triangular numbers and complete graphs.*

Proof. For every complete graph K_n, the size of K_n, $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2} = \Delta_n$, by Definition 2.1. ☑

By applying the quadratic formula to solve for n where $\frac{n(n-1)}{2} = \Delta_n$, we have, for every triangular number Δ_n, $\frac{1+\sqrt{1+8\Delta_n}}{2} = n = |V(K_n)|$.

Corollary 2.4. *Every triangular number partition of the integer k has a corresponding partition of the graph K_n of size k into subgraphs isomorphic to K_i where $i < n$.*

Notation. We denote the j^{th} triangular number partition of the integer k by $\pi_j(k) = \sum \Delta_i = k$ where Δ_i is not necessarily distinct and i is an integer such that $1 \leq i \leq k$.

Thus each partition $\pi_j(k)$ has a corresponding vertex partition of $V(K_n)$ into subgraphs isomorphic to K_i for some $i < n$. Combining Lemma 2.3, Corollary 2.4 and Proposition 1.2 lead us to use integer partitions of k to investigate the number of flats of size k of $M(K_n)$. Without loss of generality we proceed with an example which gives an insight to the counting problem.

Example 2.5. We compute triangular number partitions, $\pi_j(6)$, and their corresponding vertex partition of $V(K_n)$.

(a) $\pi_1(6) = 1+1+1+1+1+1 = \Delta_2+\Delta_2+\Delta_2+\Delta_2+\Delta_2+\Delta_2 = 6$.

(b) $\pi_2(6) = 3+1+1+1 = \Delta_3+\Delta_2+\Delta_2+\Delta_2$.

(c) $\pi_3(6) = 3+3 = \Delta_3+\Delta_3$.

(d) $\pi_4(6) = 6 = \Delta_4$.

On the other hand, using the fact that $\frac{1+\sqrt{1+8\Delta_i}}{2} = |V(K_i)|$, we let $\Delta_i \to M(K_i)$.

(a) $\pi_1(6) = \Delta_2+\Delta_2+\Delta_2+\Delta_2+\Delta_2+\Delta_2 \to M(K_2)\cup M(K_2)\cup M(K_2)\cup M(K_2)\cup M(K_2)\cup M(K_2)$.

(b) $\pi_2(6) = \Delta_3+\Delta_2+\Delta_2+\Delta_2 \to M(K_3)\cup M(K_2)\cup M(K_2)\cup M(K_2)$.

(c) $\pi_3(6) = \Delta_3+\Delta_3 \to M(K_3)\cup M(K_3)$.

(d) $\pi_4(6) = \Delta_4 \to M(K_4)$.

But flats X_6 of size 6 in $M(K_n)$ are of the form

(a) $X_{6_1} \cong M(K_2)\cup M(K_2)\cup M(K_2)\cup M(K_2)\cup M(K_2)\cup M(K_2)$,

(b) $X_{6_2} \cong M(K_3)\cup M(K_2)\cup M(K_2)\cup M(K_2)$,

(c) $X_{6_3} \cong M(K_3)\cup M(K_3)$,

(d) $X_{6_4} \cong M(K_4)$.

Thus $\pi_j(6) \to X_{6_j}$.

Furthermore, to get a flat of size 6 in $M(K_n)$ of the form X_{6_1}, six disjoint copies of $M(K_2)$, we need $n \geq 12$. Similarly, for X_{6_2}, X_{6_3}, and X_{6_4} we need $n \geq 9$, $n \geq 6$ and $n \geq 4$ respectively. For example, if $n = 6$, i.e. in $M(K_6)$, the only possible flats are of the form X_{6_3} and X_{6_4}. This highlights a crucial point in the counting of flats of $M(K_n)$: not all integer partitions of k into triangular numbers translate to flats of size k in $M(K_n)$.

We now generalise the identification and enumeration problem of flats of size k in $M(K_n)$ as follows.

To ease notation, we label each block in the vertex partition as i_β, where i is the number of vertices in the block and β the number of the block. We list the blocks in decreasing size and increasing values of β. Set the value $i_\beta = i$ and let $|\beta_i|$ be the number of blocks in the partition that have the same number of vertices i. Let B_j be the number of blocks in the vertex partition corresponding to $\pi_j(k)$.

Thus, for example, we will label the block in the corresponding vertex partition of $\pi_4(6)$ as 4_1, $B_4 = 1$ and $|\beta_4| = 1$. In the partition corresponding to $\pi_3(6)$ we label the blocks 3_1 and 3_2, $B_3 = 2$ and $|\beta_3| = 2$.

Then the number of flats of size k corresponding to each $\pi_j(k)$ in K_n, $|X_{k_j}|$, is given by

$$|X_{k_j}| = \frac{\binom{n}{i_1}\binom{n-i_1}{i_2}\binom{n-(i_1+i_2)}{i_3}\cdots\binom{n-(i_1+i_2+\ldots+i_{B_j-1})}{i_{B_j}}}{\prod |\beta_i|!}$$

where $|\beta_i|$ is the number of blocks with size i in the partition and division by $|\beta_i|!$ is to ensure that we don't double count.

Example 2.6. To clarify the notation we use $\pi_2(6)$ from Example 2.5 and calculate X_{6_2} in $M(K_9)$.

The triangular number partition $\pi_2(6) = \Delta_3 + \Delta_2 + \Delta_2 + \Delta_2$ yields the vertex partition labeled $3_1, 2_2, 2_3, 2_4$. Thus,

$$\begin{aligned} |X_{6_2}| &= \frac{\binom{9}{3_1}\binom{9-3_1=6}{2_2}\binom{6-2_2=4}{2_3}\binom{4-2_3=2}{2_4}}{(|\beta_2| = 3)!} \\ &= \frac{\binom{9}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!}. \end{aligned}$$

We note that the numerator in the above example can be written as the multinomial coefficient $\binom{n}{3,2,2,2} = \frac{9!}{3!2!2!2!}$.

Proposition 2.7. *Let X_k be a flat of size k corresponding to π, a triangular number partition of $k = \Delta_{i_1} + \cdots + \Delta_{i_a}$. Then there are exactly*

$$|X_k| = \binom{n}{i_1, \ldots, i_a,\, b} / \prod |\beta_i|!$$

flats of $K_n = K_{i_1+\cdots+i_a+b}$ of size k corresponding to π, where $i_1+\cdots+i_a+b = n$ and $|\beta_i|$ is the number of blocks in the partition that have the same number of vertices, i.

The answer to Proposition 1.2 (ii) can be obtained by adding the answer above over all triangular number partitions of k with $i_1 + \cdots + i_a + b = n$.

This leads us to two more problems: when do we get $\sum_i |V(K_i)| > n$, that is, determining the values of k such that $|X_k| = 0$ and when do we get $\sum_i |V(K_i)| < n$ i.e determining the values of k such that $|X_k| \neq 0$?

3. Values for which there are no flats of size k in $M(K_n)$

Next, we look at values of k for which there are no flats of size k of $M(K_n)$. According to Corollary 2.4 these are values of k for which every triangular number partition of k has a corresponding disjoint union of complete graphs $\pi_j(|V(K_n)|)$ with $\sum_i |V(K_i)| > n$.

We will argue from the graph theoretic perspective in this section. In an improper colouring of a graph G, call an edge *bad* if it connects two vertices of the same colour. In an improper colouring of a graph G, a set of k bad edges is a flat of size k.

Clearly, since there is a one-to-one correspondence between triangular numbers and complete graphs, there is a complete subgraph of K_n, that is, a flat $X_k \subseteq E(K_n)$ exists, for every triangular number $k \leq \binom{n}{2}$.

Now consider two successive triangular numbers $\Delta_{n-p} = \binom{n-p}{2}$ and $\Delta_{n-p+1} = \binom{n-p+1}{2}$ such that the two corresponding complete graphs are both subgraphs of K_n. We know by definition that $\Delta_{n-p+1} - \Delta_{n-p} = n-p$ and from the previous paragraph that the number of flats for $k = \Delta_{n-p}$ and $k = \Delta_{n-p+1}$ are non-zero. To generate flats for $\binom{n-p}{2} < k < \binom{n-p+1}{2}$ we need to add bad edges by choosing complete subgraphs from the remaining p vertices in the partition $\{K_{n-p}, K_p\}$. Once all p vertices have been chosen, the only way to add more edges is to add edges between the two partitions, but this means that all vertices will be the same colour. Thus, the number of flats for all $\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2}$ will be zero. We proceed to prove this statement in what follows.

Lemma 3.1. *For integers n and p, if $\dfrac{p^2+p+4}{2} \leq n$, then there is at least one integer between $\binom{n-p}{2} + \binom{p}{2}$ and $\binom{n-p+1}{2}$.*

Proof. With some manipulation $\frac{p^2+p+4}{2} \leq n$ can be written as $\binom{n-p}{2} + \binom{p}{2} + 2 \leq \binom{n-p+1}{2}$. Furthermore, $\binom{n-p}{2}$ is a triangular number and $\binom{n-p+1}{2}$ is the next triangular number, so $\binom{n-p+1}{2} > \binom{n-p}{2}$.

If the difference between $\binom{n-p}{2} + \binom{p}{2}$ and $\binom{n-p+1}{2}$ is 1, then $\binom{n-p+1}{2}$ is the integer after $\binom{n-p}{2} + \binom{p}{2}$. Thus, since the difference is greater than or equal to 2, we must have at least one integer between the two integers $\binom{n-p}{2} + \binom{p}{2}$ and $\binom{n-p+1}{2}$. ☑

From the inequality on p and n in Lemma 3.1, given n, we can calculate the values of p as $p \leq \lfloor \dfrac{-1+\sqrt{8n-15}}{2} \rfloor$.

The following Lemma 3.2 is stated in [6].

Lemma 3.2. $\Delta_{n-i} = \Delta_n + \Delta_i - i(n-1)$.

We are now in a position to state and prove a theorem on some of the integer intervals for which there are no flats of size k in $M(K_n)$. Recall that, by definition, $\binom{n}{r} = 0$ for $n < r$.

Theorem 3.3. *Let K_n be a complete graph of order n and for integers p and k, $1 \leq p \leq \lfloor \dfrac{-1+\sqrt{8n-15}}{2} \rfloor$, let $\dbinom{n-p}{2} + \dbinom{p}{2} < k < \dbinom{n-p+1}{2}$. Then the number of closed sets of edges of size k of K_n (flats of size k of $M(K_n)$) is equal to zero.*

Proof. Note that a set X_k of k bad edges in an improper colouring of a complete graph K_n partitions K_n into a disjoint union of complete subgraphs, K_i, such that $|\cup E(K_i)| = k$. Also note that any subgraph isomorphic to K_1 in the vertex partition contributes no edges to X_k.

Thus, we need to show that there is no disjoint union of complete subgraphs of K_n with size k and order n. In other words, we need to show that once we have partitioned K_n into two subgraphs K_{n-p} and K_p, it is not possible to get more than $\binom{n-p}{2} + \binom{p}{2}$ bad edges from any other partition until we choose K_{n-p+1} as a subgraph, thus giving us $\binom{n-p+1}{2}$ bad edges. The proof is in two parts. For ease of reference we will write the two parts as separate propositions. Theorem 3.3 follows directly from Propositions 3.4 and 3.5. ☑

We recall that our use of *block* refers to the elements of a set partition and the size of a *block* B is given by $|B|$, the number of elements in B.

Proposition 3.4. *Let K_n be a complete graph and p an integer such that $1 \leq p \leq \lfloor \dfrac{-1+\sqrt{8n-15}}{2} \rfloor$. Then there is no partition of K_n into two complete subgraphs such that there will be more than $\binom{n-p}{2} + \binom{p}{2}$ and less than $\binom{n-p+1}{2}$ bad edges.*

Proof. We have $p \geq 1$ and $\frac{p^2+p+4}{2} \leq n$. The latter inequality guarantees by Lemma 3.1 that the interval $\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2}$ is not empty.

Suppose there are two subgraphs K_{n-r} and K_r such that $k = \binom{n-r}{2} + \binom{r}{2}$ and $r \neq p$ such that $\binom{n-r}{2} + \binom{r}{2} > \binom{n-p}{2} + \binom{p}{2}$. We use all n vertices in the supposed partition in order to maximise the number of bad edges. We must have $(n-r) < (n-p)$, otherwise $\binom{n-r}{2} \geq \binom{n-p+1}{2}$, giving a value for k outside our proposition statement. Hence we have $r > p$. Also, $r, p \leq \lfloor \frac{n}{2} \rfloor$, otherwise, the two parts in each partition simply reverse their places and our proof is the same by symmetry. We use the identified inequalities and Lemma 3.2 to prove that $\binom{n-p}{2} + \binom{p}{2} > \binom{n-r}{2} + \binom{r}{2}$. Using the correspondence between complete graphs and triangular numbers

$$\binom{n-p}{2} + \binom{p}{2} > \binom{n-r}{2} + \binom{r}{2} \Rightarrow \Delta_{n-p} + \Delta_p > \Delta_{n-r} + \Delta_r.$$

Hence, we need to show that

$$(\Delta_{n-p} + \Delta_p) - (\Delta_{n-r} + \Delta_r) > 0.$$

Using Lemma 3.2 we can show that

$$\begin{aligned}(\Delta_{n-p} + \Delta_p) - (\Delta_{n-r} + \Delta_r) &= (\Delta_n + \Delta_p - p(n-1) + \Delta_p)\\ &\quad - (\Delta_n + \Delta_r - r(n-1) + \Delta_r)\\ &= (p-r)(p+r-n).\end{aligned}$$

Since $r > p$, $(p-r) < 0$. Also, $r, p \leq \lfloor \frac{n}{2} \rfloor$. But $p < r$, so $p \leq \lfloor \frac{n}{2} \rfloor - 1$.

Hence,

$$\begin{aligned}r + p &\leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{2} \rfloor - 1\\ &\leq \frac{n}{2} + \frac{n}{2} - 1 \leq n - 1\\ &< n.\end{aligned}$$

Thus, $(r + p - n) < 0$ and $(p - r)(p + r - n) > 0$.

We conclude that there is no partition of k into two triangular numbers such that $\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2}$, given the bounds identified, and therefore there is no partition of K_n into two complete subgraphs such that there will be more than $\binom{n-p}{2} + \binom{p}{2}$ and less than $\binom{n-p+1}{2}$ bad edges. ☑

Proposition 3.5. *Let K_n be a complete graph and p an integer such that $1 \leq p \leq \lfloor \frac{-1 + \sqrt{8n - 15}}{2} \rfloor$. Then there is no partition of K_n into three or more complete subgraphs such that there will be more than $\binom{n-p}{2} + \binom{p}{2}$ and less than $\binom{n-p+1}{2}$ bad edges.*

Proof. We proceed to prove that there is no disjoint union of three or more complete subgraphs that will give us more bad edges on n vertices than we get from the partition $\{K_{n-p}, K_p\}$ and fewer bad edges than when we choose K_{n-p+1} as the induced closed subgraph with $\binom{n-p+1}{2}$ edges.

There is a total of $\binom{n}{2}$ edges in K_n. Choosing a flat of edges as bad edges partitions the edge set, one part of the total set of edges will be bad and the remainder will be 'good'. Hence, in order to prove $\binom{n-p}{2} + \binom{p}{2}$ yields more bad edges than any partition of K_n into three (or more) complete subgraphs, it is sufficient to show that the number of 'good' edges in the partition with larger number of blocks is greater than the number of good edges in the partition $\{K_{n-p}, K_p\}$, given certain bounds which arise naturally from the proposition statement.

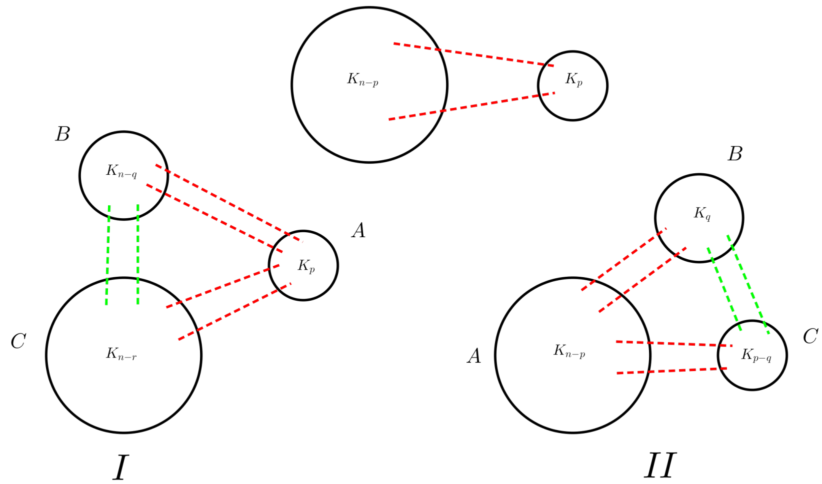

FIGURE 1. We divide K_{n-p} into two subgraphs (I) and K_p into two subgraphs (II).

We start with two complete subgraphs K_{n-p} and K_p. We have $(n-p)p$ good edges, call these red edges. We partition K_{n-p} into two complete subgraphs K_{n-r} and K_{n-q}. This gives the partition I in Figure 1. Since this is a partition of the vertices of K_{n-p}, we have $(n-r)+(n-q)=(n-p)$.

There are $(n-q)p$ and $(n-r)p$ good edges between K_p and K_{n-q} and K_{n-r} respectively and $(n-q)(n-r)$ good edges between K_{n-q} and K_{n-r}. Call the good edges between the latter two graphs green edges. Clearly we have more good edges in our partition $\{K_{n-r}, K_{n-q}, K_p\}$ than in our partition $\{K_{n-p}, K_p\}$. Also $(n-q)p+(n-r)p=(n-p)p$ since the good edges between K_p and the other two graphs remain constant no matter how we split the vertices in K_{n-p} (recall that these are red edges). Similarly we partition K_p into two subgraphs K_q and K_{p-q} as in partition II in Figure 1. What should be clear is that the 'red' edges remain the same as in the original $\{K_{n-p}, K_p\}$ partition and the 'green' edges resulting from our partitions I and II are extra. So we have $(n-q)(n-r)+(n-q)p+(n-r)p>(n-p)p$ in Partition I and $(p-q)q+(n-p)(p-q)+(n-p)q>(n-p)p$ in Partition II.

Now partition K_n into three complete subgraphs, with vertex partitions A, B and C. The induced subgraphs are all complete graphs and the edges in each of the partitions are bad edges and the vertices all the same colour. The edges between the partitions are good edges. We must have $|A|$, $|B|$ and $|C| \leq n-p$ otherwise we have at least $\binom{n-p+1}{2}$ bad edges and this falls outside the proposition statement. If $|A|$, $|B|$ or $|C|=n-p$ or p then we have partition I or II and the proposition holds, so we will assume that $|A|$, $|B|$ or $|C| \neq$

$n-p$ nor p. Hence we have $|A|$, $|B|$ and $|C| < n-p$. We use all the n vertices in the partition in order to get as many bad edges as possible.

Suppose $|A| \geq |B|, |C|$ and let $|A| = n-r$ for some integer $r > p$, then $|B| + |C| = r$. We know that there is no partition $\{K_{n-r}, K_r\}$ such that there are $\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2}$ bad edges. Since B and C partition K_r, the partition $\{K_{|B|}, K_{|C|}\}$ has fewer bad edges than K_r and, hence, the number of bad edges resulting from the vertex partition A, B and C is less than from the partition $\{K_{n-r}, K_r\}$ and, hence, less than $\{K_{n-p}, K_p\}$ from Proposition 3.4.

By the same reasoning the proposition holds even if we are using a larger number of blocks in our partition. ☑

4. A lower bound on the number of flats of size k such that $M(K_n)$ has no flat of size k.

Finally, we state and prove a lower bound for the number of flats of size k such that $M(K_n)$ has no flat of size k. We need the following two identities on triangular numbers which we state as lemmas. We note that Lemma 4.1 has been known since at least 1261, see [3], and Lemma 3.2, listed in [6], is equally easy to prove.

Lemma 4.1. $\displaystyle\sum_{i=1}^{p} \Delta_i = \frac{(p-1)p(p+1)}{6}.$

Theorem 4.2. *The number of integers k such that $M(K_n)$ has no flat of size k, is at least $p(n-1) - \frac{p(p+1)(p+2)}{6}$, where $p = \lfloor \frac{-1+\sqrt{8n-15}}{2} \rfloor$. Asymptotically this number is at least $\sim \frac{2\sqrt{2}}{3} n\sqrt{n}$*

Proof. We know from Theorem 3.3 that for all integers k such that $\binom{n-p}{2} + \binom{p}{2} < k < \binom{n-p+1}{2}$ for $1 \leq p \leq \lfloor \frac{-1+\sqrt{8n-15}}{2} \rfloor$, there are no flats of size k in $M(K_n)$, so we will count this number of integers over all p on the interval.

There are $\binom{n-p+1}{2} - \left(\binom{n-p}{2} + \binom{p}{2}\right) - 1$ integers between $\binom{n-p+1}{2}$ and $\binom{n-p}{2} + \binom{p}{2}$ so the total number of integers is given by

$$\sum_{i=1}^{p} \left(\binom{n-i+1}{2} - \binom{n-i}{2} - \binom{i}{2} - 1 \right).$$

By our definition of triangular numbers this is equivalent to

$$\sum_{i=1}^{p} (\Delta_{n+1-i} - \Delta_{n-i} - \Delta_i - 1),$$

where Δ_i is the i-th triangular number.

We recall from Definition 2.1 that $\Delta_{n+1} = \Delta_n + n$ and use the identities from Lemmas 4.1 and 3.2 to evaluate the sum.

$$
\begin{aligned}
&\sum_{i=1}^{p} (\Delta_{n+1-i} - \Delta_{n-i} - \Delta_i - 1) \\
&= \sum_{i=1}^{p} (\Delta_{n+1} + \Delta_i - in - (\Delta_n + \Delta_i - i(n-1)) - \Delta_i - 1) \\
&= \sum_{i=1}^{p} (\Delta_{n+1} - \Delta_n - \Delta_i - i - 1) \\
&= \sum_{i=1}^{p} (n - \Delta_i - i - 1) \\
&= pn - \frac{(p-1)p(p+1)}{6} - \frac{p(p+1)}{2} - p \\
&= p(n-1) - \frac{(p)(p+1)(p+2)}{6}.
\end{aligned}
$$

☑

Note, we know that there are some values for k outside the intervals identified in Theorem 3.3 such that there are no flats of size k in $M(K_n)$. Without loss of generality, let $n = 6$ then $p = 2$. By Theorem 3.3 this means that there are no flats of size k for $7 < k < 10$ as well as $10 < k < 15$. Evaluating $p(n-1) - \frac{p(p+1)(p+2)}{6}$ this gives 6 values of k such that $M(K_n)$ has no flats of size k. However, there is also no flat of size $k = 5$ in $M(K_6)$, since the triangular number partitions for 5 are $1+1+1+1+1$ and $3+1+1$, requiring a complete graph of order at least ten and seven, respectively. But 5 falls outside the intervals determined by Theorem 3.3. Thus $p(n-1) - \frac{p(p+1)(p+2)}{6}$ gives a lower bound on the number of values for k such that there are no such flats in $M(K_n)$.

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School of Mathematics
University of the Witwatersrand
P/Bag 3,WITS, 2050, RSA.
e-mail: `christo.kriel@wits.ac.za`
e-mail: `Eunice.mphako-banda@wits.ac.za`