https://doi.org/10.15446/recolma.v56n1.105621

Revista Colombiana de Matemáticas
Volumen 56(2022)1, páginas 93-111

On the invariant rational curves of a certain family of polynomial differential equations

Sobre las curvas racionales invariantes de cierta familia de ecuaciones diferenciales

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Abstract. In this work, we present sufficient conditions to determine if the limit cycles of certain differential systems in the plane are algebraic or not. In particular, we obtain criteria such that the limit cycles of equations derived from predatory prey models with rational functional response are necessarily transcendental ovals.

Key words and phrases. algebraic limit-cycles, Puiseux series, Newton polygon, predator-prey models, functional-response.

2020 Mathematics Subject Classification. 34C07, 34C08, 92D25.

Resumen. En este trabajo presentamos condiciones necesarias y suficientes para determinar si los ciclos límite de ciertas ecuaciones diferenciales en el plano son algebraicos o no. Particularmente, obtenemos criterios para que ciclos límite de ciertas ecuaciones derivadas de modelos depredador - presa con ciertos funcionales racionales de respuesta sean necesariamente óvalos trascendentes.

Palabras y frases clave. ciclos límte algebraicos, series de Puiseux, polígono de Newton, modelos depredador-presa, funcional de respuesta.

1. Introduction

Since K. Odani's remarkable work [15], many authors have studied the question of whether the limit cycles of some polynomial differential systems in the plane are algebraic or not. Recall that if a limit cycle is contained in the zero set of a polynomial in two variables this is called an algebraic limit cycle. Deciding whether periodic orbits of a system are algebraic or not, can be quite involving.

Some kind of results describe conditions for the existence of limit cycles, see for instance [6], [8] and [7]. Other kind of results deal with conditions for the existence and full description of algebraic limit cycles or algebraic solutions in general, see for instance [12], [5], [16] and [2]. Finally, there are examples showing the coexistence of both algebraic and trascendental limit cycles, see [9].

There are several tools that have been used in this area. Recently, [2] describes a bounding technique for the degree of invariant algebraic curves. This is done by a rather classical approach which consists of bounding the number of branches of Puiseux series of solutions of the differential equation passing through singular points either at the affine plane or at infinity. See [13], [14] and [1] for a detailed standard description of the use of Puiseux series in differential equations.

Recent developments such as [10] and [11] also use Puiseux series for solutions of polynomial planar differential equations. They study the problem of Liouville integrability as well as the so called Weierstrass integrability, describing the integrating factors as well as their cofactors in terms of Puiseux–Weierstrass polynomials.

In this work we present criteria for polynomial planar systems of the form

$$\begin{aligned} \dot{x} &= x(a_0 + a_1 y + \cdots + a_{n-1} y^{n-1}), \\ \dot{y} &= y(x + b_0 + b_1 y + \cdots + b_{n-1} y^{n-1} + b_n y^n), \ b_n \neq 0. \end{aligned} \tag{1}$$

This class of equations appear in several systems of mathematical biology. We discuss some examples to illustrate our theoretical results in the context of models of two-species interactions. Similar results with the same techniques for systems of two species interactions will also appear in [4].

2. Rational functions as invariant algebraic curves

Throughout this section we present our main results about boundedness of the degrees $\deg_x F(x,y)$, $\deg_x F(x,y)$, of invariant algebraic curves $F(x,y) = 0$ for plane polynomial vector fields of the form (1).

We first study a particular case of system (1) to introduce the main ideas for analyzing the general case. We consider the following system

$$\begin{aligned} \dot{x} &= x(a_0 + a_1 y + a_2 y^2), \\ \dot{y} &= y(x + b_0 + b_1 y + b_2 y^2 + b_3 y^3), \qquad b_3 \neq 0. \end{aligned} \tag{2}$$

System (2) yields the following ODE in the complex domain

$$\frac{dx}{dy} = \frac{x(a_0 + a_1 y + a_2 y^2)}{y(x + b_0 + b_1 y + b_2 y^2 + b_3 y^3)} = \frac{P(y,x)}{Q(y,x)} \tag{3}$$

whose solutions are Riemann surfaces immersed in $\overline{\mathbb{C}}_y \times \overline{\mathbb{C}}_x$, where $\mathbb{C}_y \simeq \mathbb{C}$ with the compactification $\overline{\mathbb{C}}_y = \mathbb{C}_y \cup \{\infty\} \simeq \mathbb{CP}^1$. The role of x and y as dependent/independent variables may be interchanged.

Under the change of coordinates $\eta = \frac{1}{y}$, equation (3)yields

$$\frac{dx}{d\eta} = -\frac{x(a_0\eta^2 + a_1\eta + a_2)}{(x + b_0)\eta^3 + b_1\eta^2 + b_2\eta + b_3}, \tag{4}$$

which has the trivial solution $x \equiv 0$. Furthermore, as $\eta \to 0$ the ODE (4) has regular points at the infinitude $y = \infty$ whenever $b_3 \neq 0$. Therefore, the only branch point at infinity can arise at the fixed singular point $x = 0$.

Let us mention that if we consider a rational function, $\phi(y) = f(y)/g(y)$ with $\alpha = \deg f, \beta = \deg g$, then being a solution implies, by straight forward calculations, that $\alpha = \beta + 2$. Hence $\lim_{y\to\infty} \phi(y) = \infty$ and therefore we could only expect a branching pole at $y = \infty$ for algebraic rational solutions. In fact, ϕ will have a regular pole at $y = \infty$.

On the other hand, at the infinitude $x = \infty$, if we set $\xi = 0$ with $\xi = 1/x$, then we have the equation:

$$\frac{dy}{d\xi} = -\frac{y + \xi y(b_0 + b_1 y + b_2 y^2 + b_3 y^3)}{\xi^2(a_0 + a_1 y + a_2 y^2)}. \tag{5}$$

Since in the denominator we have $\xi^2 = 0$, then we have a pole, besides the singularity arising from the trivial solution $y \equiv 0$.

The main result regarding equation (2) is the following assertion.

Theorem 2.1. *Suppose that:*

$$b_3 \neq 0. \tag{6}$$

If there exists an invariant algebraic curve $F(x, y) = 0$ *of equation* (3) *with* $x, y \nmid F(x, y)$*, then* $\deg_x F = 1$ *and* $\deg_y F \leq 3$*. In particular, any algebraic (possibly multivaluated) solution should also be a rational (univaluated) solution,* $x = \phi(y) = f(y)/g(y)$*, with polynomials* $f(y), g(y)$ *of degree at most 3. Provided we exclude the trivial solution,* $x(y) \equiv 0$*.*

In the proof of Theorem 2.1, which will be given in Section 4, we will study the Laurent-Puisseux series of branches of solutions, $x = \phi(y)$, near infinity, $y = \infty$,

$$c_0\eta^{-3} + c_1\eta^{-2} + c_2\eta^{-1} + c_3 + c_4\eta + \dots.$$

The bound $\deg_x F = 1$ given in Theorem 2.1 corresponds to the unique determination of the coefficients of solutions along $y = \infty$ for (4). Meanwhile the bound $\deg_y F \leq 3$ is related to the possible branching number of solutions $y(x)$ at infinity $x = \infty$.

A more general result can be proved with the same tools in systems of higher degree. Namely, as we will see in subsection 4.2, we can prove an analogous of Theorem 2.1 for higher order systems.

Theorem 2.2. *Let us consider the ODE in the complex domain,*

$$\frac{dy}{dx} = \frac{y(x + b_0 + \cdots + b_{n-1}y^{n-1} + b_n y^n)}{x(a_0 + \cdots + a_{n-1}y^{n-1})}, \qquad b_n \neq 0. \tag{7}$$

If there exists an invariant algebraic curve $F(x,y) = 0$ of equation (7) with $x, y \nmid F(x,y)$, then $\deg_x F = 1$ and $\deg_y F \leq n$. In particular, any algebraic (possibly multivaluated) solution should also be a rational (univaluated) solution, $x = \phi(y) = f(y)/g(y)$, with polynomials $f(y), g(y)$ of degree at most n. Provided we exclude the trivial solution, $x(y) \equiv 0$.

The proofs of these results will be given in section 4.

3. Predatory-prey models with rational functional response

Equation (7) appears in some models of mathematical biology. Namely, in [5] the authors analyze non-algebraic nature of limit cycles in general predator-prey models in the spirit of the seminal work on the subject [15]. They consider the Rosenzweig–MacArthur predator-prey models

$$\begin{aligned} \dot{u} &= ru\left(1 - \frac{u}{K}\right) - vp(u), \\ \dot{v} &= v\left(-D + \gamma p(u)\right), \end{aligned} \tag{8}$$

with Holling's Type I functional-response

$$p(u) = mu/(a + u)$$

and also a Monod–Aldane model with Holling's Type IV functional-response,

$$p(u) = \frac{mu}{au^2 + bu + 1}.$$

They claim that whenever it has limit cycles, they should be transcendental ovals. The proof in [5] reduces the model to a polynomial system of the form (2). Then the main result states that there can not exist algebraic invariant curves under some hypotheses. Accordingly, these hypotheses are unavoidable for their arguments due to examples where there are rational invariant curves of the form $x = \phi(y)$ where ϕ is a rational function.

We present a generalization of [5] as follows.

Corollary 3.1. *Let us consider the predator-prey model* (8) *with functional-response*

$$p(u) = \frac{mu}{a + u^{n-1}}, \, n \geq 3.$$

If $m, K, D > 0$, then any algebraic invariant curve, $F(x, y) = 0$, is a rational function $x = \phi(y) = f(y)/g(y)$, with polynomials $f(y), g(y)$ of degree at most n.

Take for instance the case $n = 4$, which is not contained in [5]. By a suitable change of variables and time reparametrization

$$u = Ky, \qquad v = \frac{rK^2}{m}x, \qquad \frac{ds}{dt} = \frac{K^3}{r(a + u^3)},$$

the system (8) becomes

$$\begin{aligned} \dot{x} &= x\left(-A + y - Cy^3\right), \\ \dot{y} &= y\left(-x + A - Ay + y^3 - y^4\right), \end{aligned} \tag{9}$$

with $A = a/K^3$, $C = DK^2/m$. Notice that this system has the form

$$\begin{aligned} \dot{x} &= x(a_0 + a_1 y + a_2 y^2 + a_3 y^3), \\ \dot{y} &= y(x + b_0 + b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4), \quad b_4 \neq 0. \end{aligned} \tag{10}$$

Thys system can be referred to Theorem 2.2. Therefore the only invariant algebraic curve $F(x, y) = 0$ should be a graph of a rational function. Moreover, $\deg_y F(x, y) \leq 4$.

On the other hand, there are results that guarantying the existence and uniqueness of a limit cycle. Such a limit cycle should therefore be trascendental. The origin is always a saddle point, there is also an equilibrium in $(0, 1)$. There are two other equilibria, $(x_1, y_1), (x_2, y_2)$, with $0 < y_1 \leq y_2$. They are obtained by solving simultaneously the following equations:

$$\begin{aligned} (1 - y)(A + y^3) &= x, \\ y^3 - \frac{1}{C}y + A &= 0. \end{aligned}$$

According to Theorem 6 in [17], under certain conditions, namely

$$1 - \frac{1}{C} > A > \frac{2}{(3C)^{3/2}}, \qquad 5y_1^4 + 4y_1^3 - 2Ay_1 + A > x_1,$$

these equilibria remain within the invariant region $0 \leq y \leq 1$ and there exists a limit cycle surrounding (x_1, y_1).

Conditions hold in [17] for $C = 2$, and $A = 0.1$. Numerical evidence suggests that there is a limit cycle, see Fig. 1. This limit cycle is trascendental according to Corollary 3.1.

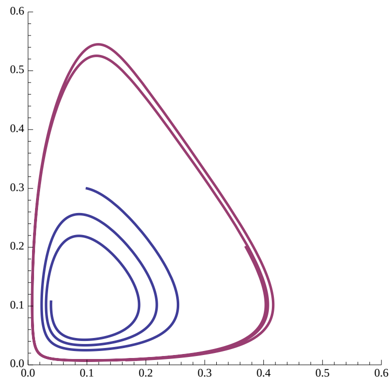

FIGURE 1. Solutions of (9) spiraling out and in for initial conditions $(0.37, 0.2), (0.1, 0.3)$, respectively, seem to emanate from an unstable limit cycle.

4. Proofs of our results

We start by considering the Newton-Puiseux algorithm to describe explicitly the nature of solutions at the infinitudes $x = \infty$ and $y = \infty$. For further explanation of the Newton-Puiseux method for ODE, see [1, 13, 14]. The crucial step of the proof is to apply the following result.

Lemma 4.1 (Theorem 1.4 in [3])**.** Let $G(z, w) = 0$ be an invariant algebraic curve, $\partial_w G \neq 0$ of the polynomial ODE

$$P(z, w)\frac{dw}{dz} - Q(z, w) = 0. \tag{11}$$

Then $\deg_w G$ is at most the number of Puiseux series

$$w(z) = c_0 z^{\mu} + \sum_{k=1}^{\infty} c_k z^{\frac{k}{n}+\mu}, \tag{12}$$

solving (11), whenever the number of these series is finite. Here $\mu_0 = l_0/n$ with n, l_0 relatively prime integers $n \geq 0$.

4.1. Proof of Theorem 2.1

To find an expansion of non-trivial solutions at $y = \infty$, i.e., along $\eta = 0$, with $\eta = 1/y$, we regard equation (4). Take the following Puiseux series:

$$x(\eta) = c_0 \eta^{\mu_0} + \sum_{k=1}^{\infty} c_k \eta^{\frac{k}{n}+\mu_0} \tag{13}$$

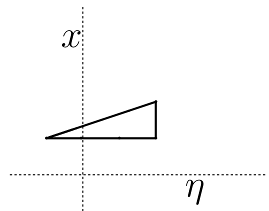

FIGURE 2. Newton polygon for (4)

For equation (4) the corresponding Newton polygon is a triangle rectangle whose only oblique side is the hypotenuse, see Fig. 2. Therefore, the only slope to consider is $-1/\mu_0 = 1/3$. Accordingly, $\mu_0 = -3$. Under substitution

$$\eta = \eta_1, \qquad x = c_0\eta_1^{\mu_0} + x_1,$$

we regard the least degree coefficient. Then c_0 is determined by the following quadratic relation

$$3(b_3c_0 + c_0^2) = 0 \quad \Rightarrow \quad c_0 = 0,\, -b_3 \in \mathbb{C}. \tag{14}$$

Assumption 1. *We choose the non-vanishing value*

$$c_0 = -b_3 \neq 0. \tag{15}$$

In the second step, under substitution $\eta_1 = \eta, x = (-b_3)\eta_1^{-3} + x_1$, we obtain a Newton polygon for

$$\begin{aligned} b_3(3b_2 - a_2) + b_3(3b_1 - a_1)\eta_1 + b_3(3b_0 - a_0)\eta_1^2 + \\ +x_1(3b_3\eta_1^2 + a_2\eta_1^3 + a_1\eta_1^4 + a_0\eta_1^5) + \\ +(b_2\eta_1^4 + b_1\eta_1^5 + b_0\eta_1^6)\frac{dx_1}{d\eta_1} + \\ +\eta_1^6 x_1\frac{dx_1}{d\eta_1} = 0 \end{aligned} \tag{16}$$

which has two possible slopes with $\mu_1 = -3, -2$. See the corresponding Newton polygon in Fig. 3.

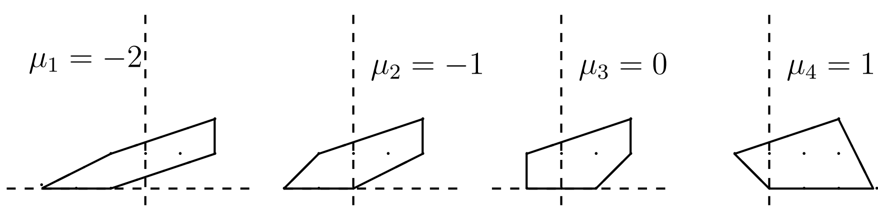

FIGURE 3. Successive Newton polygons yielding $\mu_1, \mu_2, \mu_3, \mu_4$.

We rewrite the ODE (16) as

$$A(\eta_1, x_1) + B(\eta_1, x_1)\frac{dx_1}{d\eta_1} = 0,$$

where

$$\begin{aligned}\frac{dx_1}{d\eta_1} \times \big[B_{(5,2)}\eta_1^6 x_1 + \\ B_{(3,1)}\eta_1^4 + B_{(4,1)}\eta_1^5 + B_{(5,1)}\eta_1^6\big] + \\ x_1\left[A_{(2,1)}\eta_1^2 + A_{(3,1)}\eta_1^3 + A_{(4,1)}\eta_1^4 + A_{(5,1)}\eta_1^5\right] + \\ A_{(0,0)} + A_{(1,0)}\eta_1 + A_{(2,0)}\eta_1^2 = 0.\end{aligned}$$

We do not consider the *bad side* with slope $1/3$. We only consider the *good side* of slope $-1/\mu_1 = 1/2$, since we require $\mu_1 > \mu_0$ in order to construct the Puiseux series.

By substitution, $\eta_1 = \eta_2$, $x_1 = c_1\eta_2^{\mu_1} + x_2$, the vanishing condition for the least order term yields,

$$3b_3c_1 + b_3(3b_2 - a_2) = 0 \Rightarrow c_1 = \frac{a_2}{3} - b_2 \neq 0. \tag{17}$$

In the following step, we have again two possible sides. We do not choose the bad side side of slope $1/3$. We rather consider, $\mu_2 = -1$, and the corresponding ODE

$$\begin{aligned}\frac{dx_2}{d\eta_2} \times \Big[B^{(2)}_{(4,2)}\eta_2^5 x_2 + \\ B^{(2)}_{(2,1)}\eta_2^3 + B^{(2)}_{(3,1)}\eta_2^4 + B^{(2)}_{(4,1)}\eta_2^5\Big] + \\ x_2\Big[A^{(2)}_{(1,1)}\eta_2 + A^{(2)}_{(2,1)}\eta_2^2 + A^{(2)}_{(3,1)}\eta_2^3 + A^{(2)}_{(4,1)}\eta_2^4\Big] + \\ A^{(2)}_{(0,0)} + A^{(2)}_{(1,0)}\eta_2 + A^{(2)}_{(2,0)}\eta_2^2 = 0,\end{aligned}$$

whose Newton Polygon is shown in Fig. 3. By substitution, $\eta_2 = \eta_3, x_2 = c_2\eta_3^{\mu_2} + x_3$, by imposing a vanishing least order term we obtain

$$\begin{aligned} 0 &= \frac{a_2}{3}\left(\frac{a_2}{3} - b_2\right) - b_3(a_1 - 3b_1) + 3b_3c_2 \\ &\Rightarrow \\ c_2 &= \frac{a_1}{3} - b_1 + \frac{a_2}{27b_3}(3b_2 - a_2). \end{aligned} \tag{18}$$

For each inductive step we choose one of two sides, avoiding the bad side whose slope is $1/3$, because a Puiseux series can only be produced by an increasing sequence if exponents $\mu_{i+1} > \mu_i$. See a similar terminology in [1]. Therefore, we choose the side with slope $-1/\mu_i$ where,

$$\mu_i = i - 3. \tag{19}$$

This good side has always lowest vertex $A^{(i)}_{(i,0)}$.

We substitute

$$\eta_i = \eta_{i+1}, \quad x_i = c_i\eta_{i+1}^{i-3} + x_{i+1},$$

in

$$\frac{dx_i}{d\eta_i} \times \Big[B^{(i)}_{(4,2)}\eta_i^4 x_i + B^{(i)}_{(1,1)}\eta_i^2 + B^{(i)}_{(2,1)}\eta_i^3 + B^{(i)}_{(3,1)}\eta_i^4 + \cdots + B^{(i)}_{(i+3,1)}\eta_i^{i+4}\Big] + x_i\Big[A^{(i)}_{(0,1)} + A^{(i)}_{(1,1)}\eta_i + A^{(i)}_{(2,1)}\eta_i^2 + A^{(i)}_{(3,1)}\eta_i^3 + \cdots + A^{(i)}_{(6-i+3,1)}\eta_i^{i+3}\Big] + A^{(i)}_{(i,0)}\eta_i^i + \cdots + A^{(i)}_{(2i+3,0)}\eta_i^{2i+3} = 0.$$

Arising from $A^i_{(i,0)} = 0$, we get a linear relation which determines that for then we get a explicitly c_i, as follows:

$$3b_3c_i + R_i = 0 \Rightarrow c_i = -\frac{R_i}{3b_3},$$

where R_i is a sum of rational relations on the system coefficients a_0, a_1, a_2, b_0, b_1, b_2, b_3 whose denominators are b_3^n.

After substitution we get, we get

$$\frac{dx_{i+1}}{d\eta_{i+1}} \times \Big[B^{(i+1)}_{(4,2)}\eta_{i+1}^4 x_{i+1} + B^{(i+1)}_{(1,1)}\eta_i^2 + B^{(i+1)}_{(2,1)}\eta_{i+1}^3 + B^{(i+1)}_{(3,1)}\eta_{i+1}^4 + \cdots + B^{(i+1)}_{(i+4,1)}\eta_{i+1}^{i+5}\Big] + x_{i+1}\Big[A^{(i+1)}_{(0,1)} + A^{(i+1)}_{(1,1)}\eta_i + A^{(i+1)}_{(2,1)}\eta_{i+1}^2 + A^{(i+1)}_{(3,1)}\eta_{i+1}^3 + \cdots + A^{(i+4)}_{(i+4,1)}\eta_i^{i+4}\Big] + A^{(i+1)}_{(i+1,0)}\eta_{i+1}^{i+1} + \cdots + A^{(i+1)}_{(2i+5,0)}\eta_{i+1}^{2i+5} = 0.$$

The process of construction of the $(i+1)-$th Newton polygon from the $i-$th polygon can be illustrated schematically as a directed graph shown in Fig. 4. Here, the new vertices that must be added in the $(i+1)-$th Newton polygon, appear as squares. The vertex $(i,0)$ should be suppressed. Circled vertices indicate the presence of the coefficient $B^{(i)}_{(\cdot,\cdot)}$ in the corresponding monomial. Arrows are marked between two vertices (a,b) and (d,c) if during the substitution the monomial with powers (a,b) contributes with new terms for the monomial corresponding to (c,d). The arrow from $(0,1)$ to $(i,0)$ corresponds to the good side. Take for instance, $i=3$. There is only good side corresponding to $\mu_3 = 3-3 = 0$. See the Newton polygon in Fig. 3. Hence, $\eta_3 = \eta_4, x_3 = c_3\eta_4^0 + x_4$.

$$\begin{aligned} 0 &= 3b_3c_3 + R_3 \\ &\Rightarrow \\ c_3 &= \frac{a_0}{3} - b_0 - \frac{a_1}{9b_3}(a_2 - b_2) - \frac{a_2}{81b_3^2}\left(-18b_1b_3 - 2a_2b_2 + 6b_2^2\right). \end{aligned}$$

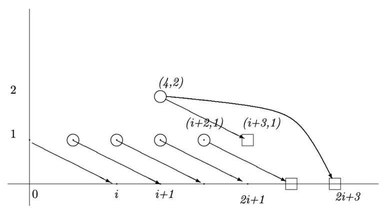

FIGURE 4. Inductive construction of Newton polygons.

Let us justify Assumption 1. If we had chosen $c_0 = 0$ instead of $c_0 = -b_3$, then straightforward calculations of the successive least order term yields,

$$b_3 c_1 = 0 \Rightarrow c_1 = 0 \Rightarrow b_3 c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow \dots,$$

which lead to the Puiseux series of the trivial solution $x \equiv 0$.

We can apply Lemma 4.1. Hence $\deg_x F \leq 1$.

Now we calculate the order in y of a suitable invariant algebraic curve. To find an expansion of non-trivial solutions along $\xi = 0$, with $\xi = 1/x$, in equation (5), we adopt the following Puiseux series expansion:

$$y(\xi) = c_0 \xi^{\mu} + \sum_{k=1}^{\infty} c_k \xi^{\frac{k}{n} + \mu}, \tag{20}$$

where $\mu = n/l_0$ and $1/\mu$ is one of many possible slopes of the corresponding Newton polygon, and l_0, n are relatively prime integers. For equation (5) the Newton polygon is a triangle rectangle whose only oblique side is the hypothenuse, see Fig. 5. When we consider equation (5), the only slope to be taking into account is $-1/\mu_0 = 3$. See Fig. 6. Under substitution $\xi = \xi_1$, $y = c_0 \xi_1^{-1/3} + y_1$, by imposing a vanishing condition the least order term, then we get

$$c_0 + b_3 c_0^4 = 0, \qquad c_0 = 0, \ \frac{-1}{\sqrt[3]{b_3}}, \ \frac{\omega}{\sqrt[3]{b_3}}, \ \frac{\overline{\omega}}{\sqrt[3]{b_3}} \in \mathbb{C} \tag{21}$$

where ω is a generating cubic root of -1. The value $c_0 = 0$ corresponds to the value at infinity of the trivial solution $y(x) \equiv 0$. Furthermore, we obtain the

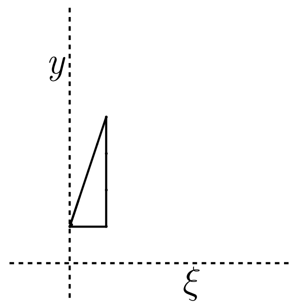

FIGURE 5. Newton Polygon of (5)

ODE, $i=1$

$$\begin{aligned}\left(A_{(0,0)}+A_{(0,1)}y_1\right)+\left(A_{(1/3,0)}+A_{(1/3,1)}y_1+A_{(1/3,2)}y_1^2\right)\xi_1^{1/3}+\\+\left(A_{(2/3,0)}+A_{(2/3,1)}y_1+A_{(2/3,2)}y_1^2+A_{(2/3,3)}y_1^3\right)\xi_1^{2/3}+\\+\left(A_{(1,1)}y_1+A_{(1,2)}y_1^2+A_{(1,3)}y_1^3+A_{(1,4)}y_1^4\right)\xi_1+\\+\left[B_{(1/3,1)}\xi_1^{4/3}+B_{(2/3,1)}\xi_1^{5/3}+B_{(1,1)}\xi_1^2\right]\frac{dy_1}{d\xi_1}+\\+\left[B_{(2/3,2)}\xi_1^{5/3}+B_{(1,2)}\xi_1^2\right]y_1\frac{dy_1}{d\xi_1}+B_{(1,3)}\xi_1^2y_1^2\frac{dy_1}{d\xi_1}=0\end{aligned} \tag{22}$$

whose Newton polygon is shown in Fig. 6. The calculation of the unique good side yields $\mu_1=0$. The substitution $\xi_1=\xi_2$, $y_1=c_1\xi_2^0+y_2$ yields

$$3b_3c_1+b_2-\frac{a}{3}\ \Rightarrow\ c_1=\frac{a_2/3-b_2}{3b_3}.$$

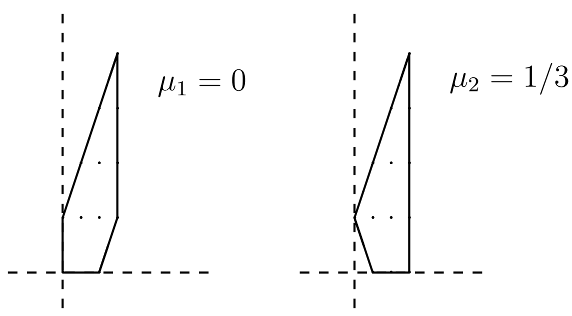

FIGURE 6. First Newton polygons for equation (5).

In general for $i \geq 2$, the only good side of the Newton polygon yields

$$\mu_i = \frac{i-1}{3}.$$

Thus, we substitute $\xi_i = \xi_{i+1}$, $y_i = c_i \xi_{i+1}^{\mu_i} + y_{i+1}$. For the least order term

$$(A_{(0,1)} c_i + R_i) \xi_{i+1}^{\mu_i} = 0 \Rightarrow c_i = -R_i/3.$$

We again show schematically the process to obtain the $(i+1)-$th Newton polygon from the $i-$th polygon as a directed graph shown in Fig. 7. The new vertices that must be added in the $(i+1)-$th Newton polygon, appear as squares. The vertex that must be suppressed is marked with a cross. Circled vertices indicate the presence of the coefficient $B^{(i)}_{(\cdot,\cdot)}$ in the corresponding monomial. Arrows are marked between two vertices (a, b) and (d, c) if during the substitution the monomial with powers (a, b) contributes with new terms for the monomial corresponding to (c, d). The arrow from $(0, 1)$ to $((i-1)/3, 0)$ corresponds to the good side. We have deleted intermediate arrows so that our drawing is clean.

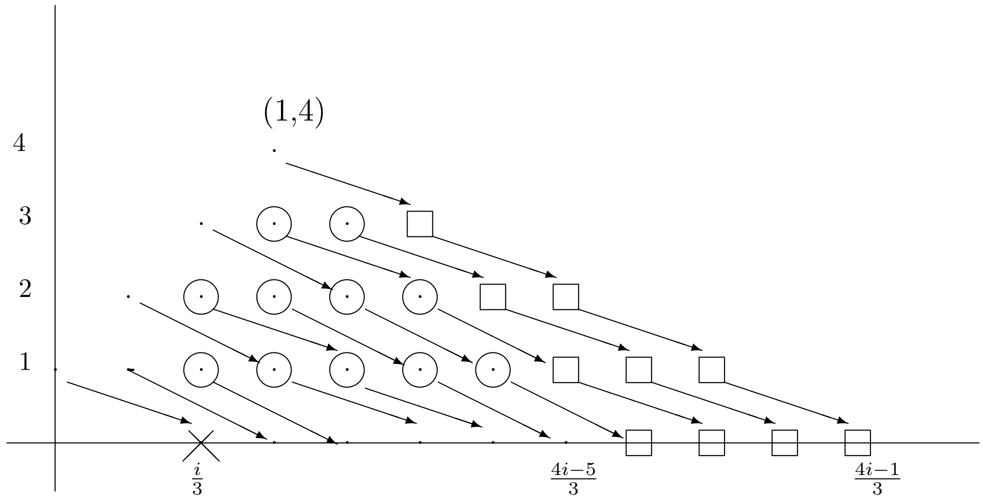

FIGURE 7. Constructive procedure of Newton polygons for equation (5).

Therefore, the 3 different values of c_0 give rise to three different branches or Puiseux series.

Therefore, by Lemma 4.1 we have $\deg_y F \leq 3$.

This ends the proof of Theorem 2.1

4.2. Rational solution for higher degree ODE

Let us consider the ODE (7). At $y = \infty$ we adopt the rational change of variable, $\eta = 1/y$. Thus (7) becomes

$$\left(x\eta^n + b_0\eta^{n-1} + \cdots + b_{n-1}\eta + b_n\right)\frac{dx}{d\eta} + x\left(a_0\eta^{n-1} + \cdots + a_{n-2}\eta + a_{n-1}\right) = 0, \tag{23}$$

whose Newton polygon is shown in Fig. 8.

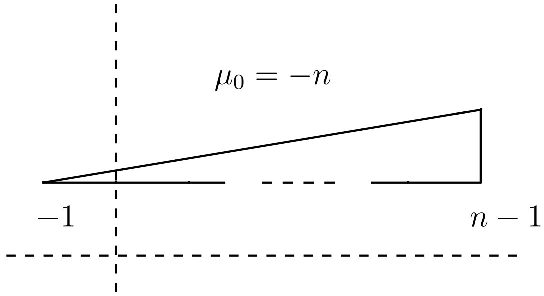

FIGURE 8. Newton polygon for (23)

Here $\mu_0 = -n$, and by the change of coordinates $\eta = \eta_1, x = c_0\eta_1^{-n} + x_1$ we get

$$\begin{aligned} &\eta_1^n x_1 \frac{dx_1}{d\eta_1} + \\ &\eta_1 \frac{dx_1}{d\eta_1}\left(\beta_{n-1}^{(1)}\eta_1^{n-1} + \dots \beta_1^{(1)}\eta_1 + \beta_0^{(1)}\right) + \\ &+ x_1\left(\alpha_0^{(1)}\eta_1^{n-1} + \dots + \alpha_{n-1}^{(1)} + \frac{\alpha_n^{(1)}}{\eta_1}\right) + \\ &+ \left(\gamma_0^{(1)}\eta_1^{-1} + \dots + \gamma_{n-1}^{(1)}\eta_1^{-n}\right) = 0. \end{aligned}$$

where

$$\begin{aligned} \alpha_k^{(1)} =& a_k, \quad \beta_k^{(1)} = b_k, \quad \gamma_k^{(1)} = a_k - nb_k, \qquad k = 0, \dots, n-1; \\ \alpha_n^{(1)} =& b_n. \end{aligned}$$

In general, the successive Newton polygons have a basis that is translated towards the right. We get $\mu_1 = -n+1, \mu_2 = -n+2, \dots$. See Fig. 9. Under the change of coordinates:

$$\eta_{i-1} = \eta_i, x_{i-1} = c_{i-1}\eta_i^{\mu_{i-1}} + x_i$$

for $i = 2, 3, \dots, n-1$, we get

$$\begin{aligned} \eta_i^n \frac{dx_i}{d\eta_i} + \eta_i \frac{dx_i}{d\eta_i}&\left(\beta_{n-1}^{(i)}\eta_i^{n-1} + \dots \beta_1^{(i)}\eta_i + \beta_0^{(i)}\right) + \\ +x_i&\left(\alpha_0^{(i)}\eta_i^{n-1} + \dots + \alpha_{n-1}^{(i)} + \alpha_n^{(i)}\eta_i^{-1}\right) + \\ &+ \left(\gamma_0^{(i)}\eta_i^{i-1} + \dots + \gamma_{n-1}^{(i)}\eta_i^{i-n}\right) = 0. \end{aligned} \tag{24}$$

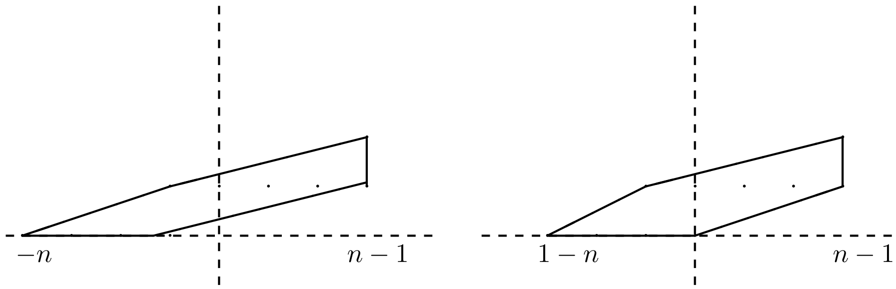

FIGURE 9. Successive Newton pentagons for $i = 1, 2$ in (24).

We have unique determination for the coefficients, c_i, of the Laurent-Puiseux series of x at $\eta = 0$ or $y = \infty$, as follows,

$$c_i \alpha_n^{(i)} + \gamma_{n-1}^{(i)} = 0. \tag{25}$$

For instance, in order to calculate the first coefficients $c_0, \dots, c_n$, the coefficients $\alpha_n^{(i)}, \gamma_{n-1}^{(i)}$, are defined recursively by the following relations:

$$\begin{aligned}
\alpha_k^{(2)} =& \alpha_k^{(1)}, \quad \beta_k^{(2)} = \beta_k^{(1)}, \\
\gamma_k^{(2)} =& c_1 \alpha_k^{(1)} - (n-1) c_1 \beta_k^{(1)} + \gamma_{k-1}^{(1)}, \quad k = 0, \dots, n-2; \\
\alpha_n^{(2)} =& \alpha_n^{(1)} - (n-1) c_1, \qquad \alpha_{n-1}^{(2)} = \alpha_{n-1}^{(1)}, \qquad \beta_{n-1}^{(2)} = \beta_{n-1}^{(1)} + c_1; \\
\gamma_{n-1}^{(2)} =& c_1 \alpha_0 - (n-1) c_1 \left(\beta_{n-1}^{(1)} + c_1 \right) + \gamma_{n-2}^{(1)}, \qquad \gamma_0^{(2)} = c_1 \alpha_0^{(1)} - (1-n) c_1 \beta_0^{(1)};
\end{aligned}$$

and for $i = 2, \dots, n$:

$$\begin{aligned}
\alpha_{n-i}^{(i+1)} =& \alpha_{n-i}^{(i)} - c_i, \qquad \beta_{n-i}^{(i+1)} = \beta_{n-i}^{(i)} - c_i (n-i), \\
\alpha_n^{(i+1)} =& \alpha_n^{(i)}, \qquad \alpha_k^{(i+1)} = \alpha_k^{(i)}, \quad \beta_k^{(i+1)} = \beta_k^{(i)}, \quad k = 2, \dots, \widehat{n-i}, \dots, n-1, \\
\gamma_{n-1}^{(i+1)} =& c_i \left((i-n) \beta_{n-1}^{(i)} + \alpha_{n-1}^{(i)} \right) + \gamma_{n-2}^{(i)} \\
\gamma_0^{(i+1)} =& c_i \left(\alpha_0^{(i)} - (n-i) \beta_0^{(i)} \right).
\end{aligned} ,$$

The unique determination (25) implies that $\deg F(x, y) \leq 1$.

We estimate the degree in y of an algebraic solution of the ODE (7). At $x = \infty$ we adopt the rational change of variable, $\xi = 1/x$. Thus (7) becomes

$$\xi^2 \frac{dy}{d\xi} (a_0 + a_1 y + \cdots + a_{n-1} y^{n-1}) y + \xi (b_0 + b_1 y + \cdots + b_n y^n) = 0 \tag{26}$$

which, under the substitution $y = c_0 \xi_1^{-1/n} + y_1$, gives the least order term

$$c_0 (b_n + c_0^n) = 0 \quad \Rightarrow \quad c_0 = 0, \, c_0 = \frac{\omega_n^i}{\sqrt[n]{|b_n|}}, \, i = 1 \dots, n,$$

where ω_n^i are the complex $n-$roots of sign (b_n). Furthermore, we get an equation in ξ_1, y_1, whose Newton polygon is shown in Fig. 10.

Along the following steps there is a unique determination of the coefficients c_i under the following substitution:

$$\xi_i = \xi_{i+a}, \qquad y_i = c_i \xi_{i+1}^{(i-1)/n} + y_{i+1},$$

which is similar to (25).

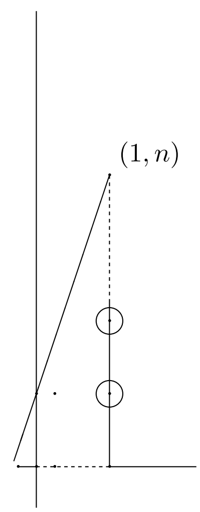

FIGURE 10. Newton polygon for (26).

The successive Newton polygons are shown in Fig. 11. We avoid detailed description since the main ideas of the proof where already exposed in the case $n = 3$.

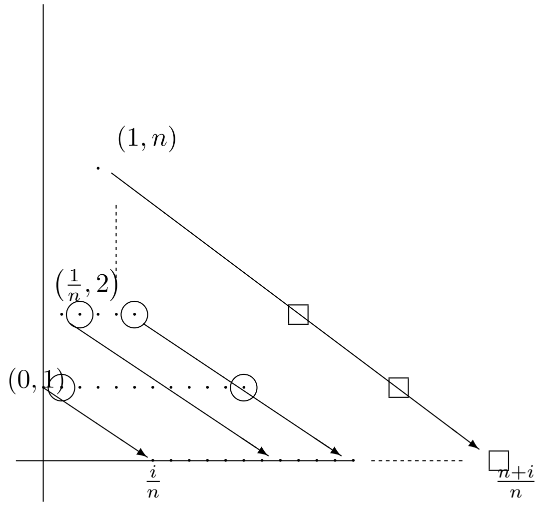

FIGURE 11. $(i+1)-$st Newton polygon fron $i-$th polygon.

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(Recibido en junio de 2021. Aceptado en julio de 2022)

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