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The problem of the first return attached to a pseudodifferential operator in dimension 3

Oscar F. Casas-Sánchez[a*], Jeanneth Galeano-Peñaloza[b], John J. Rodríguez-Vega[b]

[a]Universidad de los Andes, Departamento de Matemáticas, Bogotá, Colombia.

[b]Universidad Nacional de Colombia, Departamento de Matemáticas, Bogotá, Colombia.

Abstract. In this article we study the problem of first return associated to an elliptic pseudodifferential operator with non-radial symbol of dimension 3 over the p-adics.
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El problema del primer retorno asociado a un operador seudodiferencial en dimensión 3

Resumen. En este artículo estudiamos el problema del primer retorno asociado a un operador seudodiferencial elíptico con símbolo no radial de dimensión 3 sobre el cuerpo de los números p-ádicos.
Palabras clave: Caminatas aleatorias, ultradifusión, números p-ádicos, análisis no arquimediano.

1. Introduction

Avetisov et al. have constructed a wide variety of models of ultrametric diffusion constrained by hierarchical energy landscapes (see [2],[3]). From a mathematical point of view, in these models the time-evolution of a complex system is described by a p-adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition function of a random walk on an ultrametric space, and the random walk describes the dynamics of the system in the space of configurational states which is approximated by an ultrametric space ($\mathbb{Q}_p$).

* E-mail: oscasas@uniandes.edu.co
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The problem of the first return in dimension 1 was studied in [4], and in arbitrary dimension in [7]. In both articles, pseudodifferential operators with radial symbols were considered. More recently, Chacón-Cortés [6] considers pseudodifferential operators over $\mathbb{Q}_p^4$ with non-radial symbol; he studies the problem of first return for a random walk $X(t,w)$ whose density distribution satisfies certain diffusion equation.

In [5], the authors study elliptic pseudodifferential operators in dimension 3 and find a function, $Z(x,t)$, $x \in \mathbb{Q}_p^3$, $t \in \mathbb{R}_+$, that satisfies the following equation

$$\frac{\partial u(x,t)}{\partial t} = -\int_{\mathbb{Q}_p^3} K_{-\alpha}(y)(u(x-y,t)-u(x,t))d^3y, \tag{1}$$

where $K_{-\alpha}(x)$ is the Riesz kernel associated to the elliptic quadratic form $f^{\circ}(\xi) = p\xi_1^2 + \xi_2^2 - \epsilon\xi_3^2$.

Using the same techniques as in [6] we prove that the random walk $X(t,w)$ whose density distribution satisfies the equation (1) is recurrent if $\alpha \geq \frac{3}{2}$ and transient when $\alpha < \frac{3}{2}$. This result is analog to the one showed in [7], in the sense that 2α represent the degree of the symbol, and in this case the process is recurrent if 2α is greater that the dimension, 3.

The article is organized as follows. In Section 2 we write some facts about p-adics. In Section 3 we define the symbol for the pseudodifferential operator and its Fourier transform. In Section 4 we study the Cauchy problem and give some properties of its fundamental solution, and define a Markov process over $\mathbb{Q}_p^3$. In Section 5 we determine the probability density function for a path of $X(t,\omega)$ goes back to $\mathbb{Z}_p^3$, and we show that the process is recurrent when $\alpha \geq \frac{3}{2}$, and otherwise is transient (see Theorem 5.7).

2. Preliminars

For the sake of completeness we include some preliminars. For more details the reader may consult [1],[9],[10].

2.1. The field of p-adic numbers

Along this article p will denote a prime number different from 2. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-\gamma} & \text{if } x = p^{\gamma}\dfrac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := \operatorname{ord}(x)$, with $\operatorname{ord}(0) := +\infty$, is called the *$p$-adic order of x*. We extend the p-adic norm to $\mathbb{Q}_p^n$ by taking

$$||x||_p := \max_{1\leq i\leq n} |x_i|_p, \qquad \text{for } x = (x_1,\dots,x_n) \in \mathbb{Q}_p^n.$$

We define $\operatorname{ord}(x) = \min_{1\leq i\leq n}\{\operatorname{ord}(x_i)\}$; then $||x||_p = p^{-\operatorname{ord}(x)}$. Any p-adic number $x \neq 0$ has a unique expansion $x = p^{\operatorname{ord}(x)}\sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \{0,1,2,\dots,p-1\}$ and $x_0 \neq 0$.

By using this expansion, we define *the fractional part of* $x \in \mathbb{Q}_p$, denoted by $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \operatorname{ord}(x) \geq 0, \\ p^{\operatorname{ord}(x)} \sum_{j=0}^{-\operatorname{ord}(x)-1} x_j p^j & \text{if } \operatorname{ord}(x) < 0. \end{cases}$$

For $\gamma \in \mathbb{Z}$, denote by $B_\gamma^n(a) = \{x \in \mathbb{Q}_p^n : ||x - a||_p \leq p^\gamma\}$ *the ball of radius* p^γ *with center at* $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B_\gamma^n(0) := B_\gamma^n$. Note that $B_\gamma^n(a) = B_\gamma(a_1) \times \cdots \times B_\gamma(a_n)$, where $B_\gamma(a_i) := \{x \in \mathbb{Q}_p : |x_i - a_i|_p \leq p^\gamma\}$ is the one-dimensional ball of radius p^γ with center at $a_i \in \mathbb{Q}_p$. The ball $B_0^n(0)$ is equal to the product of n copies of $B_0(0) := \mathbb{Z}_p$, *the ring of* p*-adic integers.*

2.2. The Bruhat-Schwartz space

A complex-valued function φ defined on $\mathbb{Q}_p^n$ is *called locally constant* if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \text{ for } x' \in B_{l(x)}^n. \tag{2}$$

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a *Bruhat-Schwartz function (or a test function)* if it is locally constant with compact support. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathbf{S}(\mathbb{Q}_p^n)$. For $\varphi \in \mathbf{S}(\mathbb{Q}_p^n)$, the largest of such numbers $l = l(\varphi)$ satisfying (2) is called *the exponent of local constancy of* φ.

Let $\mathbf{S}'(\mathbb{Q}_p^n)$ denote the set of all functionals (distributions) on $\mathbf{S}(\mathbb{Q}_p^n)$. All functionals on $\mathbf{S}(\mathbb{Q}_p^n)$ are continuous.

Set $\chi(y) = \exp(2\pi i\{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuos map from $\mathbb{Q}_p$ into S (the unit circle) satisfying $\chi(y_0 + y_1) = \chi(y_0)\chi(y_1)$, $y_0, y_1 \in \mathbb{Q}_p$.

2.3. Fourier transform

Given $\xi = (\xi_1, \ldots, \xi_n)$ and $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{j=1}^n \xi_j x_j$. The Fourier transform of $\varphi \in \mathbf{S}(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi(\xi \cdot x)\varphi(x) d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where $d^n x$ is the Haar measure on $\mathbb{Q}_p^n$ normalized by the condition $vol(B_0^n) = 1$. The Fourier transform is a linear isomorphism from $\mathbf{S}(\mathbb{Q}_p^n)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$. We will also use the notation $\mathcal{F}_{x \to \xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

The Fourier transform $\mathcal{F}[f]$ of a distribution $f \in \mathbf{S}'(\mathbb{Q}_p^n)$ is defined by

$$(\mathcal{F}[f], \varphi) = (f, \mathcal{F}[\varphi]) \text{ for all } \varphi \in \mathbf{S}(\mathbb{Q}_p^n).$$

The Fourier transform $f \to \mathcal{F}[f]$ is a linear isomorphism from $\mathbf{S}'(\mathbb{Q}_p^n)$ onto $\mathbf{S}'(\mathbb{Q}_p^n)$. Furthermore, $f = \mathcal{F}[\mathcal{F}[f](-\xi)]$.

2.4. The space $\mathfrak{M}_\lambda$

We denote by $\mathfrak{M}_\lambda$, $\lambda \geq 0$, the $\mathbb{C}$-vector space of locally constant functions $\varphi(x)$ on $\mathbb{Q}_p^n$ such that $|\varphi(x)| \leq C(1+||x||_p^\lambda)$, where C is a positive constant. If the function φ depends also on a parameter t, we shall say that $\varphi \in \mathfrak{M}_\lambda$ *uniformly with respect to* t, if its constant C and its exponent of local constancy do not depend on t.

3. Pseudodifferential operators

We take $f(\xi) = \epsilon\xi_1^2 + p\epsilon\xi_2^2 - p\xi_3^2$ and $f^\circ(\xi) = p\xi_1^2 + \xi_2^2 - \epsilon\xi_3^2$, with $\epsilon \in \mathbb{Z}$ a quadratic non-residue module p. Given $\alpha > 0$, we define the pseudodifferential operator with symbol $|f(\xi)|_p^\alpha$ by

$$\begin{array}{ccc} \mathbf{S}\left(\mathbb{Q}_p^3\right) & \longrightarrow & C\left(\mathbb{Q}_p^3\right) \cap L^2\left(\mathbb{Q}_p^3\right) \\ \varphi & \longrightarrow & \left(\boldsymbol{f}\left(\partial, \alpha\right)\varphi\right)(x) := \mathcal{F}^{-1}_{\xi\to x}\left(|f(\xi)|_p^\alpha \mathcal{F}_{x\to\xi}\varphi\right). \end{array}$$

In [5] the authors show that the Fourier transform of the symbol is given by

$$\mathcal{F}\left[|f(x)|_p^{-\alpha}\right] = \frac{1-p^{-\alpha}}{1-p^{2\alpha-3}}\left[(1+p^{\alpha-1})I_{V_1\cup V_\epsilon}(x) + p^{\alpha-\frac{3}{2}}(p^{2-\alpha}+1)I_{V_p}(x)\right]|f^\circ(x)|^{\alpha-\frac{3}{2}}, \tag{3}$$

where

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

and for $\delta = 1, \epsilon, p, p\epsilon$ put $V_\delta := \{x \in \mathbb{Q}_p^3 \mid f^\circ(x) \in \delta[\mathbb{Q}_p^\times]^2\}$. Observe that $V_{p\epsilon} = \emptyset$, otherwise the elliptic form $px_1^2 + x_2^2 - \epsilon x_3^2 - p\epsilon x_4^2 = 0$ would have non-trivial solution in $\mathbb{Q}_p^4$.

If we consider

$$K_\alpha(x) := \frac{1-p^{-\alpha}}{1-p^{2\alpha-3}}\left[(1+p^{\alpha-1})I_{V_1\cup V_\epsilon}(x) + p^{\alpha-\frac{3}{2}}(p^{2-\alpha}+1)I_{V_p}(x)\right]|f^\circ(x)|^{\alpha-\frac{3}{2}},$$

then equation (3) can be written as

$$\hat{K}_\alpha(x) = |f(x)|^{-\alpha}, \quad \alpha \neq \frac{3}{2} + \frac{2\pi\sqrt{-1}}{\ln p}\mathbb{Z}, \tag{4}$$

and as a distribution on $\sigma(\mathbb{Q}_p^3)$, $K_\alpha(x)$ possesses a meromorphic continuation to all $\alpha \neq \frac{3}{2} + \frac{2\pi\sqrt{-1}}{\ln p}\mathbb{Z}$ (see [5, Lemma 5]).

Since $\mathcal{F}^{-1}\left(|f|_p^\alpha\right) = K_{-\alpha}$ we have $|f(\xi)|_p^\alpha \mathcal{F}_{x\to\xi}\varphi \in L^1\left(\mathbb{Q}_p^3\right) \cap L^2\left(\mathbb{Q}_p^3\right)$, and the operator is well-defined. Therefore it is possible to write the operator as a convolution

$$\boldsymbol{f}\left(\partial, \alpha\right)\varphi = K_{-\alpha} * \varphi = \int_{\mathbb{Q}_p^3} K_{-\alpha}(y)(\varphi(x-y) - \varphi(x))d^3y, \tag{5}$$

for $\varphi \in \mathbf{S}\left(\mathbb{Q}_p^3\right)$. Actually, the domain of the operator can be extended to the locally constant functions $u(x)$ such that

$$\int_{||x||_p \geq 1} K_{-\alpha}(x)|u(x)|d^3x < \infty. \tag{6}$$

There exists an important inequality for elliptic polynomials, which is essential to do all the calculations (see [11]). In our case, for f and f°, the inequalities are given in the next lemma.

Lemma 3.1 ([6, Lemma 3]). *Let f, f° be as above. Then*

(i) $p^{-1}\|x\|_p^2 \leq |f(x)|_p \leq \|x\|_p^2, \quad for \;\; every \; x \in \mathbb{Q}_p^3,$

(ii) $p^{-1}\|x\|_p^2 \leq |f^\circ(x)|_p \leq \|x\|_p^2, \quad for \;\; every \; x \in \mathbb{Q}_p^3.$

4. The Cauchy problem

The Cauchy problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -\boldsymbol{f}(\partial,\alpha)u(x,t), \quad x \in \mathbb{Q}_p^3, \quad 0 < t \leq T, \\ \\ u(x,0) = \varphi(x), \end{cases} \tag{7}$$

where $\alpha > 0$, $T > 0$, $\varphi \in \mathfrak{M}_{2\lambda}$, $0 \leq \lambda < \alpha$ has a solution $u : \mathbb{Q}_p^3 \times [0,T] \to \mathbb{C}$ satisfying $u(x,t) \in \mathfrak{M}_{2\lambda}$ and

$$u(x,t) := Z(x,t) * \varphi(x) = \int_{\mathbb{Q}_p^3} Z(x-\eta,t)\varphi(\eta)d^3\eta, \tag{8}$$

where *the heat kernel* $Z(x,t)$ attached to $f(x)$ is

$$Z(x,t) := Z(x,t;f,\alpha) = \int_{\mathbb{Q}_p^3} \chi(-\xi\cdot x)e^{-t|f(\xi)|_p^\alpha}d^3\xi,$$

for $x \in \mathbb{Q}_p^3$, $t > 0$ and $\alpha > 0$ (see [5]).

Theorem 4.1. *The function $Z(x,t)$ has the following properties:*

(i) $Z(x,t) \geq 0$ *for any* $t > 0$.

(ii) $\displaystyle\int_{\mathbb{Q}_p^3} Z(x,t)d^3x = 1$ *for any* $t > 0$.

(iii) $Z(x,t) \leq Ct\left(\|x\|_p + t^{\frac{1}{2\alpha}}\right)^{-3-2\alpha}$, *where C is a positive constant, for any $t > 0$ and any $x \in \mathbb{Q}_p^3$.*

(iv) $Z(x,t) * Z(x,t') = Z(x,t+t')$ *for any* $t,\ t' > 0$.

(v) $\lim_{t\to 0^+} Z(x,t) = \delta(x)$ *in* $S'(\mathbb{Q}_p^3)$.

(vi) $Z(x,t) \in C(\mathbb{Q}_p^3, \mathbb{R}) \cap L^1(\mathbb{Q}_p^3) \cap L^2(\mathbb{Q}_p^3)$ *for any* $t > 0$.

Proof. See Theorems 1, 2, Proposition 2 and Corollary 1 of [11]. ☑

4.1. Markov processes over $\mathbb{Q}_p^3$

The space $\left(\mathbb{Q}_p^3, \|\cdot\|_p\right)$ is a complete non-Archimedean metric space. Let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{Q}_p^3$; thus $\left(\mathbb{Q}_p^3, \mathcal{B}, d^3x\right)$ is a measure space. By using the terminology and results of [8, Chapters 2, 3], we set

$$p(t,x,y) := Z(x-y,t) \text{ for } t>0,\, x,y \in \mathbb{Q}_p^3,$$

and

$$P(t,x,B) = \begin{cases} \int_B p(t,y,x) d^3y & \text{for } t>0, \quad x \in \mathbb{Q}_p^3, \quad B \in \mathcal{B}, \\ \mathbf{1}_B(x) & \text{for } t = 0. \end{cases}$$

Lemma 4.2. *With the above notation the following assertions hold:*

(i) $p(t,x,y)$ *is a normal transition density.*

(ii) $P(t,x,B)$ *is a normal transition function.*

Proof. The result follows from Theorem 4.1 (see [8, Section 2.1] for further details). ☑

Lemma 4.3. *The transition function* $P(t,x,B)$ *satisfies the following two conditions:*

L(B) *For each* $u \geq 0$ *and compact* B*,*

$$\lim_{x\to\infty} \sup_{t\leq u} P(t,x,B) = 0.$$

M(B) *For each* $\epsilon > 0$ *and compact* B*,*

$$\lim_{t\to 0^+} \sup_{x\in B} P(t,x,\mathbb{Q}_p^3 \setminus B_\epsilon^3(x)) = 0.$$

Proof. (i) By Theorem 4.1 (iii) and the fact that $\|\cdot\|_p$ is an ultranorm, we have

$$\begin{aligned} P(t,x,B) &\leq Ct \int_B \left(\|x-y\|_p + t^{\frac{1}{2\alpha}}\right)^{-3-2\alpha} d^3y \\ &= t\left(\|x\|_p + t^{\frac{1}{2\alpha}}\right)^{-3-2\alpha} vol\,(B) \text{ for } x \in \mathbb{Q}_p^3 \setminus B. \end{aligned}$$

Therefore, $\lim_{x\to\infty} \sup_{t\leq u} P(t,x,B) = 0$.

(ii) By using Theorem 4.1 (iii), $\alpha > 0$, and the fact that $\|\cdot\|_p$ is an ultranorm, we have

$$\begin{aligned} P(t,x,\mathbb{Q}_p^3 \setminus B_\epsilon^3(x)) &\leq Ct \int\limits_{\|x-y\|_p>\epsilon} \left(\|x-y\|_p + t^{\frac{1}{2\alpha}}\right)^{-3-2\alpha} d^3y \\ &= Ct \int\limits_{\|z\|_p>\epsilon} \left(\|z\|_p + t^{\frac{1}{2\alpha}}\right)^{-3-2\alpha} d^3z \\ &\leq Ct \int\limits_{\|z\|_p>\epsilon} \|z\|_p^{-3-2\alpha}\, d^3z \\ &= C'\left(\alpha,\epsilon\right) t. \end{aligned}$$

Therefore,

$$\lim_{t\to 0^+} \sup_{x\in B} P(t,x,\mathbb{Q}_p^3 \setminus B_\epsilon^3(x)) \leq \lim_{t\to 0^+} \sup_{x\in B} C'\left(\alpha,\epsilon\right) t = 0. \qquad ☑$$

Theorem 4.4. *$Z(x,t)$ is the transition density of a time and space homogeneous Markov process which is bounded, right-continuous and has no discontinuities other than jumps.*

Proof. The result follows from [8, Theorem 3.6] by using that $(\mathbb{Q}_p^3, \|x\|_p)$ is a semi-compact space, i.e., a locally compact Hausdorff space with a countable base, and $P(t,x,B)$ is a normal transition function satisfying conditions $L(B)$ and $M(B)$ (cf. Lemmas 4.2 and 4.3). ☑

5. The first passage time

The solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -\displaystyle\int_{\mathbb{Q}_p^3} K_{-\alpha}(y)[u(x-y)-u(x,t)]d^3y, \quad x\in\mathbb{Q}_p^3, \quad 0<t\leq T, \\ \\ u(x,0) = \Omega(||x||_p), \end{cases} \tag{9}$$

is given by

$$u(x,t) = \int_{\mathbb{Q}_p^3} \chi(-\xi\cdot x)\Omega(\|\xi\|_p) e^{-t|f(\xi)|_p^\alpha} d^3\xi. \tag{10}$$

Among other properties, the solution (10) is infinitely differentiable with respect to $t \geq 0$, and for $m \in \mathbb{N}$,

$$\frac{\partial^m u}{\partial t^m}(x,t) = (-1)^m \int_{\mathbb{Q}_p^3} |f(\xi)|_p^{\alpha m}\, \chi(-\xi\cdot x)\Omega(\|\xi\|_p) e^{-t|f(\xi)|_p} d^3\xi. \tag{11}$$

Lemma 5.1 ([5, Lemma 6]). *For $Re(\alpha) > 0$ we have*

$$-\left(\int_{||x||_p>1} K_{-\alpha}(x) \quad d^3x\right) = \frac{p^{-\alpha}(1-p^{-1})(p^{\alpha}+p^{-1}+p^{-2})}{1-p^{-2\alpha-3}} \leq 1.$$

Definition 5.2. The random variable $\tau_{\mathbb{Z}_p^3}(\omega) := \tau(\omega) : \mathbb{Q}_p^3 \to \mathbb{R}^+$ defined by

$$\inf\{t > 0; X(t,\omega) \in \mathbb{Z}_p^3 \mid \text{there exists } t' \text{ such that } 0 < t' < t \text{ and } X(t',\omega) \notin \mathbb{Z}_p^3\}$$

is called the *first passage time* of a path of the random process $X(t,\omega)$ entering the domain $\mathbb{Z}_p^3$.

Note that the initial condition in (9) implies that

$$\Pr\left(\{\omega \in \mathbb{Q}_p^3; X(0,\omega) \in \mathbb{Z}_p^3\}\right) = 1.$$

Definition 5.3. We say that $X(t,\omega)$ is recurrent with respect to $\mathbb{Z}_p^3$ if

$$\Pr\left(\{\omega \in \mathbb{Q}_p^3; \tau(\omega) < \infty\}\right) = 1. \tag{12}$$

Otherwise we say that $X(t,\omega)$ is transient with respect to $\mathbb{Z}_p^3$.

The meaning of (12) is that every path of $X(t,\omega)$ is sure to return to $\mathbb{Z}_p^3$. If (12) does not hold, then there exist paths of $X(t,\omega)$ that abandon $\mathbb{Z}_p^3$ and never go back.

By using the same arguments given by Chacón in [6] we define the survival probability as

$$S(t) := S_{\mathbb{Z}_p^3}(t) = \int_{\mathbb{Z}_p^3} \varphi(x,t) d^3x,$$

which is the probability that a path of $X(t,\omega)$ remains in $\mathbb{Z}_p^3$ at the time t. Because there are no external forces acting on the random walk, we have

$$S'(t) = \left(\begin{array}{l}\text{Probability that a path of } X(t,\omega)\\ \text{goes back to } \mathbb{Z}_p^3 \text{ at the time } t\end{array}\right) - \left(\begin{array}{l}\text{Probability that a path of } X(t,\omega)\\ \text{exits } \mathbb{Z}_p^3 \text{ at the time } t\end{array}\right) \tag{13}$$

$$= g(t) - C \cdot S(t) \text{ with } 0 < C \leq 1.$$

In order to determine the probability density function $g(t)$ we compute $S'(t)$.

$$
\begin{aligned}
S'(t) &= \int\limits_{\mathbb{Z}_p^3} \frac{\partial \varphi(x,t)}{\partial t} d^3x = -\int\limits_{\mathbb{Z}_p^3} \int\limits_{\mathbb{Q}_p^3} K_{-\alpha}(y)[u(x-y,t) - u(x,t)] d^3y d^3x \\
&= -\int\limits_{\mathbb{Z}_p^3} \int\limits_{||y||_p>1} K_{-\alpha}(y)[u(x-y,t) - u(x,t)] d^3y d^3x \\
&= -\int\limits_{\mathbb{Z}_p^3} \int\limits_{||y||_p>1} K_{-\alpha}(y) u(x-y,t) d^3y d^3x + \int\limits_{\mathbb{Z}_p^3} \int\limits_{||y||_p>1} K_{-\alpha}(y) u(x,t) d^3y d^3x \\
&= -\int\limits_{||y||_p>1} K_{-\alpha}(y) u(y,t) d^3y + \int\limits_{||y||_p>1} K_{-\alpha}(y) d^3y \int\limits_{\mathbb{Z}_p^3} u(x,t) d^3x \\
&= -\int\limits_{||y||_p>1} K_{-\alpha}(y) u(y,t) d^3y - \left(\frac{p^{-\alpha}(1-p^{-1})(p^{\alpha}+p^{-1}+p^{-2})}{1-p^{-2\alpha-3}} \right) S(t).
\end{aligned}
$$

Therefore,

$$
g(t) = -\int\limits_{||y||_p>1} K_{-\alpha}(y) u(y,t) d^3y, \tag{14}
$$

and the constant $C := \frac{p^{-\alpha}(1-p^{-1})(p^{\alpha}+p^{-1}+p^{-2})}{1-p^{-2\alpha-3}}$ satisfies $0 < C \leq 1$.

Proposition 5.4. *The probability density function $f(t)$ of the random variable $\tau(\omega)$ satisfies the non-homogeneous Volterra equation of second kind*

$$
g(t) = \int_0^{\infty} g(t-\tau) f(\tau) d\tau + f(t). \tag{15}
$$

Proof. The result follows from (14) by using the argument given in the proof of Theorem 1 in [4]. ☑

Lemma 5.5. *For $f(x) = \epsilon x_1^2 + p\epsilon x_2^2 - p x_3^2$ and $Re(s) > 0$ the following formulas hold:*

(i)
$$
\int_{||x||_p=1} \frac{1}{s + p^{-2\gamma\alpha} |f(y)|_p^{\alpha}} d^3y = \frac{1-p^{-1}}{s+p^{-2\gamma\alpha}} + \frac{p^{-1}(1-p^{-2})}{s+p^{-2\gamma\alpha-\alpha}}.
$$

(ii) If $||\xi||_p \geq p$, then there exist constants C_1 and C_2 such that

$$
\int_{||x||_p=1} \frac{\chi(y\cdot\xi)}{s + p^{-2\gamma\alpha} |f(y)|_p^{\alpha}} d^3y = \begin{cases} \dfrac{C_1}{s+p^{-2\gamma\alpha}} - \dfrac{C_2}{s+p^{-2\gamma\alpha-\alpha}} & if \quad \|\xi\|_p = p, \\ 0 & if \quad \|\xi\|_p > p. \end{cases}
$$

Proof. By using the same technique as in [6, Lemma 15] we write $U := \sqcup_i U^{(i)}$, where $U^{(i)} = U_1^{(i)} \times U_2^{(i)} \times U_3^{(i)}$ and

$$
U_j^{(i)} := \begin{cases} p^{i_j}\mathbb{Z}_p & if \ i_j = 1, \\ \mathbb{Z}_p^* & if \ i_j = 0, \end{cases}
$$

for $(i) = (i_1, i_2, i_3) \in \{0,1\}^3 \setminus \{(1,1,1)\}$. Since $|f(y)|_p \in \{1, p^{-1}\}$, it is enough to compute the volume $\mu(U^{(i)})$. The result follows from the following table.

(i)	$\lvert f(y)\rvert_p$	$\mu(U^{(i)})$
$(0,0,0)$	1	$(1-p^{-1})^3$
$(0,0,1)$	1	$(1-p^{-1})^2 p^{-1}$
$(0,1,0)$	1	$(1-p^{-1})^2 p^{-1}$
$(0,1,1)$	1	$(1-p^{-1}) p^{-2}$
$(1,0,0)$	p^{-1}	$(1-p^{-1})^2 p^{-1}$
$(1,0,1)$	p^{-1}	$(1-p^{-1}) p^{-2}$
$(1,1,0)$	p^{-1}	$(1-p^{-1}) p^{-2}$

☑

Proposition 5.6. *The Laplace transform $G(s)$ of $g(t)$ is given by $G(s) = G_1(s) + G_2(s)$, where*

$$G_1(s) = -\frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^{\alpha})}{1-p^{-2\alpha-3}} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \sum_{\gamma=\nu}^{\infty} p^{-3\gamma} \left(\frac{1-p^{-1}}{s+p^{-2\gamma\alpha}} + \frac{(1-p^{-2})p^{-1}}{s+p^{-2\gamma\alpha-\alpha}} \right),$$

and

$$G_2(s) = -\frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^{\alpha})}{1-p^{-2\alpha-3}} \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} p^{-3(\nu-1)} \left(\frac{C_1}{s+p^{-2(\nu-1)\alpha}} - \frac{C_2}{s+p^{-2(\nu-1)\alpha-\alpha}} \right).$$

Proof. We first note that, if $\mathrm{Re}\,(s) > 0$, then

$$K_{-\alpha}(x) e^{-st} e^{-t|f(\xi)|_p^{\alpha}} \Omega\left(\|\xi\|_p\right) \in L^1\left((0,\infty) \times \mathbb{Q}_p^3 \times \mathbb{Q}_p^3 \setminus \mathbb{Z}_p^3, dt d^3\xi d^3x\right). \tag{16}$$

Therefore, by using (16) and Fubini's Theorem we have

$$\begin{aligned} G(s) &= \int_0^{\infty} e^{-st} g(t) dt \\ &= -\int_0^{\infty} e^{-st} \int_{||x||_p>1} K_{-\alpha}(x) u(x,t) d^3x dt \\ &= -\int_0^{\infty} e^{-st} \int_{||x||_p>1} K_{-\alpha}(x) \int_{\mathbb{Q}_p^3} \chi(-\xi\cdot x) \Omega(||\xi||_p) e^{-t|f(\xi)|_p^{\alpha}} d^3\xi d^3x dt \\ &= -\int_{||x||_p>1} K_{-\alpha}(x) \int_{\mathbb{Z}_p^3} \chi(-\xi\cdot x) \int_0^{\infty} e^{-t(s+|f(\xi)|_p^{\alpha})} dt d^3\xi d^3x \\ &= -\int_{||x||_p>1} K_{-\alpha}(x) \int_{\mathbb{Z}_p^3} \frac{\chi(-\xi\cdot x)}{s+|f(\xi)|_p^{\alpha}} d^3\xi d^3x. \end{aligned}$$

After the change of variables $x = p^{-\nu}y$ and $\xi = p^{\gamma}y'$, and due to the fact that $K_{-\alpha}(p^{-\nu}y) = p^{-2\nu\alpha-3\nu}K_{-\alpha}(y)$, we obtain

$$G(s) = -\sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \int_{||y||_p=1} K_{-\alpha}(y) \sum_{\gamma=0}^{\infty} p^{-3\gamma} \int_{||y'||_p=1} \frac{\chi(-p^{-\nu+\gamma}y \cdot y')}{s + p^{-2\alpha\gamma}|f(y')|_p^{\alpha}} d^3y' d^3y.$$

In order to calculate the interior integral we split the set $||y'||_p = 1$ into two parts, when $||p^{-\nu+\gamma}y \cdot y'||_p \leq 1$, and when $||p^{-\nu+\gamma}y \cdot y'||_p > 1$. The first case occurs when $\gamma \geq \nu$, and then $\chi(-p^{-\nu+\gamma}y \cdot y') = 1$. The second case occurs when $\gamma = 0, \ldots, \nu - 1$. By Lemma 5.5 $G(s)$ takes the form $G(s) = G_1(s) + G_2(s)$, where

$$\begin{aligned} G_1(s) &:= -\sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \int_{||y||_p=1} K_{-\alpha}(y) \sum_{\gamma=\nu}^{\infty} p^{-3\gamma} \int_{||y'||_p=1} \frac{1}{s + p^{-2\alpha\gamma}|f(y')|_p^{\alpha}} d^3y' d^3y \\ &= -\frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^{\alpha})}{1-p^{-2\alpha-3}} \\ &\quad \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \sum_{\gamma=\nu}^{\infty} p^{-3\gamma} \left(\frac{1-p^{-1}}{s+p^{-2\gamma\alpha}} + \frac{(1-p^{-2})p^{-1}}{s+p^{-2\gamma\alpha-\alpha}} \right), \end{aligned}$$

and

$$\begin{aligned} G_2(s) &:= -\sum_{\nu=1}^{\infty} p^{-2\nu\alpha} \int_{||y||_p=1} K_{-\alpha}(y) \sum_{\gamma=0}^{\nu-1} p^{-3\gamma} \int_{||y'||_p=1} \frac{\chi(-p^{-\nu+\gamma}y \cdot y')}{s + p^{-2\alpha\gamma}|f(y')|_p^{\alpha}} d^3y' d^3y \\ &= -\frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^{\alpha})}{1-p^{-2\alpha-3}} \\ &\quad \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha} p^{-3(\nu-1)} \left(\frac{C_1}{s+p^{-2(\nu-1)\alpha}} - \frac{C_2}{s+p^{-2(\nu-1)\alpha-\alpha}} \right). \end{aligned}$$

☑

Theorem 5.7. *(i) If $\alpha \geq \frac{3}{2}$, then $X(t,\omega;\boldsymbol{W})$ is recurrent with respect to $\mathbb{Z}_p^3$.*

(ii) If $\alpha < \frac{3}{2}$, then $X(t,\omega;\boldsymbol{W})$ is transient with respect to $\mathbb{Z}_p^3$.

Proof. By Proposition 5.4, the Laplace transform $F(s)$ of $f(t)$ equals $\frac{G(s)}{1+G(s)}$, where $G(s)$ is the Laplace transform of $g(t)$, and thus

$$F(0) = \int_0^{\infty} f(t)\, dt = 1 - \frac{1}{1+G(0)}.$$

Hence, in order to prove that $X(t,\omega;\boldsymbol{W})$ is recurrent is sufficient to show that $G(0) = \lim_{s\to 0} G(s) = \infty$, and to prove that it is transient, that $G(0) = \lim_{s\to 0} G(s) < \infty$.

(i) Take $s \in \mathbb{R}$, $s > 0$ and set $s = p^{-2\nu\alpha} = p^{-2\gamma\alpha}$; note that $s \to 0^+ \Leftrightarrow v \to \infty$ $(v = \gamma)$. Now, taking only the first term of $G_1(s)$ we have

$$\begin{aligned} G(s) &> -\frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^{\alpha})}{1-p^{-2\alpha-3}} \\ &\quad \times \sum_{\gamma=1}^{\infty} p^{-3\gamma}\left(\frac{1-p^{-1}}{s+p^{-2\gamma\alpha}}+\frac{(1-p^{-2})p^{-1}}{s+p^{-2\gamma\alpha-\alpha}}\right)+G_2(s). \end{aligned}$$

We get $G_2(p^{-2\nu\alpha})<\infty$, but the first sum diverges if $\alpha \geq \frac{3}{2}$. Then,

$$\lim_{s\to 0^+} G(s)=\infty.$$

(ii) Now

$$\begin{aligned} |G(s)| &\leq -\frac{p^{-\alpha}(1-p^{2\alpha})(1-p^{-1})(p^{-1}+p^{-2}+p^{\alpha})}{1-p^{-2\alpha-3}} \\ &\quad \times \sum_{\nu=1}^{\infty} p^{-2\nu\alpha}\sum_{\gamma=\nu}^{\infty} p^{-3\gamma}\left(\frac{1-p^{-1}}{p^{-2\gamma\alpha}}+\frac{(1-p^{-2})p^{-1}}{p^{-2\gamma\alpha-\alpha}}\right)+G_2(0). \end{aligned}$$

One sees easily that $G_2(0)$ converges, and that the double series converges if $\alpha > \frac{3}{2}$. Therefore $\lim_{s\to 0^+} G(s)<\infty$. ☑

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