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A Note on the Range of a Derivation

Una nota sobre el rango de una derivada

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Abstract. Let H be a separable infinite dimensional complex Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators on H into itself. Given $A, B \in L(H)$, define the generalized derivation $\delta_{A,B} \in L(L(H))$ by $\delta_{A,B}(X) = AX - XB$. An operator $A \in L(H)$ is P-symmetric if $AT = TA$ implies $AT^* = T^*A$ for all $T \in C_1(H)$ (trace class operators). In this paper, we give a generalization of P-symmetric operators. We initiate the study of the pairs (A, B) of operators $A, B \in L(H)$ such that $\overline{R(\delta_{A,B})}^{W^*} = \overline{R(\delta_{A^*,B^*})}^{W^*}$, where $\overline{R(\delta_{A,B})}^{W^*}$ denotes the ultraweak closure of the range of $\delta_{A,B}$. Such pairs of operators are called generalized P-symmetric. We establish a characterization of those pairs of operators. Related properties of P-symmetric operators are also given.

Key words and phrases. Generalized derivation, Fuglede-Putnam property, D-symmetric operator, P-symmetric operator, Compact operator.

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Resumen. Sea H un espacio de Hilbert separable sobre los complejos y denote por $L(H)$ al álgebra de los operadores acotados de H es sí mismo. Dados $A, B \in L(H)$, defina la derivada generalizada $\delta_{A,B} \in L(L(H))$ como $\delta_{A,B}(X) = AX - XB$. Un operador $A \in L(H)$ es P-simétrico si la condición $AT = TA$ implica que $AT^* = T^*A$ para todo $T \in C_1(H)$ (los operadores de clase de traza). En este artículo presentamos una generalización de los operadores P-simétricos. En este artículo estudiamos pares (A, B) de operadores $A, B \in L(H)$ tales que $\overline{R(\delta_{A,B})}^{W^*} = \overline{R(\delta_{A^*,B^*})}^{W^*}$, donde $\overline{R(\delta_{A,B})}^{W^*}$ denota la clausura ultradébil del rango $\delta_{A,B}$. A esta clase de operadores los llamamos operadores P-simétricos generalizados. En este artículo damos una caracterización de esta clase de pares de operadores y presentamos propiedades de los operadores P simétricos generalizados.

Palabras y frases clave. Derivada generalizada, propiedad de Fuglede-Putnam, operador D-simétrico, operador P-simétrico, operador compacto.

1. Introduction and Notation

Let H be a separable infinite dimensional complex Hilbert space, and let $L(H)$ denote the algebra of all bounded linear operators acting on H into itself. Given $A, B \in L(H)$, we define the generalized derivation $\delta_{A,B} : L(H) \longrightarrow L(H)$ by $\delta_{A,B}(X) = AX - XB$, we simply write δ_A for $\delta_{A,A}$.

An operator $A \in L(H)$ is called D-symmetric if $\overline{R(\delta_A)} = \overline{R(\delta_{A^*})}$, where $\overline{R(\delta_A)}$ denotes the norm closure of the range $R(\delta_A)$ of δ_A. Clearly A is D-symmetric if and only if $\overline{R(\delta_A)}$ is a self-adjoint subspace of $L(H)$. Examples of D-symmetric operators include the normal operators, isometries and hyponormal weighted shifts. The properties of D-symmetric operators have been considered in a number of papers (see for example [1], [5], [6], [3], [4], [7], [9], [10] and [11]).

In [2] it is proved that if A is D-symmetric, then A has the property $(FP)_{C_1(H)}$, that is, $AT = TA$ implies $A^*T = TA^*$ for every $T \in C_1(H)$ (trace class operators). Operators A satisfying the property $(FP)_{C_1(H)}$ are termed P-symmetric.

S. Bouali and J. Charles introduced P-symmetric operators, and they gave some basic properties of this class of operators ([5], [6]). In this paper, we study the pairs (A, B) of operators $A, B \in L(H)$ with the property that $\overline{R(\delta_{A,B})}^{W^*} = \overline{R(\delta_{A^*,B^*})}^{W^*}$, where $\overline{R(\delta_{A,B})}^{W^*}$ denotes the ultraweak closure of $R(\delta_{A,B})$. We call such pairs of operators generalized P-symmetric. We give a characterization and some properties of P-symmetric pairs (A, B) of bounded linear operators A and B.

The present paper investigates also the class of P-symmetric operators. We prove that if A is a rationally cyclic subnormal operator, then A is D-symmetric. A well-known result of S. Bouali and J. Charles [5] says that an operator $A \in L(H)$ is P-symmetric if and only if $\overline{R(\delta_A)}^{W^*}$ is self-adjoint. So, for a P-symmetric operator A we consider the following sets: $\mathcal{C}_\circ(A) = \{C \in L(H) : CL(H) + L(H)C \subset \overline{R(\delta_A)}^{W^*}\}$, $\mathcal{I}_\circ(A) = \{Z \in L(H) : ZR(\delta_A) + R(\delta_A)Z \subset \overline{R(\delta_A)}^{W^*}\}$, $\mathcal{B}_\circ(A) = \{B \in L(H) : R(\delta_B) \subset \overline{R(\delta_A)}^{W^*}\}$. We establish some new properties concerning these subalgebras of $L(H)$. We present some examples and counterexamples of P-symmetric and essentially D-symmetric operators.

We conclude this section with some notation and terminology. An operator on H will always be understood to be a bounded linear transformation from H

into itself. The algebra of all bounded linear operators on H will be dented by $L(H)$. Given $A \in L(H)$, we shall denote the kernel, the orthogonal complement of the kernel and the closure of the range of A by $\ker(A)$, $\ker^{\perp}(A)$ and $\overline{R(A)}$, respectively. The spectrum of A will be denoted by $\sigma(A)$, and the restriction of A to an invariant subspace M will be denoted by $A|M$. A closed subspace M of H is said to reduce A if $AM \subseteq M$ and $AM^{\perp} \subseteq M^{\perp}$, that is, if M and $M^{\perp}$ are both invariant under A. For $\lambda \in \mathbb{C}$, let $\overline{\lambda}$ denote the complex conjugate of λ. A complex number λ is said to be a reducing eigenvalue for A if $\ker(A - \lambda I)$ reduces A, where I is the identity operator. For vectors x and y in H we denote by $x \otimes y$ the rank-one operator defined by $x \otimes y(z) = \langle z, y\rangle x$ for all $z \in H$.

Let $K(H)$, $C_1(H)$ and $F(H)$ be respectively the ideal of compact operators, the ideal of trace class operators and the ideal of finite rank on H. The trace function is defined on $C_1(H)$ by $tr(T) = \sum_n \langle Te_n, e_n\rangle$, where (e_n) is any complete orthonormal sequence in H. The weakly continuous linear functionals on $L(H)$ are those of the form $f_T(X) = tr(XT)$, where $T \in F(H)$. The ultraweakly continuous linear functionals on $L(H)$ are those of the form $f_T(X) = tr(XT)$, where $T \in C_1(H)$.

For $A \in L(H)$, let $[A]$ denote the image of A under the canonical projection of $L(H)$ onto the Calkin algebra $L(H)|K(H)$. An operator $A \in L(H)$ is said to be essentially normal if $A^*A - AA^*$ is compact, equivalently, if $[A]$ is a normal element of the Calkin algebra.

Let $\mathcal{B}$ be a Banach space and let $\mathcal{S}$ be a subspace of $\mathcal{B}$. Denote by $\mathcal{B}'$ the set of all bounded linear functionals. We define the annihilator of $\mathcal{S}$ by

$$\mathcal{S}^{\circ} = \{f \in \mathcal{B}' : f(s) = 0 \text{ for all } s \in \mathcal{S}\}.$$

Any other notation will be explained as and when required.

2. Preliminaries

Definition 2.1. An operator $A \in L(H)$ is called D-symmetric, if $\overline{R(\delta_A)} = \overline{R(\delta_{A^*})}$.

Theorem 2.2. [1] *If $A \in L(H)$, then the following two statements are equivalent.*

(1) A is D-symmetric.

(2) (i) $[A]$, the corresponding element of the Calkin algebra, is D-symmetric and

(ii) $AT = TA$ and $T \in C_1(H)$ implies $A^*T = TA^*$.

Definition 2.3. Let $A \in L(H)$. Then A is called P-symmetric if $AT = TA$ and $T \in C_1(H)$ implies $A^*T = TA^*$.

Theorem 2.4. [3] *Let $A \in L(H)$, then*

(1) A is P-symmetric if and only if $\overline{R(\delta_A)}^{W^*}$ *is self-adjoint.*

(2) $P(H)$ *(the set of P-symmetric operators) is self-adjoint.*

Proposition 2.5. [3] *Let* $A \in L(H)$*. If there exist nonzero vectors* $f, g \in H$ *such that*

(1) $Af = \lambda f$ *and* $A^* f \neq \overline{\lambda} f$*,*

(2) $A^* g = \overline{\lambda} g$*,*

then A is not P-symmetric.

Example 2.6. Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis for H. Let $H_\circ = \vee\{e_1, e_2, e_3\}$ denote the linear subspace of H generated by the set $\{e_1, e_2, e_3\}$. Let $A_\circ$ be defined by

$$A_\circ = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \in L(H_\circ).$$

We next define the operator $A = A_\circ \oplus I$, with respect to the decomposition $H = H_\circ \oplus H_\circ^\perp$, where I is the identity operator. It is easily verified that

$$Ae_2 = -e_2 \ , \ A^* e_2 = e_1 - e_2 + e_3 \neq -e_2 \text{ and } A^* e_3 = -e_3.$$

It follows that A is not P-symmetric.

Definition 2.7. An operator $A \in L(H)$ is called essentially D-symmetric, if $[A]$, its corresponding element of the Calkin algebra, is D-symmetric.

The following result is an immediate consequence of Theorem 2.1.

Corollary 2.8. *(i) An operator A on H is D-symmetric if and only if A is essentially D-symmetric and P-symmetric.*

(ii) An essentially normal operator A is D-symmetric if and only if A is P-symmetric.

(iii) An operator in the trace class is P-symmetric if and only if it is normal.

***Remark* 2.9.** Let $(e_n)_{n\in\mathbb{N}}$ (respectively $(e_n)_{n\in\mathbb{Z}}$) be an orthonormal basis for H. Let S be the unilateral (respectively bilateral) shift $Se_n = \omega_n e_{n+1}$ where $n \in \mathbb{N}$ (respectively $n \in \mathbb{Z}$) with nonzero weights ω_n. By taking a unitarily equivalent weighted shift, we may assume that $\omega_n = |\omega_n| > 0$.

In [6] it is shown that S is P-symmetric if and only if S satisfies the total products condition, that is,

$$\Sigma_{k=1}^{\infty} \omega_k . \omega_{k+1} . \cdots . \omega_{k+n} = \infty \quad \text{for all} \ \ n \in \mathbb{N}.$$

Example 2.10. We now present an example of an essentially D-symmetric which is not P-symmetric. We define our operator A as follows.

Let $(e_n)_{n\in\mathbb{Z}}$ be an orthonormal basis for H. Set

$$Ae_k = \frac{1}{k^2+1} e_{k+1} \text{ for all } k \in \mathbb{Z}.$$

It is obvious that A is essentially normal. Then it follows from Theorem 2.1 in [1] (which is valid in any C^*-algebra), that A is essentially D-symmetric. However the weights of A don't satisfy the total products condition, and so A is not P-symmetric.

Example 2.11. Let $(e_n)_{n\in\mathbb{Z}}$ be an orthonormal basis for H. Define $T \in L(H)$ by

$$\begin{cases} Te_{2n} = \frac{1}{2}e_{2n+1}, \\ Te_{2n+1} = 2e_{2n+2}. \end{cases}$$

Since the weights of T satisfy the total products condition, then T is P-symmetric. It follows from Lemma 2 in [11] that T is not essentially D-symmetric. On the other hand, we have that T^2 is unitary, and hence T^2 is D-symmetric. But T is not D-symmetric.

3. Main Results

Definition 3.1. Let $A, B \in L(H)$ and $\mathcal{J}$ be a two-sided ideal of $L(H)$. The pair (A, B) is said to possess the Fuglede-Putnam property $(FP)_{\mathcal{J}}$ if $AT = TB$ and $T \in \mathcal{J}$ implies $A^*T = TB^*$. i.e. $\ker(\delta_{A,B}|\mathcal{J}) \subseteq \ker(\delta_{A^*,B^*}|\mathcal{J})$.

Definition 3.2. Let $A, B \in L(H)$. The pair (A, B) of operators A and B is called P-symmetric if

$$\overline{R(\delta_{A,B})}^{W^*} = \overline{R(\delta_{A^*,B^*})}^{W^*}.$$

For $A, B \in L(H)$, let $R(\delta_{A,B})^\circ$ denotes the set of all norm-continuous linear functionals that vanish on the range $R(\delta_{A,B})$ of $\delta_{A,B}$. Note also that ${L(H)'}^{W^*}$ is the set of all ultraweakly continuous linear functionals on $L(H)$.

Lemma 3.3. *Let $A, B \in L(H)$. Then*

$$R(\delta_{A,B})^\circ = (R(\delta_{A,B})^\circ \cap K(H)^\circ) \oplus (\ker(\delta_{B,A}) \cap C_1(H)).$$

The proof of the preceding lemma is the same as the proof of Theorem 3 [12].

Theorem 3.4. *Let $A, B \in L(H)$. The pair (A, B) is P-symmetric if and only if $(A, B) \in (FP)_{C_1(H)}$ and $(B, A) \in (FP)_{C_1(H)}$.*

Proof. Observe that the assertion $\overline{R(\delta_{A,B})}^{W^*} = \overline{R(\delta_{A^*,B^*})}^{W^*}$ is equivalent to

$$R(\delta_{A,B})^\circ \cap L(H)'^{W^*} = R(\delta_{A^*,B^*})^\circ \cap L(H)'^{W^*}.$$

We get from lemma 3.1 that

$$R(\delta_{A,B})^\circ = (R(\delta_{A,B})^\circ \cap K(H)^\circ) \oplus (\ker(\delta_{B,A}) \cap C_1(H)).$$

Hence, it follows that

$$R(\delta_{A,B})^\circ \cap L(H)'^{W^*} \simeq \ker(\delta_{B,A}) \cap C_1(H).$$

Consequently, we have $\overline{R(\delta_{A,B})}^{W^*} = \overline{R(\delta_{A^*,B^*})}^{W^*}$ if and only if

$$\ker(\delta_{B,A}) \cap C_1(H) = \ker(\delta_{B^*,A^*}) \cap C_1(H).$$

This completes the proof. ☑

Theorem 3.5. *Let $A, B \in L(H)$. If there exist nonzero vectors $x, y \in H$, and some $\lambda \in \mathbb{C}$ such that*

(1) $Bx = \lambda x$ *and* $B^*x \neq \overline{\lambda}x$,

(2) $A^*y = \overline{\lambda}y$, *then the pair (A, B) is not P-symmetric.*

Proof. We must show that $\overline{R(\delta_{A,B})}^{W^*} \neq \overline{R(\delta_{A^*,B^*})}^{W^*}$. Clearly we have that $\overline{R(\delta_{A,B})}^{W^*} = \overline{R(\delta_{A^*,B^*})}^{W^*}$ if and only if

$$f_T \in R(\delta_{A,B})^\circ \Longleftrightarrow f_T \in R(\delta_{A^*,B^*})^\circ,$$

for every $T \in C_1(H)$.

It suffices to exhibit a trace class operator T for which $f_T \in R(\delta_{A,B})^\circ$ but $f_T \notin R(\delta_{A^*,B^*})^\circ$. Let us define the rank one operator $T = x \otimes y$.

Then for any $Y \in L(H)$ we have

$$\begin{aligned} f_T(\delta_{A,B}(Y)) &= tr[(AY - YB)T] = tr(YTA) - tr(YBT) \\ &= tr(Y \circ (x \otimes A^*y)) - tr(Y \circ (Bx \otimes y)) \\ &= tr(\lambda YT) - tr(\lambda YT) = 0. \end{aligned}$$

Define an operator $X \in L(H)$ by $X = y \otimes (B - \lambda)^*x$. Then, it follows that

$$\begin{aligned} f_T(\delta_{A^*,B^*}(X)) &= tr[(A^*X - XB^*)T] \\ &= tr[\{(A^* - \overline{\lambda})X - X(B^* - \overline{\lambda})\}T] \\ &= tr[(A^* - \overline{\lambda})XT] - tr[X(B^* - \overline{\lambda})T] \\ &= tr[(A^* - \overline{\lambda})y \otimes T^*(B - \lambda)^*x] - tr[(y \otimes (B - \lambda)^*x)(B - \lambda)^*(x \otimes y)]. \\ &= -\|(B - \lambda)^*x\|^2.\|y\|^2 \neq 0, \end{aligned}$$

which completes the proof. ☑

Example 3.6. Let $(e_n)_{n\geq 1}$ be an orthonormal basis for H. Let $H_\circ = \vee\{e_1, e_2, e_3\}$, and set

$$B_\circ = \begin{pmatrix} -i & i & 0 \\ 0 & 1 & 0 \\ 0 & -1 & i \end{pmatrix} \in L(H_\circ).$$

Define the operator $B \in L(H)$ by $B = B_\circ \oplus I$ with respect to the decomposition $H = H_\circ \oplus H_\circ^\perp$, and let $A = ie_2 \otimes e_2$. A straightforward computation gives $Be_3 = ie_3,\ B^*e_3 = -e_2 - ie_3 \neq -ie_3$ and $A^*e_2 = -ie_2$. Then it follows from Theorem 3.2 that (A, B) is not P-symmetric.

Theorem 3.7. *Let $A, B \in L(H)$. If A and B are P-symmetric operators such that $\sigma(A) \cap \sigma(B) = \phi$, then the pair (A, B) is P-symmetric.*

Proof. Assume that A and B are P-symmetric operators with disjoint spectra. Let $T \in \overline{R(\delta_{A,B})}^{W^*}$, then there exists a sequence $(X_\alpha)_\alpha$ of elements in $L(H)$ such that $(AX_\alpha - X_\alpha B)_\alpha$ converges in the ultra-weak topology (or the weak * operator topology) to T, in symbols we write $AX_\alpha - X_\alpha B \xrightarrow{W^*} T$.

On $H \oplus H$ consider the operators L, S and Y_α defined as

$$L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad S = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}, \quad Y_\alpha = \begin{pmatrix} 0 & X_\alpha \\ 0 & 0 \end{pmatrix}.$$

It follows that

$$\delta_L(Y_\alpha) = \begin{pmatrix} 0 & \delta_{A,B}(X_\alpha) \\ 0 & 0 \end{pmatrix} \xrightarrow{W^*} \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix} = S.$$

Hence, we get $S \in \overline{R(\delta_L)}^{W^*}$. Since A and B are P-symmetric with disjoint spectra, then we obtain from Theorem 2.6 in [3], that L is P-symmetric. Thus, there exists a sequence $(Z_\alpha)_\alpha$ in $L(H \oplus H)$ for which $\delta_{L^*}(Z_\alpha) \xrightarrow{W^*} S$. A simple calculation shows that there exists a sequence $(U_\alpha)_\alpha$ in $L(H)$ such that $\delta_{A^*,B^*}(U_\alpha) \xrightarrow{W^*} T$. Then we conclude that $\overline{R(\delta_{A,B})}^{W^*} \subseteq \overline{R(\delta_{A^*,B^*})}^{W^*}$.

The argument to verify the reverse inclusion is identical to the above and thus the proof is complete. ☑

Corollary 3.8. *Let $A, B \in L(H)$. If (A, B) is P-symmetric, then*

$$A^* R(\delta_{A,B}) + R(\delta_{A,B}) B^* \subset \overline{R(\delta_{A,B})}^{W^*}.$$

Proof. Suppose that (A, B) is P-symmetric. Then we have

$$\overline{R(\delta_{A,B})}^{W^*} = \overline{R(\delta_{A^*,B^*})}^{W^*}.$$

If X is any operator in $L(H)$ then

$$A^*\delta_{A^*,B^*}(X) = \delta_{A^*,B^*}(A^*X),$$

and

$$\delta_{A^*,B^*}(X)B^* = \delta_{A^*,B^*}(XB^*).$$

From this it follows that

$$A^*R(\delta_{A,B}) \subset A^*\overline{R(\delta_{A,B})}^{W^*} = A^*\overline{R(\delta_{A^*,B^*})}^{W^*} \subset \overline{R(\delta_{A^*,B^*})}^{W^*} = \overline{R(\delta_{A,B})}^{W^*}.$$

By the same argument as above, we prove that $R(\delta_{A,B})B^* \subset \overline{R(\delta_{A,B})}^{W^*}$. So the proof is complete. ☑

***Remark* 3.9.** Let $A, B \in L(H)$. Suppose that (A, B) is P-symmetric and let $T \in C_1(H)$ such that $AT = TB$. Then we get

$$\delta_{A,B}(AT) = 0 \Rightarrow \delta_{A^*,B^*}(AT) = 0.$$

Hence, it follows readily that

$$(A^*A - AA^*)T = T(BB^* - B^*B) = ATT^* - TT^*A = T^*TB - BT^*T = 0.$$

Then we have $\overline{R(T)}$ reduces A, and $\ker^{\perp}(T)$ reduces B, and $A|\overline{R(T)}$, $B|\ker^{\perp}(T)$ are unitarily equivalent normal operators.

Definition 3.10. An operator A on a Hilbert space H is subnormal, if there is a Hilbert space K containing H and a normal operator N on K, such that $N(H) \subseteq H$ and $A = N|H$ (the restriction of N to H).

Definition 3.11. An operator $A \in L(H)$ is called rationally cyclic if there is an $e \in H$ such that

$$H = \overline{\{r(A)e : r \in Rat(\sigma(A))\}},$$

where $Rat(\sigma(A))$ is the set of rational functions with poles off $\sigma(A)$, and e is called a rationally cyclic vector for A.

Theorem 3.12. *Let $A \in L(H)$. If A is a rationally cyclic subnormal operator, then A is D-symmetric.*

***Proof*.** Suppose that A is a rationally cyclic subnormal operator, and let $T \in C_1(H)$ be such that $AT = TA$. It follows from Yoshino's result [14] that T is subnormal. It is well-known that any compact subnormal operator is normal. We get that T is normal . Hence, it follows from Fuglede-Putnam theorem that $AT = TA$ implies $AT^* = T^*A$. Thus, A is P-symmetric.

Since A is a rationally cyclic subnormal operator, it results from Shaw and Berger's Theorem [2] that $[A]$ is normal. Then A is essentially D-symmetric. This proves that A is D-symmetric. ☑

Definition 3.13. Let $A \in L(H)$. The commutant $\{A\}^{'}$ of A is defined by:

$$\{A\}^{'} = \{B \in L(H) : AB = BA\}.$$

The bicommutant $\{A\}^{''}$ of A is defined by:

$$\{A\}^{''} = \{C \in L(H) : CB = BC \text{ ,for all } B \in \{A\}^{'}\}.$$

***Remark* 3.14.** It is known that A is P-symmetric if and only if $\overline{R(\delta_A)}^{W^*}$ is a self-adjoint subspace of $L(H)$. Hence, for a P-symmetric operator A, it is natural to introduce the following subalgebras:

$$\mathcal{C}_\circ(A) = \{C \in L(H) : CL(H) + L(H)C \subset \overline{R(\delta_A)}^{W^*}\},$$

$$\mathcal{I}_\circ(A) = \{Z \in L(H) : ZR(\delta_A) + R(\delta_A)Z \subset \overline{R(\delta_A)}^{W^*}\},$$

$$\mathcal{B}_\circ(A) = \{B \in L(H) : R(\delta_B) \subset \overline{R(\delta_A)}^{W^*}\}.$$

It is well-known that if H is of finite dimension ([13]), then

$$\mathcal{C}_\circ(A) = \{0\}, \ \ \mathcal{I}_\circ(A) = \{A\}^{'}, \ \ \mathcal{B}_\circ(A) = \{A\}^{''}.$$

In [4] S. Bouali and J. Charles proved that if A is normal with finite spectrum then

$$\mathcal{C}_\circ(A) = \{0\}, \ \ \mathcal{I}_\circ(A) = \{A\}^{'}, \ \ \mathcal{B}_\circ(A) = \{A\}^{''}.$$

Theorem 3.15. *Let $A \in L(H)$ be a P-symmetric operator. Then the following statements are equivalent:*

(i) $K(H) \subset \overline{R(\delta_A)}^{W^*}$.

(ii) *A has no reducing eigenvalues.*

(iii) $K(H) \subset \mathcal{C}_\circ(A)$.

***Proof*.** $(i) \Rightarrow (ii)$ If A has a reducing eigenvalue, then $(Sx, x) = 0$ for all $S \in \overline{R(\delta_A)}^{W^*}$, and some non-zero $x \in H$. It follows that $K(H) \not\subset \overline{R(\delta_A)}^{W^*}$.

$(ii) \Rightarrow (i)$ Suppose that $K(H) \not\subset \overline{R(\delta_A)}^{W^*}$. Then there exist a non-zero T in the trace class such that f_T vanishes on $R(\delta_A)$, that is, $AT = TA$. Since A is P-symmetric, then $AT = TA$ and $T \in C_1(H)$ implies $AT^* = T^*A$. It follows that $A(T + T^*) = (T + T^*)A$ and $A(T - T^*) = (T - T^*)A$. Hence A commutes with a non-zero trace class operator. Consequently, A has a finite dimensional reducing subspace $H_\circ$. Clearly, $A|H_\circ$ is P-symmetric, and so $A|H_\circ$ is normal by Corollary 2.1 (iii). Thus T has a reducing eigenvalue.

The remaining equivalence $(ii) \Leftrightarrow (iii)$ is obvious.

We include the following properties for the sake of completeness. We omit the proofs which are based entirely on those of C. Gupta and P. Ramanujan for D-symmetric operators [8]. ☑

Corollary 3.16. *Let $A \in L(H)$ be a P-symmetric operator with $\mathcal{C}_\circ(A) = K(H)$. Then we have the following assertions:*

- (i) *A is essentially normal.*
- (ii) *A has no reducing eigenvalue.*
- (iii) *Each projection in $\overline{R(\delta_A)}^{W^*}$ is finite dimensional.*
- (iv) $\mathcal{I}_\circ(A) = \{Z \in L(H) : AZ - ZA \in K(H)\}$.
- (v) $\mathcal{B}_\circ(A)|\mathcal{C}_\circ(A)$ *is a commutative C^*-algebra of the Calkin algebra $L(H)|K(H)$.*

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