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On Stable Sampling and Interpolation in Bernstein Spaces

Muestreo e Interpolación Estables en Espacios de Bernstein

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Abstract. We define the concepts of stable sampling set, interpolation set, uniqueness set and complete interpolation set for a quasinormed space of functions and apply these concepts to Paley-Wiener spaces and Bernstein spaces. We obtain a sufficient condition on a uniformly discrete set to be an interpolation set based on a lemma of convergence of series in Paley-Wiener spaces. We also obtain a result of transference, Kadec type, of the property of being a stable sampling set, from a set with this property to other uniformly discrete set, which we apply to Bernstein spaces.

Key words and phrases. Quasinormed spaces, stable sampling set, interpolation set, Paley-Wiener spaces, Bernstein spaces.

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Resumen. Definimos los conceptos de conjunto de muestreo estable, conjunto de interpolación, conjunto de unicidad y conjunto de interpolación completa para los espacios quasinormados de funciones, y aplicamos estos conceptos a los espacios de Paley-Wiener y a los espacios de Bernstein. También obtenemos una condición suficiente para saber cuando un conjunto uniformemente discreto es un conjunto de interpolación, estando basada esta condición en un lema de convergencia de series en espacios de Paley-Wiener. Además, obtenemos un resultado de transferencia, tipo Kadec, sobre la propiedad de ser un conjunto de muestreo estable, de un conjunto que tiene esta propiedad a otro cunjunto que sea uniformemente discreto, y aplicamos este resultado a los espacios de Bernstein.

Palabras y frases clave. Espacios quasinormados, conjunto de muestreo estable, conjunto de interpolación, espacios de Paley-Wiener, espacios de Bernstein.

1. Introduction

There is a large number of contributions in stable sampling and interpolation theory in Paley-Wiener and Bernstein spaces, directly and as consequence of the research in frame theory, and Riesz sequences and bases theory in L^p spaces. Since the celebrated work of Claude E. Shannon ([22]), considered the father of Information Theory, research in sampling and interpolation theory (regular and irregular) has been developed in an exponential way. The contributions by A. Beurling ([4] and [5]), R. Paley and N. Wiener ([15]), M. Plancherel and G. Pólya ([17]), and H. J. Landau ([9]) are classical. See [7] for more details.

More recently we have (i.a.) the spectacular works by R. H. Torres (see [23], including sampling and interpolation in Besov spaces), Y. I. Lyubarskii and K. Seip (see [11] on the characterization of the complete interpolating sequences for Paley-Wiener spaces), B. Matei and Y. Meyer ([12], as an example), A. Olevskii and A. Ulanovskii ([13], [14]), and K. Flornes (see [6] on interpolation and sampling results in quasinormed Bernstein spaces, with quasinorm of parameter $0 < p \leq 1$).

As said before, research in frame theory, and Riesz sequences and bases theory in L^p spaces has allowed to obtain advances in stable sampling and interpolation theory. We have a particularly important example of this in 1964, when M. I. Kadec proved his celebrated theorem:

Theorem 1.1 (Kadec-1/4 Theorem). *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers. Suppose that*

$$|\lambda_n - n| \leq L < \frac{1}{4} \text{ for every } n \in \mathbb{Z}.$$

Then the set of exponential functions $\left\{e^{i\,\lambda_n\,t}\right\}_{n\in\mathbb{Z}}$ is a Riesz basis in the Hilbert space $L^2\left((-\pi,\,\pi)\right)$.

This result says, in terms of sampling and interpolation theory, that $\mathbb{Z}$ is a complete interpolation set (this is, both sampling and interpolation set) for the Paley-Wiener space $E^2_{(-\pi,\,\pi)}$, and that every L-perturbation of $\mathbb{Z}$ also verifies it whenever $L < \frac{1}{4}$ (see Definition 2.2).

The bound 1/4 is sharp, and Theorem 1.1 improves a previous very important result by R. Paley and N. Wiener, where the bound is $\frac{1}{\pi^2}$ ([15], page 113). In 1974 S. A. Avdonin obtained a generalization of Theorem 1.1 using a certain type of mean of the values λ_n's ([3]).

The study of sets of exponential functions is essential, and Kadec-1/4 Theorem has been generalized in several ways to L^p spaces and to sequences $(\lambda_n)_{n\in\mathbb{Z}}$ of complex numbers, in order to prove that the familiy $\left\{e^{i\,\lambda_n\,t}\right\}_{n\in\mathbb{Z}}$, is complete (see for example [10], [19], [24] and [20]). In addition, the Riesz basis problem in the Paley-Wiener space $E^2_{(-\pi,\,\pi)}$ has been proposed for non-exponential basis.

In this sense several important results analogous to Kadec-1/4 Theorem have been obtained for sets of sinc functions, involving the Lamb-Oseen constant (see [1] and [2]).

The aim of this paper is to establish some results of transference in irregular stable sampling and interpolation theory in Paley-Wiener spaces and Bernstein spaces. We state and prove a lemma of convergence in Paley-Wiener spaces and we obtain as consequence a sufficient condition on a uniformly discrete set to be an interpolation set. We also establish a relationship, similar to Kadec 1/4-Theorem, between the stable sampling sets of a given Bernstein space, even in a more general context, obtaining explicit sampling bounds using the Bernstein-Pesenson inequality and type Nikolskii (see [16] and [21], respectively).

We now fix some notation. We will denote by $\mathfrak{F}(\mathbb{R}^n, \mathbb{C})$ (respectively, $\mathfrak{F}(\mathbb{C}^n, \mathbb{C})$) the set of the complex functions defined in $\mathbb{R}^n$ (respectively, in $\mathbb{C}^n$), by $\mathcal{H}(\mathbb{C}^n)$ the set of holomorphic functions whose domain is $\mathbb{C}^n$, by $\mathcal{S}(\mathbb{R}^n)$ the set of Schwartz functions, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. We will denote the adherence of a set A in a topological space by $\overline{A}$. Given a Lebesgue measurable set $K \subseteq \mathbb{R}^n$, we denote by $m_n(K)$ its Lebesgue measure. Given $A \subseteq \mathbb{R}^n$, we denote the indicator function of A with respect to $\mathbb{R}^n$ by χ_A. If $\| \;\|$ is a quasinorm, we denote by $\tau_{\| \;\|}$ its associated topology. A function $h : E \longrightarrow \mathbb{C}$ defined on a vector space E is even if $h(-x) = h(x)$ for all $x \in E$.

For every function $f \in L^1(\mathbb{R}^n)$ we define the Fourier transform of f by

$$\mathcal{F}(f)(t) := \hat{f}(t) := \int_{\mathbb{R}^n} f(x) \cdot e^{-itx} dx \text{ for each } t \in \mathbb{R}^n,$$

with the usual extension to tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$. If f is the Fourier transform of a certain tempered distribution, then we will denote by $\mathcal{F}^{-1}(f)$, or also by $\check{f}$, its inverse Fourier transform.

Definition 1.2 (Uniformly discrete set)**.** Let $\Lambda \subseteq \mathbb{C}^n$ be infinite countable. We say that Λ is uniformly discrete (briefly u.d.) if

$$\delta(\Lambda) := \inf_{\lambda,\, \lambda' \in \Lambda,\, \lambda \neq \lambda'} \|\lambda - \lambda'\| > 0.$$

The constant $\delta(\Lambda)$ is called the separation constant of Λ.

Definition 1.3 (Uniqueness set)**.** Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let E be a $\mathbb{K}$-vector subspace of $\mathfrak{F}(\mathbb{R}^n, \mathbb{C})$. Let $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete. We say that Λ is a uniqueness set or complete set (briefly, US) for E if for every $f \in E$ we have that

$$(\forall \lambda \in \Lambda \; f(\lambda) = 0) \Rightarrow f = 0.$$

Definition 1.4 (Sequence space $l^p(\Lambda)$)**.** Let $\Lambda \subseteq \mathbb{R}^n$ be u.d.

(1) Let $p \in (0,\ +\infty)$. We define the set

$$l^p(\Lambda) := \left\{ (a_\lambda)_{\lambda\in\Lambda} \in \mathbb{C}^\Lambda \mid \sum_{\lambda\in\Lambda} |a_\lambda|^p < \infty \right\}.$$

The mapping $\| \ \|_p : l^p(\Lambda) \to \mathbb{R}$ given by

$$\left\| (a_\lambda)_{\lambda\in\Lambda} \right\|_p := \left(\sum_{\lambda\in\Lambda} |a_\lambda|^p \right)^{\frac{1}{p}}$$

is a quasinorm for $l^p(\Lambda)$, which is a norm if $p \in [1,\ +\infty)$. With this quasinorm $l^p(\Lambda)$ is a complete space.

(2)

$$l^\infty(\Lambda) := \left\{ (a_\lambda)_{\lambda\in\Lambda} \in \mathbb{C}^\Lambda \mid \sup_{\lambda\in\Lambda} |a_\lambda| < \infty \right\}.$$

The mapping $\| \ \|_\infty : l^\infty(\Lambda) \to \mathbb{R}$ defined by

$$\left\| (a_\lambda)_{\lambda\in\Lambda} \right\|_\infty := \sup_{\lambda\in\Lambda} |a_\lambda|$$

is a norm for $l^\infty(\Lambda)$ which make this space a Banach space.

Definition 1.5. Let $(E, \| \|)$ be a quasinormed space verifying $E \subseteq \mathfrak{F}(\mathbb{R}^n,\ \mathbb{C})$. Let $p \in (0,\ +\infty]$ and $\Lambda \subseteq \mathbb{R}^n$ be u.d. Assume that

$$(f(\lambda))_{\lambda\in\Lambda} \in l^p(\Lambda) \text{ for all } f \in E.$$

- The linear mapping $S : (E,\ \| \|) \to (l^p(\Lambda), \| \ \|_p)$ given by $f \to (f(\lambda))_{\lambda\in\Lambda}$ is called the p-sampling operator of $(E,\ \| \|)$ with respect to Λ.
- We say that Λ verifies the p-Plancherel-Polya condition (briefly p-P.P.C.) for $(E,\ \| \|)$ if S is continuous, this is, if there exists a constant $C > 0$ such that
$$\|(f(\lambda))_{\lambda\in\Lambda}\|_p \le C \, \|f\| \text{ for each } f \in E.$$
- Let $\mathcal{A} \subseteq E$. Λ is said to be a p-interpolation set (in short, p-IS) for $\mathcal{A}$ if the restriction $S|_{\mathcal{A}}$ is surjective. Given $c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$ and $f \in \mathcal{A}$, we say that f interpolates c (over Λ) if $f(\lambda) = c_\lambda$ for all $\lambda \in \Lambda$.
- Let $\mathcal{A} \subseteq E$. We say that Λ is a p-stable sampling set (briefly, p-SS) for $\mathcal{A}$ if there exist constants $c,\ C > 0$, $c \le C$, such that
$$c \, \|(f(\lambda))_{\lambda\in\Lambda}\|_p \le \|f\| \le C \, \|(f(\lambda))_{\lambda\in\Lambda}\|_p \text{ for each } f \in \mathcal{A}.$$
We call C (respectively, c) an upper bound (respectively, a lower bound) of p-stable sampling for $\mathcal{A}$ with respect to Λ.

- We say that Λ is a p-complete interpolation set (briefly, p-CIS) for $(E, \|\,\|)$ if S is a topological isomorphism.

***Remark* 1.6.** In the context of Definition 1.5 we observe that:

(1) Λ is a uniqueness set for E if and only if S is injective.

(2) Λ is a p-CIS for $(E,\ \|\,\|)$ if and only if Λ is both a p-IS and a p-SS for $(E,\ \|\,\|)$.

If E is a vector subspace of $L^p(\mathbb{R}^n)$, then we will refer to the p-SS, p-IS and p-CIS for $(E,\ \|\,\|_p)$ simply as SS, IS, and CIS, respectively.

Definition 1.7. Let $S \subseteq \mathbb{R}^n$ be a bounded set and $p \in (0,\ +\infty]$. We define

$$E_S^p := \{f \in \mathcal{S}'(\mathbb{R}^n)\ :\ \operatorname{supp}(\hat{f}) \subseteq S \text{ and } \|f\|_p < \infty\},$$

which is a closed vector subspace of $(L^p(\mathbb{R}^n),\ \|\,\|_p)$. We call $(p,\ S)$-Paley-Wiener space to the complete space $(E_S^p,\ \|\,\|_p)$.

Now we recall the definition of Bernstein space.

Definition 1.8.

- Let $\sigma > 0$ and $f \in \mathcal{H}(\mathbb{C}^n)$. We say that the entire function f is of exponential type at most σ if for every $\varepsilon > 0$ there exists $A_\varepsilon > 0$ such that $|f(z)| \leq A_\varepsilon\, e^{(\sigma+\varepsilon)\|z\|_1}$ for each $z \in \mathbb{C}^n$.

- Let $\sigma > 0$. $E_\sigma := \{f \in \mathcal{H}(\mathbb{C}^n)\ :\ f \text{ is of exponential type at most } \sigma\}$ is a $\mathbb{C}$-vector subspace of $\mathcal{H}(\mathbb{C}^n)$.

- Let $\sigma > 0$ and $p \in (0,\ +\infty]$. We define the set

$$B_\sigma^p(\mathbb{R}^n) := \{f \in E_\sigma\ :\ f|_{\mathbb{R}^n} \in L^p(\mathbb{R}^n)\},$$

  which is a closed $\mathbb{C}$-vector subspace of $(L^p(\mathbb{R}^n),\ \|\,\|_p)$. We call $(\sigma,\ p)$-Bernstein space to the space $(B_\sigma^p(\mathbb{R}^n),\ \|\,\|_p)$, which is Banach if $p \in [1,\ +\infty]$ and is quasi-Banach if $p \in (0,\ 1)$.

- We call classical Bernstein spaces to the spaces $(B_\sigma^\infty(\mathbb{R}^n),\ \|\,\|_\infty)$, with $\sigma > 0$, which are Banach spaces.

For example, the entire functions defined by $z \mapsto \sin(\sigma z)$, $z \mapsto \cos(\sigma z)$ are elements of $B_\sigma^\infty(\mathbb{R})$; the entire function defined by

$$z \mapsto sinc(\sigma z) = \begin{cases} \frac{\sin(\sigma z)}{\sigma z}, & \text{if } z \neq 0 \\ 1, & \text{if } z = 0 \end{cases}$$

belongs to $B^p_\sigma(\mathbb{R})$ for all $p \in (1,\ +\infty]$.

Observe that, given $p \in (0,\ +\infty]$, the Paley-Wiener subspaces and the Bernstein subspaces of $(L^p(\mathbb{R}^n),\ \| \|_p)$ are invariant and isometric by translations. A very important result shows a closed relationship between both types of spaces. This result is the celebrated Paley-Wiener theorem.

Theorem 1.9 (Paley-Wiener (see [15])). *Let $\sigma > 0$ and $p \in [1,\ +\infty]$. Consider $S = [-\sigma,\ \sigma]^n$, n-dimensional closed interval. Then*

$$E^p_S = B^p_\sigma(\mathbb{R}^n).$$

Therefore Bernstein spaces are particular cases of Paley-Wiener spaces.

Now we wonder when the sampling operator is continuous, that is, when Λ verifies the p-P.P.C. for a given Paley-Wiener space (in particular, for a given Bernstein space), and the answer is: provided that Λ is uniformly discrete. This is what the following result says.

Theorem 1.10 (Plancherel-Polya inequality (see [17])). *Let $S \subseteq \mathbb{R}^n$ be a bounded set and $p \in (0,\ +\infty]$. Let $\Lambda \subseteq \mathbb{R}^n$ be u.d. Then Λ verifies the p-P.P.C. for $(E^p_S,\ \| \|_p)$, this is, there exists a constant $C = C(\Lambda,\ S,\ p) > 0$ such that*

$$\|(f(\lambda))_{\lambda\in\Lambda}\|_p \leq C\ \|f\|_p \text{ for each } f \in E^p_S.$$

Besides the constant C only depends on p, S and $\delta(\Lambda)$ ($\delta(\Lambda)$ is defined in Definition 1.2).

The main results of this paper are the following ones.

Theorem 1.11. *Let $p \in [1,\ +\infty)$, and $K \subseteq \mathbb{R}^n$ be a bounded and Lebesgue measurable set such that its indicator function χ_K is a Fourier multiplier for $\mathcal{F}L^q(\mathbb{R}^n)$. Let $\Lambda \subseteq \mathbb{R}^n$ be u.d. and let $h \in E^1_K$ be real valued or even, or both. For every $\lambda \in \Lambda$ we define the translation function $h_\lambda := \tau_\lambda h : \mathbb{R}^n \longrightarrow \mathbb{C}$ by $h_\lambda(x) := h(x-\lambda)$ for each $x \in \mathbb{R}^n$. Suppose that*

$$h_\lambda(\mu) = h(\mu-\lambda) = \delta_{\lambda\mu} \text{ for each } \lambda,\ \mu \in \Lambda,$$

where δ is the Kronecker delta.

Then:

(1) Λ is an IS for $(E^p_K,\ \| \|_p)$. In fact, for each $c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$ we have that $g_c := \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda \in E^p_K$ interpolates c, that is, $g_c(\mu) = c_\mu$ for every $\mu \in \Lambda$.

(2) We define the following vector subspace of E^p_K:

$$W := \left\{ \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda \mid c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda) \right\} \subseteq E^p_K.$$

The mapping

$$\psi : (l^p(\Lambda), \| \,\|_p) \to (W,\ \| \,\|_p)$$

defined by: $c = (c_\lambda)_{\lambda\in\Lambda} \to \psi(c) := g_c := \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda$, *is a topological isomorphism. Hence* W *is closed in* E_K^p.

(3) Λ *is a CIS for* W. *Furthermore, the sampling operator for* W *is* $S_W = \psi^{-1}$.

(4) $\mathcal{B} := \{h_\lambda\}_{\lambda\in\Lambda}$ *is a Schauder basis for* W.

We need the next definition for the following result.

For every $p \in (0,\ +\infty]$ we define

$$d(p) := \max\left\{1,\ 2^{\frac{1}{p}-1}\right\} = \begin{cases} 2^{\frac{1}{p}-1}, & \text{if } p \in (0,\ 1) \\ 1, & \text{if } p \in [1,\ +\infty] \end{cases}.$$

The following theorem is a property of transference for stable sampling sets of certain vector subspaces of $L^p(\mathbb{R}^m)$ similar to Kadec 1/4-Theorem (see [8], and also [25] for this and similar results).

Theorem 1.12. *Let* $p \in (0, +\infty]$, $m \in \mathbb{Z}$, $m > 0$. *Let* $\Gamma, \Lambda \subseteq \mathbb{R}^m$ *be u.d., which we may express as* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}^m}$, $\Gamma = \{\gamma_n\}_{n\in\mathbb{Z}^m}$. *Let* $E \subseteq \mathcal{C}^1(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$ *be a closed vector subspace of* $(L^p(\mathbb{R}^m),\ \| \,\|_p)$. *Suppose that the following sampling operators are continuous:* $S_{E,\Lambda} : (E,\ \| \,\|_p) \to (l^p(\Lambda),\ \| \,\|_p)$ *given by* $f \to (f(\lambda))_{\lambda\in\Lambda}$, *and* $S_{E,\Gamma} : (E,\ \| \,\|_p) \to (l^p(\Gamma),\ \| \,\|_p)$ *given by* $f \to (f(\gamma))_{\gamma\in\Gamma}$. *Let* $h : \mathbb{R}^m \to \mathbb{R}$, $h > 0$, *be a continuous function and let* $C_1 > 0$ *be. We define*

$$\mathcal{F}(h, C_1) := \left\{ f \in E \mid \left| \frac{\partial f}{\partial x_j}(y) \right| \le C_1 \cdot \frac{\|f\|_p}{h(y)}\ \forall j \in \{1, ..., m\},\ y \in \mathbb{R}^m \right\}.$$

Let $\mathcal{A} \subseteq \mathcal{F}(h, C_1) \subseteq E$. *Suppose that*

(1) *There exists a function* $g : \Gamma \to \mathbb{R}$, $g \ge 0$, *such that*

$$|\gamma_n - \lambda_n| \le g(\gamma_n) \ \textit{for all}\ n \in \mathbb{Z}^m,$$

where $|\,|$ *denotes the Euclidean norm in* $\mathbb{R}^m$.

(2)

$$h(\gamma_n) = \min h|_{[\lambda_n,\, \gamma_n]} = \\ \min\{h(x) \mid (\exists t \in [0,\ 1]\ :\ x = t\lambda_n + (1-t)\gamma_n)\} > 0,$$

for each $n \in \mathbb{Z}^m$.

(3) $S < +\infty$, *where* $S := \begin{cases} \left(\sum_{n\in\mathbb{Z}^m} \left(\frac{g(\gamma_n)}{h(\gamma_n)}\right)^p\right)^{1/p}, & \textit{if}\ p \in (0,\ +\infty) \\ \sup_{n\in\mathbb{Z}^m} \frac{g(\gamma_n)}{h(\gamma_n)}, & \textit{if}\ p = +\infty \end{cases}.$

We define $C_2 := \sqrt{m} \cdot C_1 \cdot d(p) > 0$. *Then we have:*

(1) *If* Γ *is a SS for* $\mathcal{A}$ *with upper constant of sampling* $C > 0$, *and* $S < \frac{1}{C \cdot C_2}$, *then* Λ *is a SS for* $\overline{\mathcal{A}}$ *with upper constant of sampling* $C' := \frac{C \cdot d(p)}{1 - C \cdot C_2 \cdot S}$.

(2) *If* Λ *is a SS for* $\mathcal{A}$ *with upper constant of sampling* $D > 0$, *and* $S < \frac{1}{D \cdot C_2}$, *then* Γ *is a SS for* $\overline{\mathcal{A}}$ *with upper constant of sampling* $C' := \frac{D \cdot d(p)}{1 - D \cdot C_2 \cdot S}$.

The paper is structured as follows. Section 1 contains definitions and the list of the main results. In section 2 we study the stability of SS and IS with respect to perturbations, in Bernstein spaces. Section 3 is devoted to Lemma 3.2 (the main convergence lemma) and we prove Theorem 1.11 and obtain several consequences. Section 4 contains the lemmas of continuity of immersion of Bernstein spaces in classical Bernstein spaces. In section 5 we study the transference of the property of being SS from a u.d. set to another one, and we prove Theorem 1.12 and a consequence for Bernstein spaces. In section 6 we obtain transference results of uniqueness sets between different Bernstein spaces.

2. Stability in sampling and interpolation in Banach spaces

Recall the next well known Banach theorem:

Theorem 2.1 (Stability of surjective bounded operators)**.** *Let* $A : (X, \| \, \|_X) \to (Y, \| \, \|_Y)$ *be a linear, bounded and surjective operator between Banach spaces. Then there exists* $\gamma = \gamma(A) > 0$ *such that for every linear and bounded operator* $B : (X, \| \, \|_X) \to (Y, \| \, \|_Y)$ *verifying* $\|B\| < \gamma$, *the operator* $A + B : (X, \| \, \|_X) \to (Y, \| \, \|_Y)$ *is surjective.*

Definition 2.2 (ϵ-perturbation of a set (see [14]))**.** Let $\Lambda, \ \Lambda' \subseteq \mathbb{R}^n$ be u.d. and let $\epsilon > 0$ be. Λ' is said to be an ϵ-perturbation of Λ if Λ' admits a representation

$$\Lambda' = \{\lambda + \epsilon_\lambda \ : \ \lambda \in \Lambda\}$$

where $|\epsilon_\lambda| < \epsilon$ for all $\lambda \in \Lambda$.

We will need the Bernstein inequality in several dimensions, obtained by I. Pesenson.

Theorem 2.3 (Bernstein inequality (see [16]))**.** *Assume* $p \in [1, \ +\infty]$, $\sigma > 0$, $f \in B^p_\sigma(\mathbb{R}^n)$. *Then:*

$$\frac{\partial f}{\partial x_j} \in B^p_\sigma(\mathbb{R}^n), \text{ and } \left\| \frac{\partial f}{\partial x_j} \right\|_p \leq \sigma \|f\|_p \text{ for all } j \in \{1, ..., n\}.$$

The following result give sense to the word *stable* in stable sampling and interpolation theory.

Theorem 2.4 (Principle of stability with respect to perturbations)**.** *Let $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete and $p \in [1,\ +\infty]$. Let $\sigma > 0$. If Λ is a SS (respectively, IS, CIS) for $B^p_\sigma(\mathbb{R}^n)$, there exists $\epsilon > 0$ such that every ϵ-perturbation of Λ is a SS (respectively, IS, CIS) for $B^p_\sigma(\mathbb{R}^n)$.*

Proof. Our proof is similar to that by A. Olevskii and A. Ulanovskii for the case $p = 2$, $n = 1$ (see [14]).

Let $\varepsilon \in \left(0,\ \frac{\delta(\Lambda)}{4}\right)$ and Λ' be an ϵ-perturbation of Λ. So Λ' admits a representation $\Lambda' = \{\lambda + \epsilon_\lambda\ :\ \lambda \in \Lambda\}$ where $\sup_{\lambda\in\Lambda} |\epsilon_\lambda| \le \epsilon$. Let us consider the sampling operators corresponding to Λ, Λ', respectively:

$$S_\Lambda : (B^p_\sigma(\mathbb{R}^n),\ \|\ \|_p) \to (l^p(\Lambda), \|\ \|_p),$$

$$S_{\Lambda'} : (B^p_\sigma(\mathbb{R}^n),\ \|\ \|_p) \to (l^p(\Lambda'), \|\ \|_p).$$

S_Λ and $S_{\Lambda'}$ are continuous by Theorem 1.10.

First we take $p \in [1,\ +\infty)$. We claim that there exists a constant $D_p = D\,(p, \sigma, \delta(\Lambda)) > 0$ such that

$$\|\,(S_\Lambda - S_{\Lambda'})\,(f)\|_p^p \le n \cdot \varepsilon^p \cdot D_p^p \cdot \sigma^p \cdot \|f\|_p^p \text{ for all } f \in B^p_\sigma(\mathbb{R}^n).$$

Indeed, let $f \in B^p_\sigma(\mathbb{R}^n)$. Using the Bernstein inequality (Theorem 2.3), the mean value theorem and Theorem 1.10, we have:

$$\begin{aligned}
\|\,(S_\Lambda - S_{\Lambda'})\,(f)\|_p^p &= \sum_{\lambda\in\Lambda} |f(\lambda + \epsilon_\lambda) - f(\lambda)|^p \le \varepsilon^p \sum_{\lambda\in\Lambda} \|Df(\xi_\lambda)\|^p \\
&\le \varepsilon^p \sum_{\lambda\in\Lambda} \|\nabla f(\xi_\lambda)\|_2^p \le \varepsilon^p \cdot C_p^p \sum_{\lambda\in\Lambda} \|\nabla f(\xi_\lambda)\|_p^p \\
&= \varepsilon^p \cdot C_p^p \sum_{\lambda\in\Lambda} \sum_{j=1}^{n} \left| \frac{\partial f}{\partial x_j}(\xi_\lambda) \right|^p = \varepsilon^p \cdot C_p^p \sum_{j=1}^{n} \sum_{\lambda\in\Lambda} \left| \frac{\partial f}{\partial x_j}(\xi_\lambda) \right|^p \\
&\le \varepsilon^p \cdot D_p^p \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_p^p \le n \cdot \varepsilon^p \cdot D_p^p \cdot \sigma^p \cdot \|f\|_p^p,
\end{aligned}$$

where ξ_λ belongs to the segment with extreme points λ, $\lambda + \varepsilon_\lambda$, for all $\lambda \in \Lambda$. In the penultimate inequality we have applied Theorem 1.10 to each partial derivative $\frac{\partial f}{\partial x_j} \in B^p_\sigma(\mathbb{R}^n)$, with $j \in \{1, ..., n\}$.

Observe that the set $\Gamma_\Lambda := \{\xi_\lambda\ :\ \lambda \in \Lambda\}$ is a uniformly discrete set (in fact it is an ε-perturbation of Λ), and therefore we can apply Theorem 1.10.

Indeed, let λ, $\lambda' \in \Lambda$ with $\lambda \ne \lambda'$. Then

$$\|\lambda - \xi_\lambda\| \le \varepsilon_\lambda < \varepsilon, \quad \|\lambda' - \xi_{\lambda'}\| \le \varepsilon_{\lambda'} < \varepsilon.$$

Then as Λ is uniformly discrete, we have:

$$\begin{aligned}0 < \delta(\Lambda) &= \inf_{\lambda_1,\ \lambda_2\in\Lambda,\ \lambda_1\neq\lambda_2} \|\lambda_1-\lambda_2\| \leq \|\lambda-\lambda'\| \\ &= \|\lambda-\xi_\lambda+\xi_\lambda-\xi_{\lambda'}+\xi_{\lambda'}-\lambda'\| \leq \|\lambda-\xi_\lambda\|+\|\xi_\lambda-\xi_{\lambda'}\|+\|\xi_{\lambda'}-\lambda'\| \\ &< 2\,\varepsilon+\|\xi_\lambda-\xi_{\lambda'}\|\,.\end{aligned}$$

Hence $\delta(\Lambda) < 2\,\varepsilon + \|\xi_\lambda-\xi_{\lambda'}\|$, and so $\|\xi_\lambda-\xi_{\lambda'}\| > \delta(\Lambda) - 2\,\varepsilon$.

As we are assuming that $\varepsilon < \frac{\delta(\Lambda)}{4}$, then we obtain that

$$\|\xi_\lambda-\xi_{\lambda'}\| > \frac{\delta(\Lambda)}{2}.$$

Notice that, in particular, our claim shows that

$$\|S_\Lambda - S_{\Lambda'}\| \leq n^{\frac{1}{p}}\cdot\varepsilon\cdot\sigma\cdot D_p.$$

(1) Suppose that Λ is an IS for $B^p_\sigma(\mathbb{R}^n)$. This is, S_Λ is surjective. Let $\gamma = \gamma(S_\Lambda) > 0$ given by stability theorem for bounded surjective operators. Taking ε sufficiently small we have $\|S_\Lambda - S_{\Lambda'}\| < \gamma$. So $S_{\Lambda'}$ is surjective by stability theorem; this is, Λ' is an IS for $B^p_\sigma(\mathbb{R}^n)$.

(2) Suppose that Λ is a SS for $B^p_\sigma(\mathbb{R}^n)$. Then by Theorem 1.10 there exists a constant $C = C\left(p,\sigma,\delta\left(\Lambda\right)\right) > 0$ such that

$$\|S_\Lambda(f)\|_p \leq C\,\|f\|_p \text{ for all } f\in B^p_\sigma(\mathbb{R}^n).$$

Let $f\in B^p_\sigma(\mathbb{R}^n)$. Then:

$$\begin{aligned}\|S_{\Lambda'}(f)\|_p &\leq \|\left(S_{\Lambda'}-S_\Lambda\right)(f)\|_p + \|S_\Lambda(f)\|_p \\ &\leq n^{\frac{1}{p}}\cdot\varepsilon\cdot D_p\cdot\sigma\|f\|_p + C\,\|f\|_p = \left(n^{\frac{1}{p}}\cdot\varepsilon\cdot D_p\cdot\sigma + C\right)\|f\|_p.\end{aligned}$$

On the other hand, as Λ is a SS for $B^p_\sigma(\mathbb{R}^n)$, then we have:

$$\begin{aligned}\|f\|_p &\leq R\,\|S_\Lambda(f)\|_p = R\,\|\left(S_\Lambda - S_{\Lambda'}\right)(f) + S_{\Lambda'}(f)\|_p \\ &\leq R\,\|\left(S_\Lambda - S_{\Lambda'}\right)(f)\|_p + R\,\|S_{\Lambda'}(f)\|_p \\ &\leq R\cdot n^{\frac{1}{p}}\cdot\varepsilon\cdot D_p\cdot\sigma\cdot\|f\|_p + R\,\|S_{\Lambda'}(f)\|_p.\end{aligned}$$

So that

$$\|f\|_p \leq R\cdot n^{\frac{1}{p}}\cdot\varepsilon\cdot D_p\cdot\sigma\cdot\|f\|_p + R\,\|S_{\Lambda'}(f)\|_p.$$

Hence

$$\left(1 - R\cdot n^{\frac{1}{p}}\cdot\varepsilon\cdot D_p\cdot\sigma\right)\,\|f\|_p \leq R\,\|S_{\Lambda'}(f)\|_p.$$

Taking $\epsilon > 0$ enough small we have that $1 - R \cdot n^{\frac{1}{p}} \cdot \varepsilon \cdot D_p \cdot \sigma > 0$, and then

$$\|f\|_p \leq \frac{R}{1 - R \cdot n^{\frac{1}{p}} \cdot \varepsilon \cdot D_p \cdot \sigma} \|S_{\Lambda'}(f)\|_p.$$

Therefore Λ' is a SS for $B^p_\sigma(\mathbb{R}^n)$.

For the case $p = +\infty$ we have that there exists a constant $D = D(\sigma, \delta(\Lambda)) > 0$ such that

$$\| (S_\Lambda - S_{\Lambda'})(f)\|_\infty \leq n \cdot \varepsilon \cdot D \cdot \sigma \cdot \|f\|_\infty \text{ for all } f \in B^\infty_\sigma(\mathbb{R}^n),$$

and the proof is completely analogous. ☑

3. Lemma of Convergence.

In this section we prove a result of convergence of series in Paley-Wiener spaces which allows to make sure the convergence of series under certain conditions. First we need the following auxiliary and well known result:

Lemma 3.1. *Let $r \in (0, +\infty]$ and $S \subseteq \mathbb{R}^n$. Let $g \in E^r_S$ be even. Then:*

(1) The real and imaginary parts of g and their Fourier transform, $Re(g)$, $\widehat{Re(g)}$, $Im\, g$, $\widehat{Img}$, are even.

(2) $\widehat{Re(g)}(t) \in \mathbb{R}$ and $\widehat{Im\, g}(t) \in \mathbb{R}$ for each $t \in \mathbb{R}^n$.

(3) $Re(g)$, $Im\, g \in E^r_S$.

Lemma 3.2 (Lemma of Convergence)**.** *Let $p \in [1, +\infty)$, and $K \subseteq \mathbb{R}^n$ be a bounded and Lebesgue measurable set such that its indicator function χ_K is a Fourier multiplier for $\mathcal{F}L^q(\mathbb{R}^n)$. Let $\Lambda \subseteq \mathbb{R}^n$ be a uniformly discrete set and let $h \in E^1_K$ be real valued. For every $\lambda \in \Lambda$ we define the function $h_\lambda := \tau_\lambda h : \mathbb{R}^n \longrightarrow \mathbb{R}$ by $h_\lambda(x) := (\tau_\lambda h)(x) = h(x - \lambda)$ for all $x \in \mathbb{R}^n$. Then there exists a constant $D > 0$ such that*

$$\left\| \sum_{\lambda \in \Lambda} c_\lambda \cdot h_\lambda \right\|_p \leq D \cdot \|c\|_p \text{ for each } c = (c_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda).$$

In particular, for each $c = (c_\lambda)_{\lambda \in \Lambda} \in l^p(\Lambda)$ we have that $g_c := \sum_{\lambda \in \Lambda_1} c_\lambda \cdot h_\lambda \in L^p(\mathbb{R}^n)$, and thus $g_c \in E^p_K$.

In addition, the constant D only depends on p, $\|h\|_1$, $\delta(\Lambda)$ and on K.

***Remark* 3.3.** Observe that Lemma 3.2 is also true for even functions $h \in E^1_K$. This is an immediate consequence of applying the lemma to the real and imaginary parts of h, by Lemma 3.1. In addition, note that $h_\lambda \in E^1_K \subseteq E^p_K$ for all $\lambda \in \Lambda$.

Proof. If $h = 0$, the result is obvious. Suppose that h is not the function identically zero.

Let $c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$. We define $g_c := \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda \in \mathcal{S}'(\mathbb{R}^n)$. We will prove that there exists a constant $D > 0$ independent of c such that

$$|< g_c, \varphi >| \leq D \cdot \|c\|_p \cdot \|\varphi\|_q \text{ for each } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

$$\begin{aligned} |< g_c, \varphi >| = |< \widehat{g_c}, \hat{\varphi} >| &= \left|< \sum_{\lambda\in\Lambda} c_\lambda \cdot e^{-it\lambda}\hat{h}(t), \hat{\varphi} >\right| \\ = \left|< \sum_{\lambda\in\Lambda} c_\lambda \cdot e^{-it\lambda}\hat{h}(t)\cdot\chi_K(t), \hat{\varphi} >\right| &= \left|\sum_{\lambda\in\Lambda} c_\lambda < e^{-it\lambda}\hat{h}(t)\cdot\chi_K(t), \hat{\varphi} >\right| \\ \leq \sum_{\lambda\in\Lambda} |c_\lambda| \left|< e^{-it\lambda}\hat{h}(t)\cdot\chi_K(t), \hat{\varphi} >\right| &= \sum_{\lambda\in\Lambda} |c_\lambda| \left|< \hat{\varphi}, e^{-it\lambda}\hat{h}(t)\cdot\chi_K(t) >\right|. \end{aligned}$$

Observe that

$$\begin{aligned} \overline{\hat{h}(t)} &= \overline{\int_{\mathbb{R}^n} h(x)\cdot e^{-itx}dx} = \int_{\mathbb{R}^n} \overline{h(x)}\cdot e^{itx}dx \\ &= \int_{\mathbb{R}^n} h(x)\cdot e^{itx}dx = \hat{h}(-t) \text{ for each } t \in \mathbb{R}^n. \end{aligned}$$

We define $f_h := h \circ (-1_{\mathbb{R}^n}) \in L^1(\mathbb{R}^n)$, and we have that $\|f_h\|_1 = \|h\|_1$, and $\widehat{f_h}(t) = \hat{h}(-t)$ for all $t \in \mathbb{R}^n$.

Let $\lambda \in \Lambda$.

$$\begin{aligned} &< \hat{\varphi}, e^{-it\lambda}\hat{h}(t)\cdot\chi_K(t) >= \int_{\mathbb{R}^n} \hat{\varphi}(t)\,\overline{\hat{h}(t)}\cdot\chi_K(t)e^{it\lambda}dt \\ &= \int_{\mathbb{R}^n} \hat{\varphi}(t)\cdot\widehat{f_h}(t)\cdot\chi_K(t)e^{it\lambda}dt = \mathcal{F}^{-1}\left(\hat{\varphi}\cdot\widehat{f_h}\cdot\chi_K\right)(\lambda) \\ &= \mathcal{F}^{-1}\left((\widehat{\varphi * f_h})\cdot\chi_K\right)(\lambda). \end{aligned}$$

Then:

$$|< g_c, \varphi >| \leq \sum_{\lambda\in\Lambda} |c_\lambda| \left|\mathcal{F}^{-1}\left((\widehat{\varphi * f_h})\cdot\chi_K\right)(\lambda)\right| \leq \|c\|_p \cdot \|a\|_q,$$

where $a_\lambda := \mathcal{F}^{-1}\left((\widehat{\varphi * f_h})\cdot\chi_K\right)(\lambda)$ for each $\lambda \in \Lambda$, and $a := (a_\lambda)_{\lambda\in\Lambda}$.

Since χ_K is a Fourier multiplier for $\mathcal{F}L^q(\mathbb{R}^n)$, then

$$\begin{aligned} \left\|\mathcal{F}^{-1}\left((\widehat{\varphi * f_h})\cdot\chi_K\right)\right\|_q &\leq C_q \|\varphi * f_h\|_q \leq C'_q \|\varphi\|_q \cdot \|f_h\|_1 \\ &= C'_q \|\varphi\|_q \cdot \|h\|_1 = C'_{q,h} \|\varphi\|_q, \end{aligned}$$

where $C'_{q,h} := C'_q \cdot \|h\|_1 > 0$. In addition, using the Plancherel-Polya inequality, Theorem 1.10, we have:

$$\|a\|_q = \left\|(a_\lambda)_{\lambda\in\Lambda}\right\|_q = \left\|\left(\mathcal{F}^{-1}\left(\widehat{(\varphi * f_h)}\cdot\chi_K\right)(\lambda)\right)_{\lambda\in\Lambda}\right\|_q$$
$$\leq C''_{q,K}\cdot\left\|\mathcal{F}^{-1}\left(\widehat{(\varphi * f_h)}\cdot\chi_K\right)\right\|_q \leq C''_{q,K}\cdot C'_{q,h}\cdot\|\varphi\|_q = D\cdot\|\varphi\|_q\,.$$

So that $|< g_c, \varphi >| \leq D\cdot\|c\|_p\cdot\|\varphi\|_q$, and this is true for each $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Hence $\|g_c\|_p \leq D\cdot\|c\|_p$. ☑

Now we prove Theorem 1.11.

3.1. Proof of Theorem 1.11

Proof. (1) We know by Lemma 3.2 of convergence that the series $\sum_{\lambda\in\Lambda} c_\lambda\cdot h_\lambda$ converges in $(E^p_K, \|\,\|_p)$. Let $g_c\in E^p_K$ be its limit. Since the inclusion $i:(E^p_K, \|\,\|_p)\hookrightarrow(E^\infty_K, \|\,\|_\infty)$ is continuous, then the series also converges in $(E^\infty_K, \|\,\|_\infty)$ to g_c. That is, converges uniformly in $\mathbb{R}^n$ to g_c, and thus also converges pointwise. So that

$$g_c(\mu) = \sum_{\lambda\in\Lambda} c_\lambda\cdot h_\lambda(\mu) = \sum_{\lambda\in\Lambda} c_\lambda\,\delta_{\lambda\mu} = c_\mu,$$

for each $\mu\in\Lambda$. Hence g_c interpolates c and $g_c = \sum_{\lambda\in\Lambda} g_c(\lambda)\,h_\lambda$.

(2) Obviously ψ is surjective, by definition of W. Let $c = (c_\lambda)_{\lambda\in\Lambda}\in l^p(\Lambda)$, and define $h := \sum_{\lambda\in\Lambda} c_\lambda\,h_\lambda$. Then we have:

$$\psi(c) = g_c = \sum_{\lambda\in\Lambda} c_\lambda\,h_\lambda.$$

In addition, by Lemma 3.2 of convergence there exists a constant $D_2 > 0$ independent of c such that

$$\|\sum_{\lambda\in\Lambda} c_\lambda\cdot h_\lambda\|_p \leq D_2\,\|c\|_p \text{ for each } c = (c_\lambda)_{\lambda\in\Lambda}\in l^p(\Lambda)\,.$$

Then,

$$\|\psi(c)\|_p = \|g_c\|_p = \|\sum_{\lambda\in\Lambda} c_\lambda\cdot h_\lambda\|_p \leq D_2\,\|c\|_p$$

for all $c = (c_\lambda)_{\lambda\in\Lambda}\in l^p(\Lambda)$. Thus ψ is continuous. We will prove that ψ is an isomorphism showing that there exists a constant $D_1 > 0$ such that

$$\|c\|_p \leq D_1\cdot\|\psi(c)\|_p\,.$$

Let $c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$. $g_c = \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda \in E_K^p$. We shew in the proof of the first item that

$$g_c(\mu) = \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda(\mu) = c_\mu \text{ for every } \mu \in \Lambda,$$

and therefore $g_c = \sum_{\lambda\in\Lambda} g_c(\lambda) \cdot h_\lambda \in E_K^p$. Then:

$$\|c\|_p = \|(c_\lambda)_{\lambda\in\Lambda}\|_p = \|g_c(\lambda)_{\lambda\in\Lambda}\|_p \leq D_1 \|g_c\|_p = D_1 \|\psi(c)\|_p,$$

where we have used the Plancherel-Polya inequality (Theorem 1.10).

(3) Let $g \in W \subseteq E_K^p$. By definition of W we have that there exists $c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$ such that $g = \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda$. Since ψ is injective, then c is unique, and besides we have

$$g(\mu) = \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda(\mu) = c_\mu \text{ for each } \mu \in \Lambda.$$

So that $c_\mu = g(\mu)$ for all $\mu \in \Lambda$ and hence

$$g = \sum_{\lambda\in\Lambda} g(\lambda) \cdot h_\lambda.$$

Conclusion: $\mathcal{B} = \{h_\lambda\}_{\lambda\in\Lambda}$ is a Schauder basis for $(W, \|\ \|_p)$.

(4) It has been proved in the previous item.

☑

3.2. Consequences of Theorem 1.11

Lemma 3.4. *Let $p \in (1, +\infty)$, $\upsilon > 0$. Let $\Lambda \subseteq \mathbb{R}^n$ be a uniformly discrete set. Define $\Gamma := \Lambda - \Lambda = \{\lambda_1 - \lambda_2 \mid \lambda_1, \lambda_2 \in \Lambda\}$. Assume that there exists a real valued or even (or both) function $h \in B_\upsilon^1(\mathbb{R}^n)$ such that*

(1) $h(0) = 1$.

(2) $h(\gamma) = 0$ for every $\gamma \in \Gamma \setminus \{0\}$.

Then Λ is an IS for $(B_\upsilon^p, \|\ \|_p)$. In fact, for each $c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$ we have that $g_c := \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda \in B_\upsilon^p(\mathbb{R}^n)$ interpolates c, that is, $g_c(\mu) = c_\mu$ for every $\mu \in \Lambda$, and thus $g_c = \sum_{\lambda\in\Lambda} g_c(\lambda) \cdot h_\lambda$, where $h_\lambda := \tau_\lambda h$ for every $\lambda \in \Lambda$.

Proof. By Paley-Wiener Theorem 1.9 we have that $E_{[-\upsilon,\upsilon]^n}^p = B_\upsilon^p(\mathbb{R}^n)$, and the indicator function of $[-\upsilon, \upsilon]^n$ is a Fourier multiplier for $\mathcal{F}L^q(\mathbb{R}^n)$, where q is the conjugate exponent of p. Then by Theorem 1.11 we obtain the result. ☑

An immediate consequence of Lemma 3.4 is the following result.

Corollary 3.5 (Interpolation for Bernstein spaces)**.** *Let $p \in (1, +\infty)$, $\upsilon > 0$. Let $\Lambda \subseteq \mathbb{R}^n$ be a uniformly discrete set. Define*

$$\Gamma := \Lambda - \Lambda = \{\lambda_1 - \lambda_2 \mid \lambda_1,\ \lambda_2 \in \Lambda\}\,.$$

Suppose that there exists $f \in B^1_\upsilon(\mathbb{R}^n)$ such that

(1) $f(0) = 1$.

(2) $f(\gamma) = 0$ for every $\gamma \in \Gamma \setminus \{0\}$.

Then Λ is an IS for $(B^p_\upsilon(\mathbb{R}^n),\ \|\ \|_p)$. Moreover, define $h : \mathbb{C}^n \longrightarrow \mathbb{C}$ by

$$h(z) := \frac{1}{2}\left(f(z) + f(-z)\right) \text{ for all } z \in \mathbb{C}^n.$$

Then $h \in B^1_\upsilon(\mathbb{R}^n)$ is an even function and verifies that $h(0) = 1$ and $h(\gamma) = 0$ for every $\gamma \in \Gamma \setminus \{0\}$. Thus for each $c = (c_\lambda)_{\lambda\in\Lambda} \in l^p(\Lambda)$ we have that $g_c := \sum_{\lambda\in\Lambda} c_\lambda \cdot h_\lambda \in B^p_\upsilon(\mathbb{R}^n)$ interpolates c, where $h_\lambda := \tau_\lambda h$ for every $\lambda \in \Lambda$.

4. Lemmas of continuity of immersion in Bernstein spaces

In this section we will study several results which prove that, given σ, $p > 0$, we have the inclusion of Bernstein spaces

$$B^p_\sigma(\mathbb{R}^n) \subseteq B^\infty_\sigma(\mathbb{R}^n)$$

and this inclusion is continuous. We will need these results for the next section.

Lemma 4.1. *Let $n \in \mathbb{N}$, $n \geq 2$. Let $r \in \{1, ..., n-1\}$, and $a = (a_1, ..., a_r) \in \mathbb{R}^r$. Then:*

(1) The function

$$\psi_a : (B^\infty_\pi(\mathbb{R}^n),\ \|\ \|_\infty) \to (B^\infty_\pi(\mathbb{R}^{n-r}),\ \|\ \|_\infty)$$

defined by

$$f \to f_a : \mathbb{C}^{n-r} \to \mathbb{C}$$

given by: $z = (z_{r+1}, ..., z_n) \mapsto f_a(z) := f(a, z)$, is well defined.

(2) ψ_a is continuous. In fact we have:

$$\|f_a\|_\infty = \|\psi_a(f)\|_\infty \leq \|f\|_\infty \text{ for every } f \in B^\infty_\pi(\mathbb{R}^n).$$

This lemma is clear; let us see the version for $p \in (0,\ +\infty)$.

Lemma 4.2. *Let $p \in (0,\ +\infty)$.*

(1) Let $\Omega \subseteq \mathbb{C}$ be a connected and open set. Then $|f|^p$ is subharmonic for every $f \in \mathcal{H}(\Omega)$.

(2) The inclusion

$$\varphi : (B^p_\pi(\mathbb{R}),\ \|\ \|_p) \hookrightarrow (B^\infty_\pi(\mathbb{R}),\ \|\ \|_\infty)$$

is continuous. That is: there exists $C_{\pi,\,p} > 0$ such that

$$\|f\|_\infty \leq C_{\pi,\,p} \cdot \|f\|_p \text{ for all } f \in B^p_\pi(\mathbb{R}).$$

We may take $C_{\pi,\,p} := \left(\frac{2}{p\pi^2}(e^{p\pi} - 1)\right)^{1/p}$ (see [21], Chapter 6).

(3) Let $n \in \mathbb{N},\ n > 1$. Let $r \in \{1, ..., n-1\}$, and let $a = (a_1, ..., a_r) \in \mathbb{R}^r$. Then:

(a) The function $\psi_a : (B^p_\pi(\mathbb{R}^n),\ \|\ \|_p) \to (B^p_\pi(\mathbb{R}^{n-r}),\ \|\ \|_p)$ defined by $f \to f_a : \mathbb{C}^{n-r} \to \mathbb{C}$, given by $z = (z_{r+1}, ..., z_n) \mapsto f_a(z) := f(a, z)$, is well defined.

(b) ψ_a is continuous. In fact, we have:

$$\|f_a\|_p = \|\psi_a(f)\|_p \leq (C_{\pi,\,p})^r \cdot \|f\|_p \text{ for every } f \in B^p_\pi(\mathbb{R}^n).$$

Proof. Two first items are well known (for the first item see [18], page 336, Theorem 17.3, and for the second one see [21], Chapter 6). In the third item the only non obvious thing is the continuity of ψ_a.

First, we shall prove it for $r = 1$ (step from n to $n-1$).

$C := C_{\pi,\,p},\ a \in \mathbb{R}$. ψ_a has the form:

$$\psi_a : (B^p_\pi(\mathbb{R}^n),\ \|\ \|_p) \to (B^p_\pi(\mathbb{R}^{n-1}),\ \|\ \|_p)$$

given by $f \to f_a : \mathbb{C}^{n-1} \to \mathbb{C}$ defined by: $z \mapsto f_a(z) := f(a, z)$.

Let $f \in B^p_\pi(\mathbb{R}^n)$. For each $x \in \mathbb{R}^{n-1}$ consider the function $h_x : \mathbb{C} \to \mathbb{C}$ defined by: $\omega \mapsto h_x(\omega) := f(x, \omega)$. Observe that $h_x \in B^p_\pi(\mathbb{R})$ and $h_x(a) = f_a(x)$, for every $x \in \mathbb{R}^{n-1}$. Then:

$$\begin{aligned} \|\psi_a(f)\|_p^p = \|f_a\|_p^p &= \int_{\mathbb{R}^{n-1}} |f_a(x)|^p dx = \int_{\mathbb{R}^{n-1}} |f(a, x)|^p dx \\ = \int_{\mathbb{R}^{n-1}} |h_x(a)|^p dx &\leq \int_{\mathbb{R}^{n-1}} \|h_x\|_\infty^p dx \leq C^p \cdot \int_{\mathbb{R}^{n-1}} (\int_{\mathbb{R}} |h_x(y)|^p dy) dx \\ &= C^p \cdot \|f\|_p^p, \end{aligned}$$

where we have used the one-dimensional inequality $\|h_x\|_\infty \leq C\|h_x\|_p$. So that

$$\|\psi_a(f)\|_p \leq C \cdot \|f\|_p.$$

The step from n to $n-r$ is obtained by reiteration. ☑

Corollary 4.3. *Let $p,\, \sigma \in (0,\ +\infty)$.*

(1) The inclusion

$$\varphi_\sigma : (B^p_\sigma(\mathbb{R}),\ \|\ \|_p) \hookrightarrow (B^\infty_\sigma(\mathbb{R}),\ \|\ \|_\infty)$$

is continuous. That is: There exists $C_{\sigma,\, p} > 0$ such that

$$\|f\|_\infty \le C_{\sigma,\, p} \cdot \|f\|_p \text{ for each } f \in B^p_\sigma(\mathbb{R}).$$

We may take $C_{\sigma,\, p} := C_{\pi,\, p} \cdot \left(\frac{\sigma}{\pi}\right)^{1/p} = \left(\frac{2\sigma}{p\pi^3}(e^{p\pi} - 1)\right)^{1/p}$, as said in the statement of Lemma 4.2.

(2) Let $n \in \mathbb{N},\ n \ge 2$. Let $r \in \{1, ..., n-1\}$, and let $a = (a_1, ..., a_r) \in \mathbb{R}^r$. Then:

(a) The function $\psi_a : (B^p_\sigma(\mathbb{R}^n),\ \|\ \|_p) \to (B^p_\sigma(\mathbb{R}^{n-r}),\ \|\ \|_p)$ defined by $f \to f_a : \mathbb{C}^{n-r} \to \mathbb{C}$, given by $z = (z_{r+1}, ..., z_n) \mapsto f_a(z) := f(a, z)$, is well defined.

(b) ψ_a is continuous. In fact, we have

$$\|f_a\|_p = \|\psi_a(f)\|_p \le (C_{\sigma,\, p})^r \cdot \|f\|_p \text{ for each } f \in B^p_\sigma(\mathbb{R}^n).$$

(c) Let $n \in \mathbb{Z},\ n > 0$. The inclusion

$$\varphi : (B^p_\sigma(\mathbb{R}^n),\ \|\ \|_p) \hookrightarrow (B^\infty_\sigma(\mathbb{R}^n),\ \|\ \|_\infty)$$

is continuous. In fact

$$\|f\|_\infty \le C^n_{\sigma,\, p} \cdot \|f\|_p \text{ for each } f \in B^p_\sigma(\mathbb{R}^n).$$

Proof. (1) This is a consequence of the second item of Lemma 4.2 using a change of variable. By the previous lemma we know that the inclusion

$$\varphi_\pi : (B^p_\pi(\mathbb{R}),\ \|\ \|_p) \hookrightarrow (B^\infty_\pi(\mathbb{R}),\ \|\ \|_\infty)$$

is continuous. That is,

$$\|f\|_\infty \le C_{\pi,\, p} \cdot \|f\|_p \text{ for each } f \in B^p_\pi(\mathbb{R}),$$

where $C_{\pi,\, p} = \left(\frac{2}{p\pi^2}(e^{p\pi} - 1)\right)^{1/p}$ (see [21], Chapter 6).

Let us consider the linear operators

$$P_1 : (B^p_\sigma(\mathbb{R}),\ \|\ \|_p) \to (B^p_\pi(\mathbb{R}),\ \|\ \|_p),$$

defined by $f \to P_1(f) : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto P_1(f)(z) := f\left(\frac{\pi}{\sigma} z\right)$, and

$$P_2 : (B_\pi^\infty(\mathbb{R}),\ \|\ \|_\infty) \to (B_\sigma^\infty(\mathbb{R}),\ \|\ \|_\infty),$$

defined by $f \to P_2(f) : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto P_2(f)(z) := f\left(\frac{\sigma}{\pi} z\right)$. Both operators are continuous since

$$\|P_1(f)\|_p = \left(\frac{\sigma}{\pi}\right)^{1/p} \|f\|_p \text{ for every } f \in B_\sigma^p(\mathbb{R}),$$

and

$$\|P_2(f)\|_\infty = \|f\|_\infty \text{ for every } f \in B_\pi^\infty(\mathbb{R}).$$

Since $\varphi_\sigma = P_2 \circ \varphi_\pi \circ P_1$, then φ_σ is continuous, and in fact we have

$$\begin{aligned}\|f\|_\infty = \|\varphi_\sigma(f)\|_\infty = \|P_2\left(\varphi_\pi(P_1(f))\right)\|_\infty = \|\varphi_\pi(P_1(f))\|_\infty \leq \\ \leq C_{\pi,\, p}\|P_1(f)\|_p = C_{\pi,\, p} \cdot \left(\frac{\sigma}{\pi}\right)^{1/p} \|f\|_p = C_{\sigma,\, p}\, \|f\|_p,\end{aligned}$$

for all $f \in B_\sigma^p(\mathbb{R})$.

(2) The proof is analogous to the one of the third item of Lemma 4.2.

(3) This is a consequence of the previous item using induction over the dimension n. For $n = 1$ the result is true by the first item. Assume that the result is true for $n = k \in \mathbb{Z}$, $k > 0$. We will show that the result is true for $n = k + 1$.

Let $f \in B_\sigma^p(\mathbb{R}^{k+1})$. For every $\omega \in \mathbb{C}$ we define

$$f_\omega : \mathbb{C}^k \to \mathbb{C},$$

given by $(z_1, ..., z_k) \mapsto f_\omega(z_1, ..., z_k) := f\,(z_1, ..., z_k, \omega)$, which verifies $f_\omega \in B_\sigma^p(\mathbb{R}^k)$. Then:

$$\begin{aligned}\|f\|_\infty = \sup_{x\in\mathbb{R}^{k+1}} |f(x)| = \sup_{a\in\mathbb{R}} \left(\sup_{\widetilde{x}\in\mathbb{R}^k} |f_a\,(\widetilde{x})|\right) = \sup_{a\in\mathbb{R}} \|f_a\|_\infty \leq \\ \leq C_{\sigma,\, p}^k \cdot \sup_{a\in\mathbb{R}} \|f_a\|_p \leq C_{\sigma,\, p}^k \cdot \sup_{a\in\mathbb{R}} (C_{\sigma,\, p}\|f\|_p) = C_{\sigma,\, p}^{k+1}\, \|f\|_p,\end{aligned}$$

where the first inequality follows from the hypothesis of induction and the second one follows the previous item with $r = 1$.
Conclusion: the result is true for every $n \in \mathbb{Z}$, $n > 0$.

☑

5. Transference of SS

In this section, for every $r \in (0,\ +\infty]$ we define

$$d(r) := \max\left\{1,\ 2^{\frac{1}{r}-1}\right\} = \begin{cases} 2^{\frac{1}{r}-1}, & \text{if } r \in (0,\ 1) \\ 1, & \text{if } r \in [1,\ +\infty] \end{cases}.$$

We recall the following two well known results.

Lemma 5.1. *Let $p \in (0,\ +\infty],\ m \in \mathbb{Z},\ m > 0$. Let $\Lambda \subseteq \mathbb{R}^n$ be u.d. Then*

$$\|a+b\|_p \leq d(p) \cdot \left(\|a\|_p + \|b\|_p\right) \text{ for each } a,\ b \in l^p\left(\Lambda\right).$$

Lemma 5.2. *Let $p \in (0,\ +\infty],\ m \in \mathbb{Z},\ m > 0$. Let $\Omega \subseteq \mathbb{R}^m,\ \Omega \neq \varnothing$, and $\Lambda \subseteq \Omega$ be u.d. Let $(X,\ \|\ \|)$ be a quasinormed space verifying $X \subseteq \mathfrak{F}\left(\Omega,\ \mathbb{C}\right)$. Let $\mathcal{A} \subseteq X$, and suppose that the sampling mapping $S : (X,\ \|\ \|) \to (l^p\left(\Lambda\right), \|\ \|_p)$, given by $f \to (f(\lambda))_{\lambda\in\Lambda}$, is continuous. Then the following statements are equivalent:*

(1) Λ *is a SS for* $\mathcal{A}$.

(2) There exists a constant $C > 0$ such that $\|f\| \leq C\ \|(f(\lambda))_{\lambda\in\Lambda}\|_p$ for each $f \in \mathcal{A}$.

(3) There exists a constant $C > 0$ such that $\|f\| \leq C\ \|(f(\lambda))_{\lambda\in\Lambda}\|_p$ for each $f \in \overline{\mathcal{A}}$.

(4) Λ *is a SS for* $\overline{\mathcal{A}}$.

***Remark* 5.3.** In Lemma 5.2 the constants $C > 0$ of the items 2 and 3 are the same.

Proof. As the sampling mapping $S : (X,\ \|\ \|) \to (l^p(\Lambda), \|\ \|_p)$ is continuous, then there exists a constant $D > 0$ such that

$$\|(f(\lambda))_{\lambda\in\Lambda}\|_p = \|S\left(f\right)\|_p \leq D\ \|f\| \text{ for every } f \in X.$$

Defining $c := \frac{1}{D} > 0$ we have that

$$c\ \|(f(\lambda))_{\lambda\in\Lambda}\|_p \leq \|f\| \text{ for every } f \in X.$$

Hence the first item is equivalent to the second one. By the same motive the third and fourth items are equivalent between themselves.

On the other hand the third item obviously implies the second one as $\mathcal{A} \subseteq \overline{\mathcal{A}}$. Finally we only have to prove that the second item implies the third one. Indeed, assume that there exists a constant $C > 0$ such that

$$\|f\| \leq C\ \|(f(\lambda))_{\lambda\in\Lambda}\|_p \text{ for all } f \in \mathcal{A}.$$

We will show that

$$\|f\| \le C\ \|(f(\lambda))_{\lambda\in\Lambda}\|_p \text{ for every } f\in\overline{\mathcal{A}}.$$

Let $f\in\overline{\mathcal{A}}$. There exists a sequence $(g_n)_{n\in\mathbb{Z}^+}$ in $\mathcal{A}$ which converges to f in X. As S is continuous, then $(S(g_n))_{n\in\mathbb{Z}^+}$ converges to $S(f)$ in $l^p(\Lambda)$, and consequently

$$\lim_{n\to+\infty}\left\|(g_n(\lambda))_{\lambda\in\Lambda}\right\|_p = \lim_{n\to+\infty}\|S(g_n)\|_p = \|S(f)\|_p = \|(f(\lambda))_{\lambda\in\Lambda}\|_p.$$

Let $n\in\mathbb{Z}^+$. Since $g_n\in\mathcal{A}$, then our assumption gives us

$$\|g_n\| \le C\ \|\,(g_n(\lambda))_{\lambda\in\Lambda}\,\|_p.$$

Taking limits in this inequality when $n\to+\infty$ we finally obtain

$$\|f\| \le C\ \|(f(\lambda))_{\lambda\in\Lambda}\|_p,$$

what concludes the proof. ☑

Let us prove now Theorem 1.12.

5.1. Proof of Theorem 1.12

Proof. Take $F\in\mathcal{A}\subseteq E\subseteq\mathcal{C}^1(\mathbb{R}^m)$. Let $n\in\mathbb{Z}^m$. Then

$$\sup_{x\in[\lambda_n,\,\gamma_n]}\|DF(x)\| = \max_{x\in[\lambda_n,\,\gamma_n]}\|DF(x)\| = \|DF(x_n)\|$$

for some $x_n\in[\lambda_n,\,\gamma_n]$, where $[\lambda_n,\,\gamma_n]$ denotes the segment in $\mathbb{R}^m$ with extreme points λ_n and γ_n. Hence $|F(\gamma_n)-F(\lambda_n)|\le\|DF(x_n)\|\cdot|\gamma_n-\lambda_n|$.

Let us estimate $\|DF(x)\|$ for each $x\in\mathbb{R}^m$. Let $x\in\mathbb{R}^m$.

$$\begin{aligned}\|DF(x)\| &= \max_{a\in\mathbb{R}^m\ |a|=1}\{|DF(x)(a)|\} = \\ &= \max_{a=(a_1,\dots,a_m)\in\mathbb{R}^m\ |a|=1}\left\{\left|\sum_{j=1}^m\frac{\partial F}{\partial x_j}(x)\cdot a_j\right|\right\} = \\ &= \max_{a\in\mathbb{R}^m\ |a|=1}\{|<\nabla F(x),\ a>|\} \le \max_{a\in\mathbb{R}^m\ |a|=1}\{|\nabla F(x)|\cdot|a|\} = |\nabla F(x)| = \\ &= \left(\sum_{j=1}^m\left|\frac{\partial F}{\partial x_j}(x)\right|^2\right)^{1/2} \le \left(\sum_{j=1}^m C_1^2\,\frac{\|F\|_p^2}{h(x)^2}\right)^{1/2} = \sqrt{m}\,C_1\,\frac{\|F\|_p}{h(x)},\end{aligned}$$

where the first inequality is consequence of the Cauchy-Schwarz inequality.

Therefore for each $n \in \mathbb{Z}^m$ we have:

$$\|DF(x_n)\| \leq \sqrt{m}\, C_1 \, \frac{\|F\|_p}{h(x_n)} \leq \sqrt{m}\, C_1 \, \frac{\|F\|_p}{h(\gamma_n)},$$

by hypothesis (2), since $x_n \in [\lambda_n,\, \gamma_n]$.

Thus

$$|F(\gamma_n) - F(\lambda_n)| \leq \sqrt{m}\, C_1 \, \frac{\|F\|_p}{h(\gamma_n)} \, |\gamma_n - \lambda_n| \text{ for all } n \in \mathbb{Z}^m,\ F \in \mathcal{A}.$$

We claim that

$$\left\| \left(F\left(\gamma_n\right) - F\left(\lambda_n\right)\right)_{n \in \mathbb{Z}^m} \right\|_p \leq \sqrt{m}\, C_1 \cdot S \cdot \|F\|_p \text{ for all } F \in \mathcal{A}.$$

Let $F \in \mathcal{A}$. We may distinguish two cases:

(1) Case I: $p \in (0,\, +\infty)$. Then we have:

$$\begin{aligned}
\left\| \left(F\left(\gamma_n\right) - F\left(\lambda_n\right)\right)_{n \in \mathbb{Z}^m} \right\|_p &= \left(\sum_{n \in \mathbb{Z}^m} \left|F\left(\gamma_n\right) - F\left(\lambda_n\right)\right|^p \right)^{1/p} \leq \\
&\leq \left(\sum_{n \in \mathbb{Z}^m} \left(\sqrt{m}\right)^p \, C_1^p \, \|F\|_p^p \, \frac{|\gamma_n - \lambda_n|^p}{h(\gamma_n)^p} \right)^{1/p} \leq \\
&\leq \sqrt{m}\, C_1 \, \|F\|_p \left(\sum_{n \in \mathbb{Z}^m} \frac{g(\gamma_n)^p}{h(\gamma_n)^p} \right)^{1/p} = \sqrt{m}\, C_1 \cdot S \cdot \|F\|_p \,.
\end{aligned}$$

(2) Case II: $p = +\infty$. Then we have:

$$\begin{aligned}
\left\| \left(F\left(\gamma_n\right) - F\left(\lambda_n\right)\right)_{n \in \mathbb{Z}^m} \right\|_\infty &= \sup_{n \in \mathbb{Z}^m} \left|F\left(\gamma_n\right) - F\left(\lambda_n\right)\right| \leq \\
\leq \sqrt{m}\, C_1 \, \|F\|_p \sup_{n \in \mathbb{Z}^m} \frac{|\gamma_n - \lambda_n|}{h(\gamma_n)} &\leq \sqrt{m}\, C_1 \, \|F\|_\infty \sup_{n \in \mathbb{Z}^m} \frac{g(\gamma_n)}{h(\gamma_n)} = \\
&= \sqrt{m}\, C_1 \cdot S \cdot \|F\|_\infty \,.
\end{aligned}$$

So that we have proved our claim. Now we are in conditions to prove the items of the theorem.

(1) Assume that Γ is a SS for $\mathcal{A}$ with upper constant of sampling $C > 0$, and $S < \frac{1}{C \cdot C_2}$. We will prove that Λ is a SS for $\mathcal{A}$ with upper constant of

sampling $C' = \frac{C \cdot d(p)}{1 - C \cdot C_2 \cdot S}$. Let $F \in \mathcal{A}$.

$$\begin{aligned}\|F\|_p &\leq C \cdot \left\|\left(F\left(\gamma_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p = \\ &= C \cdot \left\|\left(F\left(\gamma_n\right) - F\left(\lambda_n\right)\right)_{n\in\mathbb{Z}^m} + \left(F\left(\lambda_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p \leq \\ &\leq C \cdot d(p)\left(\left\|\left(F\left(\gamma_n\right) - F\left(\lambda_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p + \left\|\left(F\left(\lambda_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p\right) \leq \\ &\leq C \cdot d(p)\left[\sqrt{m}\, C_1 \cdot S \cdot \|F\|_p + \left\|\left(F\left(\lambda_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p\right] = \\ &= C \cdot C_2 \cdot S \cdot \|F\|_p + C \cdot d(p) \cdot \left\|\left(F\left(\lambda_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p .\end{aligned}$$

Hence $\|F\|_p - C \cdot C_2 \cdot S \cdot \|F\|_p \leq C \cdot d(p) \cdot \left\|\left(F\left(\lambda_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p$. We define $C_3 := 1 - C \cdot C_2 \cdot S$, which is independent of F. $C_3 > 0$ because $S < \frac{1}{C \cdot C_2}$. $C' = \frac{C \cdot d(p)}{C_3} > 0$, which is also independent of F. Then:

$$\|F\|_p \leq C' \cdot \left\|\left(F\left(\lambda_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p .$$

Conclusion: Λ is a SS for $\mathcal{A}$ with upper constant of sampling $C' := \frac{C \cdot d(p)}{1 - C \cdot C_2 \cdot S}$. Therefore, by Lemma 5.2, Λ is a SS for $\overline{\mathcal{A}}$ with upper constant of sampling $C' := \frac{C \cdot d(p)}{1 - C \cdot C_2 \cdot S}$.

(2) Assume that Λ is a SS for $\mathcal{A}$ with upper constant of sampling $D > 0$, and $S < \frac{1}{D \cdot C_2}$. We will prove that Γ is a SS for $\mathcal{A}$ with upper constant of sampling $C' = \frac{D \cdot d(p)}{1 - D \cdot C_2 \cdot S}$. Let $F \in \mathcal{A}$.

$$\begin{aligned}\|F\|_p &\leq D \cdot \left\|\left(F\left(\lambda_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p = \\ &= D \cdot \left\|\left(F\left(\lambda_n\right) - F\left(\gamma_n\right)\right)_{n\in\mathbb{Z}^m} + \left(F\left(\gamma_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p \leq \\ &\leq D \cdot d(p)\left(\left\|\left(F\left(\lambda_n\right) - F\left(\gamma_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p + \left\|\left(F\left(\gamma_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p\right) \leq \\ &\leq D \cdot d(p)\left[\sqrt{m}\, C_1 \cdot S \cdot \|F\|_p + \left\|\left(F\left(\gamma_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p\right] = \\ &= D \cdot C_2 \cdot S \cdot \|F\|_p + D \cdot d(p) \cdot \left\|\left(F\left(\gamma_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p .\end{aligned}$$

Hence

$$\|F\|_p - D \cdot C_2 \cdot S \cdot \|F\|_p \leq D \cdot d(p) \cdot \left\|\left(F\left(\gamma_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p .$$

We define $C_3 := 1 - D \cdot C_2 \cdot S$, which is independent of F. $C_3 > 0$ because $S < \frac{1}{D \cdot C_2}$. Define $C' := \frac{D \cdot d(p)}{C_3} > 0$, which is also independent of F. Then:

$$\|F\|_p \leq C' \cdot \left\|\left(F\left(\gamma_n\right)\right)_{n\in\mathbb{Z}^m}\right\|_p .$$

Conclusion: Γ is a SS for $\mathcal{A}$ with upper constant of sampling

$$C' = \frac{D \cdot d(p)}{1 - D \cdot C_2 \cdot S},$$

and, by Lemma 5.2, it is also a SS for $\overline{\mathcal{A}}$ with the same upper constant of sampling.

☑

Now we obtain two consequences of Theorem 1.12. The first one is the particular case $\Gamma = \mathbb{Z}^m$ and the second one is the application of Theorem 1.12 to Bernstein spaces.

Corollary 5.4. *Let $p \in (0, +\infty]$, $m \in \mathbb{Z}$, $m > 0$. Let $\Lambda \subseteq \mathbb{R}^m$ be u.d., which we may express as $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}^m}$. Let $E \subseteq \mathcal{C}^1(\mathbb{R}^m) \cap L^p(\mathbb{R}^m)$ be a closed vector subspace of $(L^p(\mathbb{R}^m), \| \|_p)$. Suppose that the following sampling operators are continuous:*

$$S_{E,\Lambda} : (E, \| \|_p) \to (l^p(\Lambda), \| \|_p) \text{ defined by } f \to (f(\lambda))_{\lambda\in\Lambda},$$

and

$$S_{E,\mathbb{Z}^m} : (E, \| \|_p) \to (l^p(\mathbb{Z}^m), \| \|_p) \text{ given by } f \to (f(n))_{n\in\mathbb{Z}^m}.$$

Let $h : \mathbb{R}^m \to \mathbb{R}$, $h > 0$, be a continuous function and let $C_1 > 0$ be. We define

$$\mathcal{F}(h, C_1) := \left\{ f \in E \mid \left| \frac{\partial f}{\partial x_j}(y) \right| \le C_1 \cdot \frac{\|f\|_p}{h(y)} \; \forall j \in \{1, ..., m\}, \; y \in \mathbb{R}^m \right\}.$$

Let $\mathcal{A} \subseteq \mathcal{F}(h, C_1) \subseteq E$. Assume that

(1) There exists a function $g : \mathbb{Z}^m \to \mathbb{R}$, $g \ge 0$, such that

$$|n - \lambda_n| \le g(n) \text{ for all } n \in \mathbb{Z}^m,$$

where $| \, |$ denotes the Euclidean norm in $\mathbb{R}^m$.

(2)

$$\min\{h(x) \mid (\exists t \in [0, 1] : x = t\lambda_n + (1-t)n)\} = h(n),$$

for all $n \in \mathbb{Z}^m$.

(3) $S < +\infty$, where $S := \begin{cases} \left(\sum_{n\in\mathbb{Z}^m} \left(\frac{g(n)}{h(n)}\right)^p\right)^{1/p}, & \text{if } p \in (0, +\infty) \\ \sup_{n\in\mathbb{Z}^m} g(n)/h(n), & \text{if } p = +\infty \end{cases}.$

We define $C_2 := \sqrt{m} \cdot C_1 \cdot d(p) > 0$. Then we have:

(1) If $\mathbb{Z}^m$ is a SS for $\mathcal{A}$ with upper constant of sampling $C > 0$, and $S < \frac{1}{C \cdot C_2}$, then Λ is a SS for $\overline{\mathcal{A}}$ with upper constant of sampling $C' = \frac{C \cdot d(p)}{1 - C \cdot C_2 \cdot S}$.

(2) If Λ is a SS for $\mathcal{A}$ with upper constant of sampling $D > 0$, and $S < \frac{1}{D \cdot C_2}$, then $\mathbb{Z}^m$ is a SS for $\overline{\mathcal{A}}$ with upper constant of sampling $C' = \frac{D \cdot d(p)}{1 - D \cdot C_2 \cdot S}$.

We may apply Theorem 1.12 to Bernstein spaces because of Theorem 2.3 and Corollary 4.3 of immersion of Bernstein spaces in classical Bernstein spaces.

Corollary 5.5. *Let $p \in [1,\ +\infty]$, $m \in \mathbb{Z}$, $m > 0$, and $\sigma > 0$. Let $\Gamma,\ \Lambda \subseteq \mathbb{R}^m$ be u.d., which we may express as $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}^m}$, $\Gamma = \{\gamma_n\}_{n\in\mathbb{Z}^m}$. Let $C_1 > 0$ be a constant verifying*

$$\left|\frac{\partial f}{\partial x_j}(y)\right| \leq C_1 \cdot \|f\|_p \ \text{ for each } f \in B^p_\sigma(\mathbb{R}^m),\ j \in \{1, ..., m\},\ y \in \mathbb{R}^m.$$

We may take $C_1 = \sigma\ \left(\frac{2\sigma}{p\pi^3}(e^{p\pi}-1)\right)^{m/p}$ if $p \in [1,\ +\infty)$ (see [21], Chapter 6); and $C_1 = \sigma$ if $p = +\infty$ (see Theorem 2.3).

Suppose that

(1) There exists a function $g : \Gamma \longrightarrow \mathbb{R}$, $g \geq 0$, such that

$$|\gamma_n - \lambda_n| \leq g(\gamma_n) \ \text{ for all } n \in \mathbb{Z}^m,$$

where $|\,|$ denotes the Euclidean norm in $\mathbb{R}^m$.

(2) $S < +\infty$, where $S := \begin{cases} \left(\sum_{n\in\mathbb{Z}^m} g(\gamma_n)^p\right)^{\frac{1}{p}}, & \text{if } p \in (0,\ +\infty) \\ \sup_{n\in\mathbb{Z}^m} g(\gamma_n), & \text{if } p = +\infty \end{cases}.$

Define $C_2 := \sqrt{m} \cdot C_1 \cdot d(p) > 0$. Then

(1) If Γ is a SS for $B^p_\sigma(\mathbb{R}^m)$ with upper constant of sampling $C > 0$, and $S < \frac{1}{C\cdot C_2}$, then Λ is a SS for $B^p_\sigma(\mathbb{R}^m)$ with upper constant of sampling $C' = \frac{C\cdot d(p)}{1-C\cdot C_2\cdot S}$.

(2) If Λ is a SS for $B^p_\sigma(\mathbb{R}^m)$ with upper constant of sampling $D > 0$, and $S < \frac{1}{D\cdot C_2}$, then Γ is a SS for $B^p_\sigma(\mathbb{R}^m)$ with upper constant of sampling $C' = \frac{D\cdot d(p)}{1-D\cdot C_2\cdot S}$.

***Proof*.** Let $f \in B^p_\sigma(\mathbb{R}^m)$, $j \in \{1, ..., m\}$, $y \in \mathbb{R}^m$. By Theorem 2.3 and Corollary 4.3 of immersion of Bernstein spaces in classical Bernstein spaces, we have:

$$\left|\frac{\partial f}{\partial x_j}(y)\right| \leq \left\|\frac{\partial f}{\partial x_j}\right\|_\infty \leq \sigma \cdot \|f\|_\infty \leq \sigma\ \left(\frac{2\sigma}{p\pi^3}(e^{p\pi}-1)\right)^{m/p} \|f\|_p\,,$$

where the second inequality is consequence of Theorem 2.3, and the last inequality is for $p \in [1,\ +\infty)$.

Theorem 1.12 concludes the proof. ☑

6. Uniqueness sets

In this section we will state and prove two results which give a relationship of transference between the uniqueness sets of a Bernstein space respect to other one. First we need an auxiliary result.

Lemma 6.1. *Let $\sigma > 0,\ p \in (0,\ +\infty]$. Let $z_0 \in \mathbb{C}^n$.*

(1) There exists $g \in B_\sigma^p(\mathbb{R}^n)$ such that $g(z_0) \neq 0$.

(2) There exists $h \in B_\sigma^p(\mathbb{R}^n) \setminus \{0\}$ such that $h(z_0) = 0$.

Proof. (1) Let $f \in B_\sigma^p(\mathbb{R}^n)$, $f \neq 0$ (that is, f is not identically zero). There exists $z_1 \in \mathbb{C}^n$ such that $f(z_1) \neq 0$. We take the translation $g := \tau_{z_1 - z_0} f$, defined by $\tau_{z_0 - z_1} f(z) := f(z - (z_0 - z_1)) = f(z - z_0 + z_1)$ for every $z \in \mathbb{C}^n$. Then $g \in B_\sigma^p(\mathbb{R}^n)$ and $g(z_0) \neq 0$.

(2) It is analogous to the previous item. Let $f \in B_\sigma^p(\mathbb{R}^n)$, $f \neq 0$, verifying that there exists $z_1 \in \mathbb{C}^n$ such that $f(z_1) = 0$. We take the translation $g := \tau_{z_0 - z_1} f$, defined as in the previous item. Then $g \in B_\sigma^p(\mathbb{R}^n) \setminus \{0\}$ and $g(z_0) = 0$.

☑

Proposition 6.2. *Let $\sigma, \epsilon > 0,\ p \in [1, +\infty]$. Let $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete. Suppose that Λ is not a US for $B_\sigma^p(\mathbb{R}^n)$. Then Λ is not a US for $B_{\sigma+\epsilon}^1(\mathbb{R}^n)$.*

Proof. Let us take $f \in B_\sigma^p(\mathbb{R}^n) \setminus \{0\}$ such that $f(\lambda) = 0$ for each $\lambda \in \Lambda$. There exists $z_f \in \mathbb{C}^n$ verifying that $f(z_f) \neq 0$. Let q be the conjugate exponent of p. We take $g \in B_\epsilon^q(\mathbb{R}^n)$ such that $g(z_f) \neq 0$. Now we consider $h := f \cdot g \in B_{\sigma+\epsilon}^1(\mathbb{R}^n)$ (in fact, $\|h\|_1 \leq \|f\|_p \cdot \|g\|_q$). Then $h \neq 0$ and $h(\lambda) = f(\lambda) \cdot g(\lambda) = 0$ for every $\lambda \in \Lambda$. Hence Λ is not a US for $B_{\sigma+\epsilon}^1(\mathbb{R}^n)$. ☑

Proposition 6.3. *Let $\sigma,\ \epsilon > 0$ and $p \in (0,\ +\infty]$. Let $\Lambda \subseteq \mathbb{R}^n$ be uniformly discrete. Suppose that Λ is not a US for $B_\sigma^\infty(\mathbb{R}^n)$. Then Λ is not a US for $B_{\sigma+\epsilon}^p(\mathbb{R}^n)$.*

Proof. Let us take $f \in B_\sigma^\infty(\mathbb{R}^n) \setminus \{0\}$ such that $f(\lambda) = 0$ for every $\lambda \in \Lambda$. There exists $z_f \in \mathbb{C}^n$ verifying that $f(z_f) \neq 0$. We take $g \in B_\epsilon^p(\mathbb{R}^n)$ such that $g(z_f) \neq 0$. Now we consider $h := f \cdot g \in B_{\sigma+\epsilon}^p(\mathbb{R}^n)$. Then $h \neq 0$ and $h(\lambda) = f(\lambda) \cdot g(\lambda) = 0$ for each $\lambda \in \Lambda$. Therefore Λ is not a US for $B_{\sigma+\epsilon}^p(\mathbb{R}^n)$. ☑

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