

## On the continuity of the map square root of nonnegative isomorphisms in Hilbert spaces

JEOVANNY DE JESUS MUENTES ACEVEDO\*

Universidade de São Paulo, Instituto de Matemática e Estatística, São Paulo, Brasil.

**Abstract.** Let  $H$  be a real (or complex) Hilbert space. Every nonnegative operator  $L \in L(H)$  admits a unique nonnegative square root  $R \in L(H)$ , i.e., a nonnegative operator  $R \in L(H)$  such that  $R^2 = L$ . Let  $GL_S^+(H)$  be the set of nonnegative isomorphisms in  $L(H)$ . First we will show that  $GL_S^+(H)$  is a convex (real) Banach manifold. Denoting by  $L^{1/2}$  the nonnegative square root of  $L$ . In [3], Richard Bouldin proves that  $L^{1/2}$  depends continuously on  $L$  (this proof is non-trivial). This result has several applications. For example, it is used to find the polar decomposition of a bounded operator. This polar decomposition allows us to determine the positive and negative spectral subspaces of any self-adjoint operator, and moreover, allows us to define the Maslov index. The autor of the paper under review provides an alternative proof (and a little more simplified) that  $L^{1/2}$  depends continuously on  $L$ , and moreover, he shows that the map

$$\begin{aligned} \mathcal{R} : GL_S^+(H) &\rightarrow GL_S^+(H) \\ L &\mapsto L^{1/2} \end{aligned}$$

is a homeomorphism.

**Keywords:** Nonnegative operators, functions of operators, Hilbert spaces, spectral theory.

**MSC2010:** 47A56, 46G20, 54C60.

## Sobre la continuidad de la aplicación raíz cuadrada de isomorfismos no negativos en espacios de Hilbert

**Resumen.** Sea  $H$  un espacio de Hilbert real (o complejo). Todo operador no negativo  $L \in L(H)$  admite una única raíz cuadrada no negativa  $R \in L(H)$ , esto es, un operador no negativo  $R \in L(H)$  tal que  $R^2 = L$ . Sea  $GL_S^+(H)$

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\* Email: jeovanny@ime.usp.br

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el conjunto de los isomorfismos no negativos en  $L(H)$ . Primero probaremos que  $GL_S^+(H)$  es una variedad de Banach (real). Denotando como  $L^{1/2}$  la raíz cuadrada no negativa de  $L$ , en [3] Richard Bouldin prueba que  $L^{1/2}$  depende continuamente de  $L$  (esta prueba es no trivial). Este resultado tiene varias aplicaciones. Por ejemplo, es usado para encontrar la descomposición polar de un operador limitado. Esta descomposición polar nos lleva a determinar los subespacios espectrales positivos y negativos de cualquier operador autoadjunto, y además, lleva a definir el índice de Máslov. El autor de este artículo da una prueba alternativa (y un poco más simplificada) de que  $L^{1/2}$  depende continuamente de  $L$ , y además, prueba que la aplicación

$$\begin{aligned} \mathcal{R} : GL_S^+(H) &\rightarrow GL_S^+(H) \\ L &\mapsto L^{1/2} \end{aligned}$$

es un homeomorfismo.

**Palabras clave:** Operadores no negativos, funciones de operadores, espacios de Hilbert, teoría espectral.

## 1. Introduction

Throughout this article,  $E$  and  $F$  shall denote complex Banach spaces and  $H$  a real or complex Hilbert space. Theorem 2.9 shows that every nonnegative bounded operator admits a unique nonnegative square root.

In [6], P.M. Fitzpatrick, J. Pejsachowicz and L. Recht use this square root to show the existence of a parametrix cogredient to a path of self-adjoint Fredholm operators, which is used to define the spectral flux for this path. The square root is also used to find the polar decomposition of a bounded operator (see T. Kato in [7], p. 334). This polar decomposition permits us to determine the positive and negative spectral subspaces of any self-adjoint operator, and moreover, to define the Maslov index (see Y. Long in [10], Chapter 2). These applications depend on the continuity of the map  $\mathcal{R}$ . We shall give an alternative proof of the continuity of  $\mathcal{R}$ .

In the next section we will remember some notions and recall several known results that will be used in the rest of the work.

We shall denote by  $GL_S^+(H)$  the subset of  $L(H)$  consisting of positive isomorphisms. In the third section we will prove that  $GL_S^+(H)$  is a convex Banach manifold.

Let  $E$  be a complex Banach space. Let  $L \in L(E)$  and  $\sigma(L)$  the spectrum of  $L$ . With the aid of the Cauchy's integral formula, in Section 4 we will see that for any holomorphic map  $f : \Delta \rightarrow \mathbb{C}$ , where  $\Delta$  is an open subset of  $\mathbb{C}$  that contains  $\sigma(L)$ , we can define the operator  $\hat{f}(L) \in L(E)$  as

$$\hat{f}(L) = -\frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(L - \lambda I)^{-1} d\lambda, \quad (1)$$

where  $\Gamma$  is a simple positively oriented closed path (or a finite number of closed paths which do not intersect), contained in  $\Delta$  and then containing  $\sigma(L)$  in its interior. When  $E$

is a real Banach space, we use the complexifications of  $E$  and of  $L$  to obtain an analogous formule as in (1).

Finally, in Section 5, we prove that if  $L \in L(H)$  is positive, the nonnegative square root of  $L$  can be expressed in the form  $\hat{\gamma}(L)$ , where  $\gamma : \Theta \rightarrow \mathbb{C}$  is a convenient holomorphic map and  $\Theta$  is an open subset of  $\mathbb{C}$ , which contains the spectrum of  $L$ . Using this expression, we shall show that the map square root  $\mathcal{R} : GL_S^+(H) \rightarrow GL_S^+(H)$  is continuous.

## 2. Preliminaries

Let  $L(E, F)$  (or  $L(E)$  if  $F = E$ ) be the Banach space of bounded linear operators  $L : E \rightarrow F$  with the norm

$$\|L\| = \sup_{x \in E} \frac{\|Lx\|}{\|x\|}.$$

The identity of  $L(E)$  is denoted by  $I$ .

**Proposition 2.1.** *Let  $L \in L(E, F)$  be an isomorphism. If  $A \in L(E, F)$  and  $\|A - L\| < 1/\|L^{-1}\|$ , then  $A$  is an isomorphism.*

*Proof.* See, for example, [7], p. 31. ☑

**Proposition 2.2.** *Let  $L \in L(E, F)$  be onto  $Y$ .*

1. *If  $L$  is one-to-one, then  $L^{-1} : F \rightarrow E$  is a bounded linear operator.*
2. *There is a constant  $M > 0$  such that for every  $y \in F$  there is  $x \in L^{-1}(y)$  satisfying  $\|x\| \leq M\|y\|$ .*
3.  *$F$  is isomorphic to  $E/\text{Ker}(L)$ .*

*Proof.* See [5], Corollary 2.25. ☑

The spectrum of a real square matrix in  $\mathbb{R}^n$  consists of the roots of its characteristic polynomial. These roots can be real or complex numbers. If  $X$  is a real infinite-dimensional Banach space and  $L \in L(X)$ , we define the spectrum of  $L$  making use of the complexifications of  $X$  and of  $L$  (see, for example, [5], p. 39, for this construction). For the reader's convenience we shall recall these notions.

**Definition 2.3.** Let  $X$  be a real vector space. The set

$$\widehat{X} = \{x_1 + ix_2 : x_1, x_2 \in X\}$$

becomes a complex vector space, setting

1.  $(x_1 + ix_2) + (y_1 + iy_2) = x_1 + y_1 + i(x_2 + y_2)$  for  $x_1 + ix_2, y_1 + iy_2 \in \widehat{X}$ .
2.  $(a + ib)(x_1 + ix_2) = ax_1 - bx_2 + i(bx_1 + ax_2)$  for  $x_1 + ix_2 \in \widehat{X}$  and  $a + ib \in \mathbb{C}$ .

We call  $\widehat{X}$  the *complexification* of  $X$ .

When  $X$  is a real Banach space with norm  $\|\cdot\|$ , then, for each  $x_1 + ix_2 \in \widehat{X}$ ,

$$\|x_1 + ix_2\|_{\widehat{X}} = \max_{\theta} (\|\cos \theta x_1 - \sin \theta x_2\|^2 + \|\sin \theta x_1 + \cos \theta x_2\|^2)^{1/2} \quad (2)$$

defines a norm in  $\widehat{X}$ . With the norm  $\|\cdot\|_{\widehat{X}}$ ,  $\widehat{X}$  becomes a complex Banach space. Furthermore, note that

$$\|x\| = \|x + i0\|_{\widehat{X}} \text{ for all } x \in X.$$

**Definition 2.4.** The *complexification* of  $L \in L(X)$  is the operator  $\widehat{L} \in L(\widehat{X})$ , defined by

$$\widehat{L}(x + iy) = L(x) + iL(y) \text{ for all } x + iy \in \widehat{X}.$$

From definition of  $\widehat{L}$ , we infer that

$$\|\widehat{L}\|_{\widehat{X}} = \|L\|.$$

Now suppose that  $H$  is a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$ . Then

$$\langle x_1 + ix_2, y_1 + iy_2 \rangle_{\widehat{H}} = \langle x_1, y_1 \rangle - i\langle x_1, y_2 \rangle + i\langle x_2, y_1 \rangle + \langle x_2, y_2 \rangle, \quad (3)$$

for  $x_1 + ix_2, y_1 + iy_2 \in \widehat{H}$ , defines an inner product in  $\widehat{H}$ . Moreover, it is easy to see that  $\|\cdot\|_{\widehat{H}}$  is induced by this inner product. Consequently,  $(\widehat{H}, \|\cdot\|_{\widehat{H}})$  is a complex Hilbert space. Furthermore,

$$\|x_1 + ix_2\|_{\widehat{H}} = (\|x_1\|^2 + \|x_2\|^2)^{1/2} \text{ for all } x_1 + ix_2 \in \widehat{H}. \quad (4)$$

Now we define the spectrum of an operator in  $L(E)$ .

**Definition 2.5.** Let  $L$  be an element of  $L(E)$ . A *regular value* of  $L$  is a number  $\lambda \in \mathbb{C}$  such that  $L - \lambda I$  is an isomorphism. The set consisting of regular values of  $L$ , denoted by  $\rho(L)$ , is called the *resolvent set* of  $L$ . Its complement  $\sigma(L) = \mathbb{C} \setminus \rho(L)$  is called the *spectrum* of  $L$ . For any  $\lambda \in \rho(L)$ , the application  $R(\lambda) = (L - \lambda I)^{-1}$  is called the *resolvent application* of  $L$  in  $\lambda$ . If  $L \in L(X)$ , where  $X$  is a real Banach space, we define the spectrum of  $L$  as the spectrum of its complexification, this is  $\sigma(L) = \sigma(\widehat{L})$ .

It is well known that the spectrum of an operator  $L \in L(E)$  is a non-empty compact subset of  $\mathbb{C}$ , and that

$$\|\lambda\| \leq \|L\|, \quad \text{for all } \lambda \in \sigma(L). \quad (5)$$

For reader's convenience, we will record some other notions that we need to obtain our principal result. We shall denote by  $H$  a real (complex) Hilbert space. Next theorem gives an important property of the spectrum of self-adjoint operators.

**Theorem 2.6** ([12], Theorem 6.11-A). *The spectrum of a self-adjoint operator  $L \in L(H)$  is contained in the interval  $[m, M] \subseteq \mathbb{R}$ , where*

$$m = \inf_{\|x\|=1} \langle Lx, x \rangle \quad \text{and} \quad M = \sup_{\|x\|=1} \langle Lx, x \rangle.$$

**Definition 2.7.** Let  $L \in L(H)$  be a self-adjoint operator. We say that  $L$  is *nonnegative* if  $\langle Lx, x \rangle \geq 0$ , for all  $x \in H$ . If  $\langle Lx, x \rangle > 0$ , for all  $x \in H$ , with  $\|x\| = 1$ , then we say that  $L$  is *positive*. The set of positive isomorphisms in  $L(H)$  is denoted by  $GL_S^+(H)$ .

Observe that if  $H$  is a real Hilbert space and  $L \in L(H)$  is nonnegative, then  $\widehat{L} \in L(\widehat{H})$  is nonnegative (see (3)).

It is not difficult to see that if  $L$  is positive, then  $L$  is injective.

**Definition 2.8.** Let  $L \in L(H)$  be a nonnegative operator. A *square root* of  $L$  is any operator  $R \in L(H)$  such that  $R^2 = L$ .

In general, a nonnegative operator may have several square roots. The following theorem, whose proof can be found, for example, in [9], Theorem 9.4-2, shows that each nonnegative operator has exactly one nonnegative square root.

**Theorem 2.9.** *Each nonnegative operator  $L \in L(H)$  has a unique nonnegative square root  $R \in L(H)$ . The operator  $R$  commutes with each operator in  $L(H)$  that commutes with  $L$ .*

In Section 5 we will give an explicit formula for the square root of a  $L \in GL_S^+(H)$ .

We end this section showing that the resolvent application is holomorphic.

**Definition 2.10.** Let  $\Delta$  be an open subset of  $\mathbb{C}$ . We say that an application  $f : \Delta \rightarrow E$  is *holomorphic* at  $\lambda_0 \in \Delta$  if the limit

$$\lim_{\lambda \rightarrow \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = f'(\lambda_0)$$

exists. If  $f$  is holomorphic at each point of  $\Delta$ , we say that  $f$  is *holomorphic* on  $\Delta$ , or simply that  $f$  is *holomorphic*.

**Proposition 2.11.** *Let  $L \in L(E)$ . The application  $R : \rho(L) \rightarrow L(E)$ , defined by  $R(\lambda) = (L - \lambda I)^{-1}$  for  $\lambda \in \rho(L)$ , is holomorphic.*

*Proof.* See [7], Chapter I, Section 5. □

### 3. Some topological properties of $GL_S^+(H)$

Recall that we denote by  $GL_S^+(H)$  the set of positive isomorphisms in  $L(H)$ , where  $H$  is a real (complex) Hilbert space. In this section we shall show that  $GL_S^+(H)$  is convex and, moreover, is a (real) Banach manifold. The proofs of the Theorem 3.2 and Proposition 3.3 are those of the author.

First, assume that  $H$  is finitely dimensional. Take  $L, T \in GL_S^+(H)$  and  $t \in [0, 1]$ . Note that  $tL + (1 - t)T$  is positive, because if  $x \neq 0 \in H$ , then

$$\langle tL + (1 - t)Tx, x \rangle = \langle tLx, x \rangle + \langle (1 - t)Tx, x \rangle > 0.$$

Thus,  $tL + (1 - t)T$  is injective. Since  $H$  is finitely dimensional,  $tL + (1 - t)T$  is an isomorphism. Therefore  $tL + (1 - t)T \in GL_S^+(H)$ . Hence  $GL_S^+(H)$  is a convex subset of  $L(H)$ .

To show that  $GL_S^+(H)$  is convex, when  $H$  is an infinite dimensional Hilbert space, we first recall a well known result in the theory of operators in Hilbert spaces.

**Lemma 3.1.** *Let  $H$  a real (or complex) Hilbert space. If  $L \in GL_S^+(H)$ , then there exists  $c > 0$  such that*

$$\inf_{\|x\|=1} \langle Lx, x \rangle \geq c.$$

**Theorem 3.2.** *The set  $GL_S^+(H)$  is convex.*

*Proof.* Let  $L$  and  $T$  be positive isomorphisms and  $t \in [0, 1]$ . Since  $tL + (1 - t)T$  is positive, we have

$$\text{Ker}(tL + (1 - t)T) = \{0\}.$$

Now, we shall show that  $tL + (1 - t)T$  is surjective. To this end, let us first see that  $\text{Im } tL + (1 - t)T$  is closed. By the Lemma 3.1, we have that there are positive real numbers  $c_1$  and  $c_2$  such that

$$c_1 \leq \inf_{\substack{\|x\|=1 \\ x \in H}} \langle Lx, x \rangle \quad \text{and} \quad c_2 \leq \inf_{\substack{\|x\|=1 \\ x \in H}} \langle Tx, x \rangle.$$

Therefore, if  $x \in H$  with  $\|x\| = 1$ , the Cauchy-Schwarz inequality implies that

$$\begin{aligned} \|(tL + (1 - t)T)x\| &\geq \langle (tL + (1 - t)T)x, x \rangle = t\langle Lx, x \rangle + (1 - t)\langle Tx, x \rangle \\ &\geq (tc_1 + (1 - t)c_2)\|x\|. \end{aligned}$$

Since  $tc_1 + (1 - t)c_2 > 0$ , it follows from Proposition 2.2 that  $\text{Im}(tL + (1 - t)T)$  is closed. Thus, using that  $tL + (1 - t)T$  is self-adjoint and knowing the fact that for any operator  $S \in L(H)$ ,  $[\text{Ker } S]^\perp = \overline{\text{Im}(S^*)}$ , we come to that

$$\begin{aligned} \text{Im}(tL + (1 - t)T) &= \overline{\text{Im}(tL + (1 - t)T)} = \overline{\text{Im}(tL + (1 - t)T)^*} \\ &= [\text{Ker}(tL + (1 - t)T)]^\perp = H, \end{aligned}$$

that is,  $tL + (1 - t)T$  is surjective. Consequently,  $tL + (1 - t)T$  is a nonnegative isomorphism.  $\square$

We denote by  $L_S(H)$  the space of self-adjoint operators in  $L(H)$ . Note that if  $H$  is a complex Hilbert space then  $L_S(H)$  is not a vector subspace  $L(H)$ , because if  $L \neq 0 \in L_S(H)$ ,  $iL \notin L_S(H)$ . However, it is not difficult to prove that  $L_S(H)$  is a vector space over  $\mathbb{R}$ . Moreover, it is easy to see that  $L_S(H)$  is a real Banach space with the topological structure induced by the one of  $L(H)$ .

We now show that  $GL_S^+(H)$  is an open subset of  $L_S(H)$ .

**Proposition 3.3.** *The set  $GL_S^+(H)$  is open in  $L_S(H)$ .*

*Proof.* Let  $L \in GL_S^+(H)$ . It follows from Lemma 3.1 that there exists  $c > 0$  such that

$$c \leq \inf_{\|x\|=1} \langle Lx, x \rangle.$$

Let  $T \in L_S(H)$  be such that  $\|L - T\| < \min\{1/\|L^{-1}\|, c\}$ . Thus,  $T$  is an isomorphism (see Lemma 2.1) and, furthermore, for  $x \neq 0$  in  $H$ , we have

$$\langle Lx, x \rangle \geq c\langle x, x \rangle > \|L - T\|\langle x, x \rangle = \|(L - T)x, x\| \geq \langle (L - T)x, x \rangle,$$

that is,

$$0 < \langle Lx, x \rangle - \langle (L - T)x, x \rangle = \langle Tx, x \rangle \quad \text{for all } x \in H \text{ with } x \neq 0.$$

So,  $T$  is a nonnegative isomorphism. □

Since  $L_S(H)$  is a real Banach space, it follows from Proposition 3.3 that  $GL_S^+(H)$  is a Banach manifold.

#### 4. Functions of operators

The notion of a complex function can be generalized to functions with values in complex Banach space. As in the complex case, the continuous functions with values in Banach spaces are integrable. Readers who want to learn about this notion may read, for example, [11], Section 6.3, or [1], Chapter VII, Section 3.

In this section, we will recall some notions and facts to obtain an analogous version to the Cauchy's integral formula in the context of operators in  $E$ . I will introduce the required concepts or theorems for this construction. The concepts and definitions of the first part of this section can be found in detail in [4], Chapter IV or in [8], Chapter 3, among other books on complex variable.

**Definition 4.1.** Let  $\Gamma : [a, b] \rightarrow \mathbb{C}$  be a path, that is, a continuous application. For a partition of  $[a, b]$  given by  $P = \{t_0, t_1, \dots, t_m\}$ , we define

$$\Lambda_\Gamma(P) = \sum_{k=1}^m \|\Gamma(t_k) - \Gamma(t_{k-1})\|.$$

Let

$$\Lambda_\Gamma = \sup\{\Lambda_\Gamma(P) : P \text{ is a partition of } [a, b]\}.$$

Then, we say that  $\Gamma$  is *rectifiable* with *length*  $\Lambda_\Gamma$ , provided that  $\Lambda_\Gamma$  is finite.

**Definition 4.2.** Let  $\Gamma : [a, b] \rightarrow \mathbb{C}$  be a path.

We say that  $\Gamma$  is *closed* if  $\Gamma(a) = \Gamma(b)$ . In this case the *interior* of  $\Gamma$ , denoted by  $\overset{\circ}{\Gamma}$ , is a region of  $\mathbb{C}$  bounded by  $\Gamma$ .

The path  $\Gamma$  is called *simple* if  $\Gamma(t_1) \neq \Gamma(t_2)$  for  $t_1, t_2 \in [a, b]$ , with  $t_1 \neq t_2$  and at least one of them is in  $(a, b)$ . If  $t_1 < t_2$ , we use the notation  $\Gamma(t_1) < \Gamma(t_2)$ , whenever  $\Gamma(t_1) \neq \Gamma(t_2)$ . For simplicity, the image  $\Gamma([t_1, t_2])$  is denoted by  $[\Gamma(t_1), \Gamma(t_2)]$ . Furthermore, we shall write  $\lambda \in \Gamma$  to denote that  $\lambda$  belongs to  $\Gamma([a, b])$ .

A closed path  $\Gamma$  is *positively oriented* if it is such that when traveling on it one always have  $\dot{\Gamma}$  on the left.

For simplicity, a closed, simple, positively oriented and rectifiable path  $\Gamma : [a, b] \rightarrow \mathbb{C}$  is called *closed path*, and moreover, it will be denoted by  $\Gamma$ .

Now we recall the definition of the integral of an application along a path contained in the complex plane. Consider a rectifiable path  $\Gamma : [a, b] \rightarrow \Delta$ . A *partition* of  $\text{Im } \Gamma$  is a subset  $P = \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\} \subseteq \text{Im } \Gamma$ , where  $\lambda_0 = \Gamma(a)$  and  $\lambda_n = \Gamma(b)$ , such that

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n.$$

The *norm* of  $P$  is defined by

$$\|P\| = \max\{|\lambda_i - \lambda_{i-1}| : i = 1, \dots, n\}.$$

Now take an application  $f : \Delta \rightarrow E$  and a rectifiable path  $\Gamma : [a, b] \rightarrow \Delta$ . For a partition  $P = \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\}$  of  $\text{Im } \Gamma$ , let  $Q = \{\zeta_1, \zeta_2, \dots, \zeta_n\}$ , where  $\zeta_i \in [\lambda_{i-1}, \lambda_i]$ , for  $i = 1, 2, \dots, n$ . Further, consider the sum

$$S(P, Q, f) = \sum_{i=1}^n (\lambda_i - \lambda_{i-1}) f(\zeta_i). \quad (6)$$

**Definition 4.3.** We say that  $f : \Delta \rightarrow E$  is *integrable* on the path  $\Gamma$  if there exists a number  $A$  with the following property: For a given  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that

$$|S(P, Q, f) - A| < \varepsilon \text{ whenever } \|P\| < \delta.$$

The number  $A$  is called *integral* of  $f$  on  $\Gamma$  and is denoted by

$$\int_{\Gamma} f(\lambda) d\lambda.$$

In the case of real continuous functions in one real variable, the analogous notion of integral above is equivalent to the classical definition of the Riemann integral.

We now proceed with the following theorem.

**Theorem 4.4.** *If  $f : \Delta \rightarrow E$  is continuous, then  $f$  is integrable in any rectifiable path contained in  $\Delta$ .*

We can find a prove of the Theorem above when  $E = \mathbb{C}$  in [4], Chapter IV, Theorem 1.4. However, it is not difficult to see that the same proof can be adapted to the case where  $E$  is any complex Banach space.

It follows from Theorem 4.4 and Definition 4.3 that, if  $f : \Delta \rightarrow E$  is continuous (and consequently integrable), then for any sequence  $P_m = \{\lambda_0^m, \lambda_1^m, \dots, \lambda_n^m\}$  of partitions of  $\Gamma$ , such that  $\lim_{m \rightarrow \infty} \|P_m\| = 0$ , and  $Q_m = \{\zeta_1^m, \zeta_2^m, \dots, \zeta_n^m\}$ , where  $\zeta_i^m \in [\lambda_{i-1}^m, \lambda_i^m]$  for  $i = 1, 2, \dots, n$ , one has

$$\int_{\Gamma} f(\lambda) d\lambda = \lim_{n \rightarrow \infty} S(P_n, Q_n, f). \quad (7)$$



In fact, because  $f$  is integrable, the limit in (7) exists, and by Definition (4.3) this limit will be the integral of  $f$  on  $\Gamma$ .

A property of the above integral is given in the following lemma (see, for example, [8], p. 45, Theorem 5).

**Lemma 4.5.** *Let  $f : \Delta \rightarrow E$  be a holomorphic application. Then for any rectifiable path  $\Gamma \subseteq \Delta$ ,*

$$\left\| \int_{\Gamma} f(\lambda) d\lambda \right\| \leq Ml,$$

where  $M = \sup_{\lambda \in \Gamma} |f(\lambda)|$  and  $l$  is the length of  $\Gamma$ .

We continue this work by presenting some classical results that will be used in this section. The following definition is an analogous to the Cauchy's integral formula for holomorphic complex applications (see, for example, [8], p. 61).

**Definition 4.6.** Suppose that  $L \in L(E)$  and let  $f : \Delta \rightarrow \mathbb{C}$  be a holomorphic application such that  $\sigma(L) \subseteq \Delta$ . Further, let  $\omega \subseteq \mathbb{C}$  be an open set such that its boundary consists of a finite number of closed paths  $\Gamma_1, \dots, \Gamma_n$ , and

$$\sigma(L) \subseteq \omega = \bigcup_{i=1}^n \overset{\circ}{\Gamma}_i \subseteq \overline{\bigcup_{i=1}^n \overset{\circ}{\Gamma}_i} \subseteq \Delta.$$

We define an operator  $\hat{f}(L)$  by the formula

$$\hat{f}(L) = -\frac{1}{2\pi i} \int_{\partial\omega} f(\lambda)(L - \lambda I)^{-1} d\lambda. \tag{8}$$

The existence of above integral follows from Theorem 4.4, because the application  $\lambda \mapsto f(\lambda)(L - \lambda I)^{-1}$  is continuous thanks to Proposition 2.11. Therefore,  $\hat{f}(L) \in L(E)$ . Moreover, we always can find  $\omega$  as in the Definition 4.6 (see [11], Lemma 6.28). Consequently, the operator  $\hat{f}(L)$  is well defined.

The following theorem, whose proof can be found in [11], Theorem 6.12, gives a property of the operator in (8).

**Theorem 4.7.** *Let  $L$  be an operator on  $L(E)$  and  $\Gamma$  be a closed path such that  $\sigma(L) \subseteq \overset{\circ}{\Gamma}$ . Then, for each positive integer  $k$ , we have*

$$L^k = -\frac{1}{2\pi i} \int_{\Gamma} \lambda^k (L - \lambda I)^{-1} d\lambda.$$

Let  $L \in L(E)$  and  $f : \mathbb{C} \rightarrow \mathbb{C}$  a polynomial function given by  $f(\lambda) = \sum_{k=0}^n a_k \lambda^k$ , where  $a_0, a_1, \dots, a_n \in \mathbb{C}$  are fixed and  $\lambda \in \mathbb{C}$ . Consider the operator  $\hat{f}(L)$  defined as follows:

$$\hat{f}(L) = \sum_{k=0}^n a_k L^k, \quad \text{where } L^0 = I.$$

As a consequence of Theorem 4.7 we have

$$\begin{aligned}\widehat{f}(L) &= \sum_{k=0}^n a_k L^k \\ &= -\frac{1}{2\pi i} \sum_{k=0}^n a_k \int_{\Gamma} \lambda^k (L - \lambda I)^{-1} d\lambda \\ &= -\frac{1}{2\pi i} \int_{\Gamma} \sum_{k=0}^n a_k \lambda^k (L - \lambda I)^{-1} d\lambda \\ &= -\frac{1}{2\pi i} \int_{\Gamma} f(\lambda) (L - \lambda I)^{-1} d\lambda.\end{aligned}$$

Thus  $\sum_{k=0}^n a_k L^k$  coincides with the definition given in formula (8).

We shall end this section by presenting some properties of  $\widehat{f}(L)$ , whose proofs can be found in [11], Lemma 6.15 and Theorem 6.17, respectively.

**Lemma 4.8.** *Let  $L$  be an operator in  $L(E)$ . Suppose that  $f : \Delta \rightarrow \mathbb{C}$  and  $g : \Delta \rightarrow \mathbb{C}$  are holomorphic applications and  $\sigma(L) \subseteq \Delta$ . If  $h : \Delta \rightarrow \mathbb{C}$  is defined by  $h(\lambda) = f(\lambda)g(\lambda)$  (product of complex applications), then*

$$\widehat{h}(L) = \widehat{f}(L)\widehat{g}(L) := \widehat{f}(L) \circ \widehat{g}(L),$$

that is,  $\widehat{h}(L)(x) = \widehat{f}(L)(\widehat{g}(L)(x))$  for all  $x \in E$ .

**Lemma 4.9.** *Let  $L$  be an operator in  $L(E)$ . If  $f : \Delta \rightarrow \mathbb{C}$  is holomorphic in an open neighborhood of  $\sigma(L)$ , then*

$$\sigma(\widehat{f}(L)) = f(\sigma(L)),$$

that is,  $\lambda \in \sigma(\widehat{f}(L))$  if and only if  $\lambda = f(\zeta)$  for some  $\zeta \in \sigma(L)$ .

## 5. On the continuity of the nonnegative square root application

As stated in the introduction, we will prove that the map  $\mathcal{R}$  sending each nonnegative isomorphism  $L \in L(H)$  to its nonnegative square root  $R \in L(H)$  is a well-defined homeomorphism. Denote by  $L^{1/2}$  the nonnegative square root of  $L$ . In [3], Richard Bouldin proves that  $L^{1/2}$  depends continuously on  $L$ . This proof is non-trivial. The proof given in this paper is elemental. We use basic tools in Complex Analysis and Functional Analysis.

First, we give alternatives proofs of Lemmas 5.1-5.3. Those are basic results of Functional Analysis and we will use them later. Theorem 5.4 (whose proof belongs to the author of the present paper) proves that  $L^{1/2}$  can be expressed by the formule (11). Finally, Theorem 5.5 shows that  $\mathcal{R}$  is a homeomorphism.

Note that we can suppose that  $H$  is a complex Hilbert space, because when  $H$  is a real Hilbert space and  $L \in L(H)$ , we use the complexification  $\widehat{L}$  of  $L$  and  $\mathcal{R}(\widehat{L}) \in GL_S^+(\widehat{H})$ . Then consider the map

$$\begin{aligned}\mathcal{R} : GL_S^+(H) &\rightarrow GL_S^+(H) \\ L &\mapsto \text{Re}(\mathcal{R}(\widehat{L})),\end{aligned}$$

where  $\text{Re}(\mathcal{R}(\widehat{L}))$  is the real part of  $\mathcal{R}(\widehat{L})$ ; this is, if  $\widehat{R} \in L(\widehat{H})$  and  $\widehat{R}(x+iy) = R_1x+iR_2y$ , for all  $x, y \in H$ , then  $\text{Re}(\widehat{R}) = R_1$ .

We shall see that there exists a holomorphic application  $\gamma : \Theta \rightarrow \mathbb{C}$ , with  $\sigma(L) \subseteq \Theta \subseteq \mathbb{C}$ , such that  $R = \hat{\gamma}(L)$ .

Indeed, let  $\Theta = \{a + ib \in \mathbb{C} : a > 0\}$  and consider the map

$$\gamma : \Theta \rightarrow \mathbb{C}, \quad \text{given by } \lambda \mapsto |\lambda|^{1/2} e^{\frac{i \text{Arg } z}{2}} \quad \text{for all } \lambda \in \mathbb{C}, \tag{9}$$

where  $\text{Arg } z$  denotes the principal argument of  $z \in \mathbb{C}$ . We can see in [2], Example 15, that  $\gamma$  is holomorphic in  $\Theta$ . It is easy to check that

$$\gamma(\bar{\lambda}) = \overline{\gamma(\lambda)} \quad \text{and} \quad \gamma(\lambda)^2 = \gamma(\lambda)\gamma(\lambda) = \lambda, \quad \text{for all } \lambda \in \Theta. \tag{10}$$

Let  $L$  be an operator in  $GL_S^+(H)$ , where  $H$  is a complex Hilbert space. It follows from Theorem 2.6 that  $\sigma(L)$  is a subset of  $\mathbb{R}^+$ . Hence  $\sigma(L) \subseteq \Theta$ , and using Definition 4.6, for an appropriate path  $\Gamma$  we may define

$$\hat{\gamma}(L) = -\frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(L - \lambda I)^{-1} d\lambda.$$

We shall show that  $\hat{\gamma}(L)$  is the nonnegative square root of  $L$ . For this purpose, we need the following results.

**Lemma 5.1.** *Let  $L \in L(H)$  be an isomorphism and  $R \in L(H)$  be an operator such that  $R^2 = L$ ; then,  $R$  is an isomorphism.*

*Proof.* If  $Rx = 0$ , for  $x \in H$ , then is  $R^2x = Lx = 0$ . Since  $L$  is an isomorphism, we have  $x = 0$ . Consequently,  $R$  is injective.

It is clear that  $R$  is surjective, because  $\text{Im } R = \text{Im } L = H$ . ☑

**Lemma 5.2.** *If  $L \in L(H)$  is a nonnegative isomorphism, then  $L$  is positive.*

*Proof.* We shall show that  $\langle Lx, x \rangle > 0$  for each  $x \in H$  with  $\|x\| = 1$ . By Theorem 2.9 we can take the nonnegative square root  $R$  of  $L$ . Since  $L$  is an isomorphism,  $R$  is an isomorphism thanks to Lemma 5.1. Let  $x \in H$  with  $\|x\| = 1$ . Using the fact that  $R^2 = L$  and that  $R$  is self-adjoint, we find that

$$\langle Lx, x \rangle = \langle R^2x, x \rangle = \langle Rx, Rx \rangle = \|Rx\|^2 > 0.$$

Consequently,  $L$  is positive. ☑

**Lemma 5.3.** *Let  $L \in L(H)$  be self-adjoint; then,*

$$\inf_{\|x\|=1} \langle Lx, x \rangle = \min_{\lambda \in \sigma(L)} \lambda.$$

*Proof.* Set

$$\lambda_0 = \min_{\lambda \in \sigma(L)} \lambda \quad \text{and} \quad m = \inf_{\|x\|=1} \langle Lx, x \rangle.$$

It follows from Theorem 2.6 that  $\lambda_0 \geq m$ . Suppose that  $\lambda_0 > m$ ; then,  $L - mI$  is an isomorphism and, for all  $x \in H$  with  $\|x\| = 1$ , one has

$$\langle (L - mI)x, x \rangle = \langle Lx, x \rangle - \langle mx, x \rangle = \langle Lx, x \rangle - m \geq 0,$$

that is  $L - mI$  is a nonnegative operator. Since  $L - mI$  is a nonnegative isomorphism,  $L - mI$  is positive by Lemma 5.2. It follows from Lemma 3.1 that there exists  $c > 0$  such that

$$\inf_{\|x\|=1} \langle (L - mI)x, x \rangle \geq c.$$

However,

$$\inf_{\|x\|=1} \langle (L - mI)x, x \rangle = \inf_{\|x\|=1} [\langle Lx, x \rangle - \langle mx, x \rangle] = \inf_{\|x\|=1} \langle Lx, x \rangle - m = 0,$$

which contradicts

$$\inf_{\|x\|=1} \langle (L - mI)x, x \rangle > 0.$$

Consequently,  $\lambda_0 = m$ . □

**Theorem 5.4.** *Let  $L \in GL_S^+(H)$ . Then there exists a closed path  $\Gamma \subseteq \Theta$  with  $\sigma(L) \subseteq \mathring{\Gamma}$ , such that*

$$L^{1/2} = \hat{\gamma}(L) = -\frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(L - \lambda I)^{-1} d\lambda. \quad (11)$$

*Proof.* It is clear that there are infinitely many paths  $\Gamma$  such that the integral in (11) is well defined (it is sufficient to find a closed path contained in  $\Theta$  containing  $\sigma(L)$  in its interior). The definition of this integral does not depend on these paths, thus we choose a path  $\Gamma$  that makes easier the rest of the proof. By Lemma 3.1 there exists  $c > 0$  such that

$$\inf_{\|x\|=1} \langle Lx, x \rangle \geq c.$$

It follows from equation (5), Theorem 2.6 and Lemma 3.1, that  $\sigma(L) \subseteq [c, \|L\|]$ . Then

$$\sigma(L) \subseteq (c/2, \|L\| + c/2).$$

Take  $\Gamma$  as the positively oriented circle centered in  $\frac{c+\|L\|}{2}$  (the middle point of  $(c/2, \|L\| + c/2)$ ) and passing through the points  $c/2$  and  $\|L\| + c/2$ . Hence,

$$\sigma(L) \subseteq \mathring{\Gamma} \subseteq \Theta.$$

We shall prove that  $\hat{\gamma}(L)$  is the nonnegative square root of  $L$ . By the uniqueness in Theorem 2.9, it is sufficient to see that  $\hat{\gamma}(L)^2 = L$  and that  $\hat{\gamma}(L)$  is nonnegative. In fact, since  $\gamma(\lambda)\gamma(\lambda) = \lambda$  for all  $\lambda \in \Theta$ , by Lemma 4.8 we obtain that

$$\hat{\gamma}(L)^2 = \hat{\gamma}(L)\hat{\gamma}(L) = L.$$

Now we see that  $\hat{\gamma}(L)$  is self-adjoint. For this purpose, we will use the definition of the integral in (11) as the limit of a sequence of sums as in (6) (see (7)). Indeed, for each  $n \in \mathbb{N}$  take a partition  $P_n = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$  of  $\Gamma$  such that, for any  $k = 0, 1, \dots, n$ ,

$$|\lambda_k - \lambda_{k-1}| \rightarrow 0 \quad \text{when } n \rightarrow \infty.$$

Moreover, for  $k = 0, 1, \dots, n$ , let  $\xi_k = \bar{\lambda}_{n-k}$ . Therefore,  $\xi_k \in \Gamma$  (by the definition of  $\Gamma$ ) and, since  $\lambda_0 < \lambda_1 < \dots < \lambda_n = \lambda_0$ , then  $\xi_0 < \xi_1 < \dots < \xi_n = \xi_0$ . Take  $x, y \in H$ . Due to the fact that  $[(L - \lambda I)^{-1}]^* = (L - \bar{\lambda}I)^{-1}$  and  $\gamma(\lambda) = \gamma(\bar{\lambda})$ , we get

$$\begin{aligned} & \left\langle \sum_{k=1}^n \gamma(\lambda_k)(\lambda_k - \lambda_{k-1})(L - \lambda_k I)^{-1} x, y \right\rangle \\ &= \left\langle x, \sum_{k=1}^n \overline{\gamma(\lambda_k)}(\bar{\lambda}_k - \bar{\lambda}_{k-1})(L - \bar{\lambda}_k I)^{-1} y \right\rangle \\ &= \left\langle x, \sum_{k=1}^n \gamma(\bar{\lambda}_k)(\bar{\lambda}_k - \bar{\lambda}_{k-1})(L - \bar{\lambda}_k I)^{-1} y \right\rangle \\ &= \left\langle x, \sum_{k=1}^n \gamma(\xi_{n-k})(\xi_{n-k} - \xi_{n-k+1})(L - \xi_{n-k} I)^{-1} y \right\rangle \\ &= \left\langle x, - \sum_{k=1}^n \gamma(\xi_{n-k})(\xi_{n-k+1} - \xi_{n-k})(L - \xi_{n-k} I)^{-1} y \right\rangle \\ &= \left\langle x, - \sum_{j=1}^n \gamma(\xi_{j-1})(\xi_j - \xi_{j-1})(L - \xi_{j-1} I)^{-1} y \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} & \left\langle \frac{1}{2\pi i} \sum_{k=1}^n \gamma(\lambda_k)(\lambda_k - \lambda_{k-1})(L - \lambda_k I)^{-1} x, y \right\rangle \\ &= \left\langle x, \frac{1}{2\pi i} \sum_{j=1}^n \gamma(\xi_{j-1})(\xi_j - \xi_{j-1})(L - \xi_{j-1} I)^{-1} y \right\rangle. \end{aligned}$$

Further, by (7),

$$\begin{aligned} & \lim_{n \rightarrow \infty} - \frac{1}{2\pi i} \sum_{k=1}^n \gamma(\lambda_k)(\lambda_k - \lambda_{k-1})(L - \lambda_k I)^{-1} = \gamma(L) \\ &= \lim_{n \rightarrow \infty} - \frac{1}{2\pi i} \sum_{j=1}^n \gamma(\xi_{j-1})(\xi_j - \xi_{j-1})(L - \xi_{j-1} I)^{-1}. \end{aligned}$$

Thus  $\langle \hat{\gamma}(L)x, y \rangle = \langle x, \hat{\gamma}(L)y \rangle$  for  $x, y \in H$ . This fact implies that  $\hat{\gamma}(L)$  is self-adjoint.

We now prove that  $\hat{\gamma}(L)$  is nonnegative. By Lemma 4.9 we obtain

$$\sigma(\hat{\gamma}(L)) = \gamma(\sigma(L)).$$

Therefore, since  $\sigma(L) \subseteq \mathbb{R}^+$ , then  $\sigma(\hat{\gamma}(L)) \subseteq \mathbb{R}^+$ , and it is a consequence of Lemma 5.3 that

$$0 \leq \min_{\lambda \in \sigma(\hat{\gamma}(L))} \lambda = \inf_{\|x\|=1} \langle \hat{\gamma}(L)x, x \rangle,$$

that is,  $\hat{\gamma}(L)$  is nonnegative, as desired.  $\square$

It follows from Lemma 5.1 that if  $L \in GL_S^+(H)$ , then  $L^{1/2} \in GL_S^+(H)$ . Hence by Theorem 2.9 the application

$$\begin{aligned} \mathcal{R} : GL_S^+(H) &\rightarrow GL_S^+(H) \\ L &\mapsto L^{1/2} \end{aligned}$$

is well defined. Using Theorem 5.4 we prove that  $\mathcal{R}$  is, in fact, a homeomorphism, obtaining our main result.

**Theorem 5.5.** *The map*

$$\begin{aligned} \mathcal{R} : GL_S^+(H) &\rightarrow GL_S^+(H) \\ L &\mapsto L^{1/2} \end{aligned}$$

*is a homeomorphism.*

*Proof.* Let  $L \in GL_S^+(H)$  and  $c$  be as in the proof of Theorem 5.4. Consider  $r = \min\{1/\|L^{-1}\|, c/2\}$  and set

$$B(L, r) = \{T \in L_S(H) : \|L - T\| < r\}.$$

Thus  $B(L, r) \subseteq GL_S^+(H)$  (see Proposition 3.3). Take  $T \in B(L, r)$ ; we shall show that

$$\sigma(T) \subseteq (c/3, \|L\| + c/2). \quad (12)$$

Since  $T$  is positive, by Theorem 2.6  $\sigma(T)$  is a subset of  $\mathbb{R}^+$ . On the other hand, if  $\lambda \in \sigma(T)$ , then

$$\lambda \leq \|T\| \leq \|L\| + \|T - L\| < \|L\| + c/2.$$

Now suppose that  $\lambda \in \mathbb{R}$  satisfies  $0 < \lambda \leq c/3$ . Given  $x \in H$  with  $\|x\| = 1$ , we have  $\|Lx\| \geq \langle Lx, x \rangle \geq c\|x\|$ . Consequently

$$\|Tx\| \geq \|Lx\| - \|Lx - Tx\| \geq c - c/2 = c/2.$$

Then,

$$\|(T - \lambda I)x\| \geq \|Tx\| - \|\lambda x\| \geq \frac{c}{2} - |\lambda| > 0.$$

This fact proves that  $\text{Ker}(T - \lambda I) = \{0\}$ . It is a consequence of Proposition 2.2 that  $T - \lambda I$  is closed; hence, since  $T - \lambda I$  is self-adjoint, one gets

$$\text{Im}(T - \lambda I) = \overline{\text{Im}(T - \lambda I)^*} = [\text{Ker}(T - \lambda I)]^\perp = H.$$

Therefore  $\lambda \in \rho(T)$ .

Let  $\Gamma$  be the positively oriented circle with center in the middle point of the interval  $(c/3, \|L\| + c/2)$  and passing through the points  $c/3$  and  $\|L\| + c/2$ . It follows from (12) that

$$\sigma(T) \subseteq \mathring{\Gamma} \subseteq \Theta, \quad \text{for all } T \in B(L, r).$$

Hence, we may define

$$\hat{\gamma}(T) = -\frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(T - \lambda I)^{-1} d\lambda, \quad \text{for all } T \in B(L, r).$$

Theorem 5.4 implies that for any  $T \in B(L, r)$ , the operator  $T^{1/2}$  coincides with  $\hat{\gamma}(T)$ , that is

$$\mathcal{R}(T) = \hat{\gamma}(T), \quad \text{for all } T \in B(L, r). \tag{13}$$

It is easy to see that, if  $T \in B(L, r)$ , then

$$(T - \lambda I)^{-1} - (L - \lambda I)^{-1} = -(T - \lambda I)^{-1}(T - L)(L - \lambda I)^{-1} \quad \text{for } \lambda \in \Gamma.$$

Thus,

$$\begin{aligned} \hat{\gamma}(T) - \hat{\gamma}(L) &= -\frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(T - \lambda I)^{-1} d\lambda + \frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(L - \lambda I)^{-1} d\lambda \\ &= -\frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)[(T - \lambda I)^{-1} - (L - \lambda I)^{-1}] d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(T - \lambda I)^{-1}(T - L)(L - \lambda I)^{-1} d\lambda. \end{aligned}$$

Consequently,

$$\hat{\gamma}(T) - \hat{\gamma}(L) = \frac{1}{2\pi i} \int_{\Gamma} \gamma(\lambda)(T - \lambda I)^{-1}(T - L)(L - \lambda I)^{-1} d\lambda.$$

Let  $M > 0$  be such that

$$\|(T - \lambda I)^{-1}\| \leq M, \quad \text{for } \lambda \in \Gamma \text{ and } T \in B(L, r).$$

Then

$$\|\gamma(\lambda)(L - \lambda I)^{-1}(L - T)(T - \lambda I)^{-1}\| \leq mM^2\|L - T\|, \quad \text{where } m = \max_{\lambda \in \Gamma} |\gamma(\lambda)|.$$

Fix  $\varepsilon > 0$ ; if  $\|L - T\| < \varepsilon$ , then

$$\|\mathcal{R}(L) - \mathcal{R}(T)\| = \|\hat{\gamma}(L) - \hat{\gamma}(T)\| < \frac{1}{2\pi} mM^2 l \varepsilon,$$

where  $l$  is the length of  $\Gamma$  (Lemma 4.5). This implies that  $\mathcal{R}$  is continuous.

On the other hand, it is not difficult to show that the application

$$\begin{aligned} \mathcal{C} : GL_S^+(H) &\rightarrow GL_S^+(H) \\ L &\mapsto L^2 \end{aligned}$$

is the inverse of  $\mathcal{R}$ . Therefore  $\mathcal{R}$  is a homeomorphism. \(\square\)

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