Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 34, No. 1, 2016, pág. 39–58

DOI: http://dx.doi.org/10.18273/revint.v34n1-2016003

On a finite moment perturbation of linear functionals and the inverse Szegő transformation

Edinson Fuentes[a*], Luis E. Garza[b]

[a] Universidad Pedagógica y Tecnológica de Colombia, Escuela de Matemáticas y Estadística, Tunja, Colombia.

[b] Universidad de Colima, Facultad de Ciencias, Colima, México.

Abstract. Given a sequence of moments $\{c_n\}_{n\in\mathbb{Z}}$ associated with an Hermitian linear functional $\mathcal{L}$ defined in the space of Laurent polynomials, we study a new functional $\mathcal{L}_\Omega$ which is a perturbation of $\mathcal{L}$ in such a way that a finite number of moments are perturbed. Necessary and sufficient conditions are given for the regularity of $\mathcal{L}_\Omega$, and a connection formula between the corresponding families of orthogonal polynomials is obtained. On the other hand, assuming $\mathcal{L}_\Omega$ is positive definite, the perturbation is analyzed through the inverse Szegő transformation.

Keywords: Orthogonal polynomials on the unit circle, perturbation of moments, inverse Szegő transformation.

MSC2010: 42C05, 33C45, 33D45, 33C47.

Sobre una perturbación finita de momentos de un funcional lineal y la transformación inversa de Szegő

Resumen. Dada una sucesión de momentos $\{c_n\}_{n\in\mathbb{Z}}$ asociada a un funcional lineal hermitiano $\mathcal{L}$ definido en el espacio de los polinomios de Laurent, estudiamos un nuevo funcional $\mathcal{L}_\Omega$ que consiste en una perturbación de $\mathcal{L}$ de tal forma que se perturba un número finito de momentos de la sucesión. Se encuentran condiciones necesarias y suficientes para la regularidad de $\mathcal{L}_\Omega$, y se obtiene una fórmula de conexión que relaciona las familias de polinomios ortogonales correspondientes. Por otro lado, suponiendo que $\mathcal{L}_\Omega$ es definido positivo, se analiza la perturbación mediante de la transformación inversa de Szegő.

Palabras clave: Polinomios ortogonales en la circunferencia unidad, perturbación de momentos, transformación de Szegő inversa.

[*] E-mail: edinson.fuentes@uptc.edu.co
Received: 13 October 2015, Accepted: 03 February 2016.
To cite this article: E. Fuentes, L.E. Garza, On a finite moment perturbation of linear functionals and the inverse Szegő transformation, *Rev. Integr. Temas Mat.* 34 (2016), No. 1, 39–58.

1. Preliminaries and introduction

Consider a linear functional $\mathcal{L}$ defined in the linear space of Laurent polynomials $\Lambda = span\{z^n\}_{n\in\mathbb{Z}}$ such that $\mathcal{L}$ is Hermitian, i.e.,

$$c_n = \langle \mathcal{L}, z^n \rangle = \overline{\langle \mathcal{L}, z^{-n} \rangle} = \overline{c}_{-n}, \quad n \in \mathbb{Z}.$$

Then, a bilinear functional can be defined in the linear space $\mathbb{P} = span\{z^n\}_{n\geq 0}$ of polynomials with complex coefficients by

$$\langle p(z), q(z) \rangle_{\mathcal{L}} = \langle \mathcal{L}, p(z)\bar{q}(z^{-1}) \rangle, \quad p, q \in \mathbb{P}.$$

The sequence of complex numbers $\{c_n\}_{n\in\mathbb{Z}}$ is called the sequence of moments associated with $\mathcal{L}$. On the other hand, the Gram matrix associated with the canonical basis $\{z^n\}_{n\geq 0}$ of $\mathbb{P}$ is

$$\mathbf{T} = \begin{bmatrix} c_0 & c_1 & \cdots & c_n & \cdots \\ c_{-1} & c_0 & \cdots & c_{n-1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ c_{-n} & c_{-n+1} & \cdots & c_0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{bmatrix}, \tag{1}$$

which is known in the literature as Toeplitz matrix [7]. A sequence of monic polynomials $\{\phi_n\}_{n\geq 0}$, with $deg\,(\phi_n) = n$, is said to be orthogonal with respect to $\mathcal{L}$ if the condition

$$\langle \phi_n, \phi_m \rangle_{\mathcal{L}} = \delta_{m,n}\mathbf{k}_n,$$

where $\mathbf{k}_n \neq 0$, holds for every $n, m \geqslant 0$. Notice that the sequence $\{\phi_n\}_{n\geq 0}$ can be obtained by using the Gram-Schmidt orthogonalization process with respect to the basis $\{z^n\}_{n\geqslant 0}$. The necessary and sufficient conditions for the existence of such a sequence can be expressed in terms of the Toeplitz matrix $\mathbf{T}$: $\{\phi_n\}_{n\geq 0}$ satisfies the orthogonality condition if and only if $\mathbf{T}_n$, the $(n+1)\times(n+1)$ principal leading submatrix of $\mathbf{T}$, is non-singular for every $n \geq 0$. In such a case, $\mathcal{L}$ is said to be quasi-definite (or regular). On the other hand, if $\det \mathbf{T}_n > 0$ for every $n \geq 0$, then $\mathcal{L}$ is said to be positive definite and it has the integral representation

$$\langle \mathcal{L}, p(z) \rangle = \int_{\mathbb{T}} p(z) d\sigma(z), \quad p \in \mathbb{P}, \tag{2}$$

where σ is a nontrivial positive Borel measure supported on the unit circle $\mathbb{T} = \{z : |z| = 1\}$. In such a case, there exists a (unique) family of polynomials $\{\varphi_n\}_{n\geq 0}$, with $deg\,\varphi_n = n$ and positive leading coefficient, such that

$$\int_{\mathbb{T}} \varphi_n(z)\overline{\varphi_m(z)} d\sigma(z) = \delta_{m,n}. \tag{3}$$

$\{\varphi_n\}_{n\geq 0}$ is said to be the sequence of orthonormal polynomials with respect to σ. If we denote by κ_n the leading coefficient of $\varphi_n(z)$, then we have $\phi_n(z) = \varphi_n(z)/\kappa_n$. These polynomials satisfy the following forward and backward recurrence relations (see [7], [10], [11]):

$$\phi_{n+1}(z) = z\phi_n(z) + \phi_{n+1}(0)\phi_n^*(z), \quad n \geq 0, \tag{4}$$

$$\phi_{n+1}(z) = \left(1 - |\phi_{n+1}(0)|^2\right) z\phi_n(z) + \phi_{n+1}(0)\phi_{n+1}^*(z), \quad n \geq 0, \tag{5}$$

where $\phi_n^*(z) = z^n \bar{\phi}_n(z^{-1})$ is the so-called reversed polynomial and the complex numbers $\{\phi_n(0)\}_{n\geq 1}$ are known as Verblunsky (Schur, reflection) parameters. It is important to notice that in the positive definite case we get $|\phi_n(0)| < 1$, $n \geq 1$, and

$$\mathbf{k}_n = \langle \phi_n, \phi_n \rangle_{\mathcal{L}} > 0,\ n \geq 0.$$

Moreover, we have

$$\mathbf{k}_n = \frac{\det \mathbf{T}_n}{\det \mathbf{T}_{n-1}}, \qquad n \geq 1, \quad \mathbf{k}_0 = c_0, \quad \mathbf{T_{-1}} \equiv 0. \tag{6}$$

The n-th kernel polynomial $K_n(z, y)$ associated with $\{\phi\}_{n\geq 0}$ is defined by

$$K_n(z,y) = \sum_{j=0}^{n} \frac{\overline{\phi_j(y)}\phi_j(z)}{\mathbf{k}_j} = \frac{\overline{\phi_{n+1}^*(y)}\phi_{n+1}^*(z) - \overline{\phi_{n+1}(y)}\phi_{n+1}(z)}{\mathbf{k}_{n+1}(1-\overline{y}z)}, \tag{7}$$

and the right hand side is known in the literature as Christoffel-Darboux formula and it holds if $\bar{y}z \neq 1$. It satisfies the so called reproducing property

$$\int_{\mathbb{T}} K_n(z,y)\overline{p(z)}d\sigma(z) = \overline{p(y)}, \tag{8}$$

for every polynomial p of degree at most n. $K_n^{(i,j)}(z, y)$ will denote the i-th and j-th partial derivative of $K_n(z, y)$ with respect to z and y, respectively. Notice that we have $\phi_n^*(z) = \mathbf{k}_n K_n(z, 0), n \geq 0$.

Furthermore, in terms of the moments, an analytic function can be defined by

$$F(z) = c_0 + 2\sum_{k=1}^{\infty} c_{-k} z^k. \tag{9}$$

If $\mathcal{L}$ is a positive definite functional, then (9) is analytic in $\mathbb{D}$ and its real part is positive in $\mathbb{D}$. In such a case, (9) is called a Carathéodory function, and can be represented by the Riesz-Herglotz transform

$$F(z) = \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\sigma(\theta),$$

where σ is the positive measure associated with $\mathcal{L}$. By extension, for a quasi-definite linear functional, (9) will denote its corresponding Carathéodory function.

On the other hand, given a positive, nontrivial Borel measure α supported in $[-1, 1]$, we can define a positive, nontrivial Borel measure σ supported in $[-\pi, \pi]$ in such a way that if $d\alpha(x) = \omega(x)dx$, then

$$d\sigma(\theta) = \frac{1}{2}\omega(\cos\theta)|\sin\theta|d\theta. \tag{10}$$

There exists a relation between the corresponding families of orthogonal polynomials (see [6]). On the other hand, since the moments $\{c_n\}_{n\geq 0}$ are real (see [6]), $F(z)$, the Carathéodory function associated with σ, has real coefficients. Therefore, we have

$$\mathfrak{Re} F(e^{i\theta}) = \mathfrak{Re} F(e^{i(2\pi-\theta)}),$$

and then $d\sigma(\theta) + d\sigma(2\pi - \theta) = 0$. Thus, there exists a simple relation between the Stieltjes function (the real line analog of the Carathédory functions, given by $S(x) = \sum_{n=0}^{\infty} \mu_n x^{-(n+1)}$, where $\{\mu_n\}_{n\geq 0}$ are the moments associated with the measure on the real line) and the Carathéodory function associated with α and σ, respectively, given by (see [9])

$$F(z) = \frac{1-z^2}{2z} \int_{-1}^{1} \frac{d\alpha(t)}{x-t} = \frac{1-z^2}{2z} S(x), \tag{11}$$

where $x = \frac{z+z^{-1}}{2}$, $z = x + \sqrt{x^2-1}$. In the literature, this relation is known as the Szegő transformation. Conversely, if σ is a positive, nontrivial Borel measure with support in the unit circle such that its moments are real, then there exists a positive, nontrivial Borel measure α, supported in $[-1, 1]$, such that (10) holds. This is called the inverse Szegő transformation.

Given a measure σ supported on the unit circle, the perturbations

(1) $d\tilde{\sigma}_C = |z-\xi|^2 d\sigma$, $|z| = 1, \xi \in \mathbb{C}$,

(2) $d\tilde{\sigma}_U = d\sigma + M_c \delta(z-\xi) + \overline{M}_c \delta(z - \overline{\xi}^{-1}), \xi \in \mathbb{C} - \{0\}, M_c \in \mathbb{C}$,

(3) $d\tilde{\sigma}_G = \frac{d\sigma}{|z-\xi|^2} + M_c \delta(z-\xi) + \overline{M}_c \delta(z - \overline{\xi}^{-1}), \xi \in \mathbb{C} - \{0\}, M_c \in \mathbb{C}, |\xi| \neq 1$,

are called Christoffel, Uvarov, and Geronimus transformations, respectively. They are the unit circle analogue of the Christoffel, Uvarov and Geronimus transformations on the real line (see [12]). In general, a linear spectral transformation of a Stieltjes function is another Stieltjes function $\tilde{S}(x)$ that has the form

$$\tilde{S}(x) = \frac{A(x)S(x) + B(z)}{D(z)},$$

where A, B and D are polynomials in x. The three transformations defined above are important due to the fact that any linear spectral transformation of a given Stieltjes function (i.e., for any polynomials A, B and D) can be obtained as a combination of Christoffel and Geronimus transformations (see [12]). A similar result holds for linear spectral transformations of Carathéodory functions, which are defined in a similar way (see [4]).

In [2], the authors studied the perturbation associated with the linear functional

$$\langle p(z), q(z) \rangle_{\tilde{\mathcal{L}}} := \langle p(z), q(z) \rangle_{\mathcal{L}} + m \int_{\mathbb{T}} p(z)\overline{q(z)} \frac{dz}{2\pi i z}, \tag{12}$$

where $m \in \mathbb{R}$, $p, q \in \mathbb{P}$, and $\mathcal{L}$ is (at least) a quasi-definite Hermitian linear functional defined in the linear space of Laurent polynomials. Notice that all moments associated with $\tilde{\mathcal{L}}$ are equal to the moments associated with $\mathcal{L}$, except for the first moment, which is $\tilde{c}_0 = c_0 + m$. The corresponding Toeplitz matrix $\tilde{\mathbf{T}}$ is the result of adding m to the main diagonal of $\mathbf{T}$. Later on, the linear functional

$$\langle p(z), q(z) \rangle_{\mathcal{L}_j} = \langle p(z), q(z) \rangle_{\mathcal{L}} + m \langle z^j p(z), q(z) \rangle_{\mathcal{L}_\theta} + \bar{m} \langle p(z), z^j q(z) \rangle_{\mathcal{L}_\theta}, \tag{13}$$

where $j \in \mathbb{N}$ is fixed and $\langle \cdot, \cdot \rangle_{\mathcal{L}_\theta}$ is the bilinear functional associated with the normalized Lebesgue measure on the unit circle was studied in [3]. It is easily seen that the moments associated with $\mathcal{L}_j$ are equal to those of $\mathcal{L}$, except for the moments of order j and $-j$, which are perturbed by adding m and $\bar{m}$, respectively. In other words, the corresponding Toeplitz matriz is perturbed on the j-th and $-j$-th subdiagonals. In both cases, the authors obtained the regularity conditions for such a linear functional and deduced connection formulas between the corresponding orthogonal sequences.

Assuming that both $\mathcal{L}$ and $\mathcal{L}_j$ are positive definite, the perturbation (13) can be expressed in terms of the corresponding measures as

$$d\tilde{\sigma} = d\sigma + mz^j \frac{d\theta}{2\pi} + \bar{m} z^{-j} \frac{d\theta}{2\pi}. \tag{14}$$

On the other hand, the connection between the measure (14) and its corresponding measure supported in $[-1, 1]$ via the inverse Szegő transformation was analyzed in [5], and it is deduced that the perturbed moments on the real line depend on the Chebyshev polynomials of the first kind.

In this contribution, we will extend those results to the case where a perturbation of a finite number of moments is introduced in (13). In Section 2, necessary and sufficient conditions for the regularity of the perturbed functional are obtained, as well as a connection formula that relates the corresponding families of monic orthogonal polynomials. For the positive definite case, the study of the perturbation through the inverse Szegő transformation will be analyzed in Section 3. An illustrative example will be presented in Section 4.

2. A perturbation on a finite number of moments associated with a linear functional $\mathcal{L}$

Let $\mathcal{L}$ be a quasi-definite linear functional on the linear space of Laurent polynomials, and let $\{c_n\}_{n\in\mathbb{Z}}$ be its associated sequence of moments.

Definition 2.1. Let Ω be a finite set of non negative integers. The linear functional $\mathcal{L}_\Omega$ is defined such that the associated bilinear functional satisfies

$$\langle p(z), q(z) \rangle_{\mathcal{L}_\Omega} = \langle p(z), q(z) \rangle_{\mathcal{L}} + \sum_{r\in\Omega} \left(M_r \langle z^r p(z), q(z) \rangle_{\mathcal{L}_\theta} + \overline{M}_r \langle p(z), z^r q(z) \rangle_{\mathcal{L}_\theta} \right), \tag{15}$$

where $M_r \in \mathbb{C}$, $p, q \in \mathbb{P}$, and $\langle \cdot, \cdot \rangle_{\mathcal{L}_\theta}$ is the bilinear functional associated with the normalized Lebesgue measure in the unit circle.

Notice that, from (15), one easily sees that

$$\tilde{c}_n = \langle \mathcal{L}_\Omega, z^n \rangle = \langle z^n, 1 \rangle_{\mathcal{L}_\Omega} = \begin{cases} c_n, & \text{if} \quad n \notin \Omega, \\ c_{-n} + M_{-n}, & \text{if} \quad n \in \Omega \text{ and } n \in \mathbb{Z}^-, \\ c_n + \overline{M}_n, & \text{if} \quad n \in \Omega \text{ and } n \notin \mathbb{Z}^-. \end{cases} \tag{16}$$

In other words, $\mathcal{L}_\Omega$ represents an additive perturbation of the moments c_r and c_{-r} of $\mathcal{L}$, with $r \in \Omega$. The rest of the moments remain unchanged. This is, the Toeplitz matrix associated with $\mathcal{L}_\Omega$ is

$$\widetilde{\mathbf{T}} = \mathbf{T} + \sum_{r\in\Omega} \underbrace{\begin{bmatrix} 0 & \cdots & \overline{M}_r & 0 & \cdots \\ \vdots & 0 & \cdots & \overline{M}_r & \cdots \\ M_r & \vdots & \ddots & \vdots & \ddots \\ 0 & M_r & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}}_{M_r \text{ is on the } r-\text{th subdiagonal}},$$

and therefore $\mathcal{L}_\Omega$ is also Hermitian. Moreover, if $\mathcal{L}$ is a positive definite functional, then the above perturbation can be expressed in terms of the corresponding measures as

$$\begin{aligned} d\tilde{\sigma}_\Omega &= d\sigma + \sum_{r\in\Omega} \left(M_r z^r \frac{d\theta}{2\pi} + \overline{M}_r z^{-r} \frac{d\theta}{2\pi} \right) \\ &= d\sigma + 2 \sum_{r\in\Omega} \mathfrak{Re}(M_r z^r) \frac{d\theta}{2\pi}. \end{aligned} \tag{17}$$

On the other hand, if $F_\Omega(z)$ is the Carathéodory function associated with $\mathcal{L}_\Omega$, then

$$F_\Omega(z) = F(z) + 2 \sum_{r\in\Omega} M_r z^r, \tag{18}$$

which is a linear spectral transformation of $F(z)$. The following notation will be used hereinafter:

- $\mathbf{A}_{(s_1,s_2;l_1,l_2;r)}$ will denote a $(s_2 - s_1 + 1) \times (l_2 - l_1 + 1)$ matrix whose entries are $a_{(s,l)_r}$, where $s_1 \le s \le s_2$ and $l_1 \le l \le l_2$,. For instance,

$$\mathbf{A}_{(2,3;4,5;6)} = \begin{bmatrix} a_{(2,4)_6} & a_{(2,5)_6} \\ a_{(3,4)_6} & a_{(3,5)_6} \end{bmatrix}.$$

- $\Psi_n(0) = [\psi_n^{(0)}(0), \cdots, \psi_n^{(n-1)}(0)]^T$ and $\Phi_n(0) = [\phi_n^{(0)}(0), \cdots, \phi_n^{(n-1)}(0)]^T$.
- $\mathbf{I}_n$ will denote the $n \times n$ identity matrix.
- Derivatives of negative order are defined as zero. For instance, $K_n^{(0,-2)}(z, y) \equiv 0$.

Necessary and sufficient conditions for the regularity of $\mathcal{L}_\Omega$, as well as the relation between the corresponding sequences of orthogonal polynomials, are given in the next result.

Proposition 2.2. *Let $\mathcal{L}$ be a quasi-definite linear functional and let $\{\phi_n\}_{n\ge0}$ be its associated monic orthogonal polynomials sequence (**MOPS**). The following statements are equivalent* [1]

[1] This result generalizes the case when Ω has a single element $k \neq 0$ (see [3]) and the case $\Omega = \{0\}$ (see [2]).

1. $\mathcal{L}_\Omega$ *is a quasi-definite linear functional.*

2. *The matrix* $\boldsymbol{I}_n + \sum_{r\in\Omega} \boldsymbol{S}_n^r$ *is nonsingular, and*

$$\tilde{\mathbf{k}}_n = \mathbf{k}_n + (\boldsymbol{Q}_n \boldsymbol{W}_n)^T \sum_{r\in\Omega} \boldsymbol{Y}_n^r + \sum_{r\in\Omega} \overline{M}_r \overline{\frac{\phi_n^{(n-r)}(0)}{(n-r)!}} \neq 0, \;\; n \geq 0, \tag{19}$$

with

$$\boldsymbol{Y}_n^r = \begin{bmatrix} M_r \overline{\frac{\phi_n^{(r)}(0)}{0!r!}} \\ \vdots \\ M_r \overline{\frac{\phi_n^{(2r-1)}(0)}{(r-1)!(2r-1)!}} \\ M_r \overline{\frac{\phi_n^{(2r)}(0)}{r!(2r)!}} + \overline{M}_r \overline{\frac{\phi_n^{(0)}(0)}{r!0!}} \\ \vdots \\ M_r \overline{\frac{\phi_n^{(n)}(0)}{(n-r)!(n)!}} + \overline{M}_r \overline{\frac{\phi_n^{(n-2r)}(0)}{(n-r)!(n-2r)!}} \\ \overline{M}_r \overline{\frac{\phi_n^{(n-2r+1)}(0)}{(n-r+1)!(n-2r+1)!}} \\ \vdots \\ \overline{M}_r \overline{\frac{\phi_n^{(n-r-1)}(0)}{(n-1)!(n-r-1)!}} \end{bmatrix}, \tag{20}$$

$$\boldsymbol{Q}_n = (\boldsymbol{I}_n + \sum_{r\in\Omega} \boldsymbol{S}_n^r)^{-1},$$

$$\boldsymbol{W}_n = \Phi_n(0) - \sum_{r\in\Omega} \overline{M}_r n! \boldsymbol{C}_{(0,n-1;n,n;r)},$$

$$\mathbb{K}_{n-1}^r(z,0) = \begin{bmatrix} M_r \frac{K_{n-1}^{(0,r)}(z,0)}{0!r!} \\ \vdots \\ M_r \frac{K_{n-1}^{(0,2r-1)}(z,0)}{(r-1)!(2r-1)!} \\ M_r \frac{K_{n-1}^{(0,2r)}(z,0)}{r!(2r)!} + \overline{M}_r \frac{K_{n-1}^{(0,0)}(z,0)}{r!0!} \\ \vdots \\ M_r \frac{K_{n-1}^{(0,n-1)}(z,0)}{(n-r-1)!(n-1)!} + \overline{M}_r \frac{K_{n-1}^{(0,n-2r-1)}(z,0)}{(n-r-1)!(n-2r-1)!} \\ \overline{M}_r \frac{K_{n-1}^{(0,n-2r)}(z,0)}{(n-r)!(n-2r)!} \\ \vdots \\ \overline{M}_r \frac{K_{n-1}^{(0,n-r-1)}(z,0)}{(n-1)!(n-r-1)!}. \end{bmatrix}, \tag{21}$$

and

$$\boldsymbol{S}_n^r = \begin{bmatrix} M_r \boldsymbol{A}_{(0,r-1;0,r-1;r)} & \boldsymbol{B}_{(0,r-1;r,n-r-1;r)} & \overline{M}_r \boldsymbol{C}_{(0,r-1;n-r,n-1;r)} \\ M_r \boldsymbol{A}_{(r,n-r-1;0,r-1;r)} & \boldsymbol{B}_{(r,n-r-1;r,n-r-1;r)} & \overline{M}_r \boldsymbol{C}_{(r,n-r-1;n-r,n-1;r)} \\ M_r \boldsymbol{A}_{(n-r,n-1;0,r-1;r)} & \boldsymbol{B}_{(n-r,n-1;r,n-r-1;r)} & \overline{M}_r \boldsymbol{C}_{(n-r,n-1;n-r,n-1;r)} \end{bmatrix},$$

where the entries of the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are given by

$$a_{(s,l)_r} = \frac{K_{n-1}^{(s,l+r)}(0,0)}{l!(l+r)!},$$

$$b_{(s,l)_r} = M_r \frac{K_{n-1}^{(s,l+r)}(0,0)}{l!(l+r)!} + \overline{M}_r \frac{K_{n-1}^{(s,l-r)}(0,0)}{l!(l-r)!},$$

$$c_{(s,l)_r} = \frac{K_{n-1}^{(s,l-r)}(0,0)}{l!(l-r)!}.$$

*Furthermore, if $\{\psi_n\}_{n\geq 0}$ denotes the **MOPS** associated with $\mathcal{L}_\Omega$, then*

$$\psi_n(z) = \phi_n(z) - (\boldsymbol{Q}_n \boldsymbol{W}_n)^T \sum_{r\in\Omega} \mathbb{K}_{n-1}^r(z,0) - \sum_{r\in\Omega} \overline{M}_r \frac{K_{n-1}^{(0,n-r)}(z,0)}{(n-r)!}, \tag{22}$$

for every $n \geq 1$.

Remark 2.3. Notice that $\mathbf{Q}_n$ and $\mathbf{S}_n^r$ are $n\times n$ matrices, whereas $\mathbf{Y}_n^r$, $\mathbf{W}_n$ and $\mathbb{K}_{n-1}^r(z,0)$ are n-th dimensional column vectors.

Proof. Assume $\mathcal{L}_\Omega$ is a quasi-definite linear functional, and denote by $\{\psi_n\}_{n\geq 0}$ its associated **MOPS**. Let us write

$$\psi_n(z) = \phi_n(z) + \sum_{k=0}^{n-1} \lambda_{n,k}\phi_k(z), \tag{23}$$

where, for $0 \leq k \leq n-1$,

$$\begin{aligned} \lambda_{n,k} &= \frac{\langle \psi_n(z), \phi_k(z)\rangle_{\mathcal{L}}}{\mathbf{k}_n} \\ &= \frac{\langle \psi_n(z), \phi_k(z)\rangle_{\mathcal{L}_\Omega} - \sum_{r\in\Omega} M_r \int_{\mathbb{T}} y^r \psi_n(y)\overline{\phi_k(y)}\frac{dy}{2\pi i y} - \sum_{r\in\Omega} \overline{M}_r \int_{\mathbb{T}} y^{-r} \psi_n(y)\overline{\phi_k(y)}\frac{dy}{2\pi i y}}{\mathbf{k}_k} \\ &= -\sum_{r\in\Omega} \frac{M_r}{\mathbf{k}_k} \int_{\mathbb{T}} y^r \psi_n(y)\overline{\phi_k(y)}\frac{dy}{2\pi i y} - \sum_{r\in\Omega} \frac{\overline{M}_r}{\mathbf{k}_k} \int_{\mathbb{T}} y^{-r} \psi_n(y)\overline{\phi_k(y)}\frac{dy}{2\pi i y}, \end{aligned}$$

and notice that $\langle \psi_n(z), \phi_k(z)\rangle_{\mathcal{L}} \neq 0$ (in general), and $\langle \psi_n(z), \phi_k(z)\rangle_{\mathcal{L}_\Omega} = 0$ for $n > k$.

Substituting in (23) and using (7), we get

$$\begin{aligned}\psi_n(z) = \phi_n(z) &- \sum_{r\in\Omega}\left(M_r\int_{\mathbb{T}} y^r\psi_n(y)\sum_{k=0}^{n-1}\frac{\phi_k(z)\overline{\phi_k(y)}}{\mathbf{k}_k}\frac{dy}{2\pi iy}\right)\\ &- \sum_{r\in\Omega}\left(\overline{M}_r\int_{\mathbb{T}} y^{-r}\psi_n(y)\sum_{k=0}^{n-1}\frac{\phi_k(z)\overline{\phi_k(y)}}{\mathbf{k}_k}\frac{dy}{2\pi iy}\right)\\ = \phi_n(z) &- \sum_{r\in\Omega}\left(M_r\int_{\mathbb{T}} y^r\psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} - \overline{M}_r\int_{\mathbb{T}} y^{-r}\psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy}\right).\end{aligned}$$

From the power series expansion of $\psi_n(y)$ and $K_{n-1}(z,y)$, we have

$$\begin{aligned}y^r\psi_n(y) &= y^r\sum_{l=0}^{n}\frac{\psi_n^{(l)}(0)}{l!}y^l\\ &= \sum_{l=r}^{n+r}\frac{\psi_n^{(l-r)}(0)}{(l-r)!}y^l,\end{aligned}$$

and for $|y|=1$,

$$K_{n-1}(z,y) = \sum_{l=0}^{n-1}\frac{K_{n-1}^{(0,l)}(z,0)}{l!}\frac{1}{y^l},$$

and since $\int_{\mathbb{T}} y^{r-t}\frac{dy}{2\pi iy} = 1$ if $r=t$ and zero otherwise, we arrive at

$$\begin{aligned}\int_{\mathbb{T}} y^r\psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} &= \int_{\mathbb{T}}\sum_{l=r}^{n+r}\frac{\psi_n^{(l-r)}(0)}{(l-r)!}y^l\sum_{l=0}^{n-1}\frac{K_{n-1}^{(0,l)}(z,0)}{l!}\frac{1}{y^l}\frac{dy}{2\pi iy}\\ &= \sum_{l=0}^{n-r-1}\frac{\psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{T}} y^{-r}\psi_n(y)K_{n-1}(z,y)\frac{dy}{2\pi iy} &= \sum_{l=0}^{n-r}\frac{\psi_n^{(l+r)}(0)}{(l+r)!}\frac{K_{n-1}^{(0,l)}(z,0)}{l!}\\ &= \sum_{l=r}^{n}\frac{\psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}.\end{aligned}$$

As a consequence, we get

$$\psi_n(z) = \phi_n(z) - \sum_{r\in\Omega}\left(M_r\sum_{l=0}^{n-r-1}\frac{\psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!} - \overline{M}_r\sum_{l=r}^{n}\frac{\psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}\right), \tag{24}$$

which after a reorganization of the terms becomes

$$\begin{aligned}\psi_n(z) = \phi_n(z) - \sum_{r\in\Omega}\left(M_r\sum_{l=0}^{r-1}\frac{\psi_n^{(l)}(0)}{l!}\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!} + \overline{M}_r\sum_{l=n-r}^{n}\frac{\psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}\right)\\ - \sum_{r\in\Omega}\left(\sum_{l=r}^{n-r-1}\frac{\psi_n^{(l)}(0)}{l!}\left(M_r\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!} + \overline{M}_r\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}\right)\right).\end{aligned} \tag{25}$$

In order to find the constant values $\psi_n^{(l)}(0)$, we take s derivatives, $0\leq s\leq n$, with respect to the variable z and evaluate at $z=0$ to obtain the $(n+1)\times(n+1)$ linear system

$$\begin{aligned}\psi_n^{(s)}(0) = \phi_n^{(s)}(0) - \sum_{r\in\Omega}\left(M_r\sum_{l=0}^{r-1}\frac{\psi_n^{(l)}(0)}{l!}\frac{K_{n-1}^{(s,l+r)}(0,0)}{(l+r)!} + \overline{M}_r\sum_{l=n-r}^{n}\frac{\psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(s,l-r)}(0,0)}{(l-r)!}\right)\\ - \sum_{r\in\Omega}\left(\sum_{l=r}^{n-r-1}\frac{\psi_n^{(l)}(0)}{l!}\left(M_r\frac{K_{n-1}^{(s,l+r)}(0,0)}{(l+r)!} + \overline{M}_r\frac{K_{n-1}^{(s,l-r)}(0,0)}{(l-r)!}\right)\right).\end{aligned}$$

If we denote

$$\begin{aligned}a_{(s,l)_r} &= \frac{K_{n-1}^{(s,l+r)}(0,0)}{l!(l+r)!},\\ b_{(s,l)_r} &= M_r\frac{K_{n-1}^{(s,l+r)}(0,0)}{l!(l+r)!} + \overline{M}_r\frac{K_{n-1}^{(s,l-r)}(0,0)}{l!(l-r)!},\\ c_{(s,l)_r} &= \frac{K_{n-1}^{(s,l-r)}(0,0)}{l!(l-r)!},\end{aligned}$$

then the linear system becomes

$$\begin{aligned}&\psi_n^{(s)}(0) = \phi_n^{(s)}(0)\\ &\quad - \sum_{r\in\Omega}\left(M_r\sum_{l=0}^{r-1}a_{(s,l)_r}\psi_n^{(l)}(0) + \sum_{l=r}^{n-r-1}b_{(s,l)_r}\psi_n^{(l)}(0) + \overline{M}_r\sum_{l=n-r}^{n}c_{(s,l)_r}\psi_n^{(l)}(0)\right).\end{aligned}$$

Notice that the last equation (i.e., when $s=n$) gives no information, since $a_{(n,l)_r} = b_{(n,l)_r} = c_{(n,l)_r} = 0$ and $\psi_n^{(n)}(0) = \phi_n^{(n)}(0) = n!$. As a consequence, the $(n+1)\times(n+1)$ linear system can be reduced to an $n\times n$ linear system that can be expressed in matrix form as

$$\begin{aligned}(\mathbf{I}_n + \sum_{r\in\Omega}\mathbf{S}_n^r)\Psi_n(0) &= \Phi_n(0) - \sum_{r\in\Omega}\overline{M}_r n!\mathbf{C}_{(0,n-1;n,n;r)}\\ &= \mathbf{W}_n.\end{aligned} \tag{26}$$

Since $\mathcal{L}_\Omega$ is quasi-definite, it has a unique **MOPS** and therefore the linear system has a unique solution. As a consequence, the matrix $\mathbf{I}_n + \sum_{r\in\Omega}\mathbf{S}_n^r$ is nonsingular and

$$\Psi_n(0) = (\mathbf{I}_n + \sum_{r\in\Omega}\mathbf{S}_n^r)^{-1}\mathbf{W}_n. \tag{27}$$

This is, we have

$$\psi_n(z) = \phi_n(z) - \sum_{r\in\Omega} \mathbf{W}_n^T \left((\mathbf{I}_n + \sum_{r\in\Omega} \mathbf{S}_n^r)^{-1} \right)^T \mathbb{K}_{n-1}^r(z,0) - \sum_{r\in\Omega} \overline{M}_r \frac{K_{n-1}^{(0,n-r)}(z,0)}{(n-r)!}, \quad (28)$$

which is (22). On the other hand, for $n \geq 0$,

$$\begin{aligned}
\tilde{\mathbf{k}}_n &= \langle \psi_n(z), \psi_n(z) \rangle_{\mathcal{L}_\Omega} = \langle \psi_n(z), \phi_n(z) \rangle_{\mathcal{L}_\Omega} \\
&= \langle \psi_n(z), \phi_n(z) \rangle_{\mathcal{L}} + \sum_{r\in\Omega} \left(M_r \langle z^r \psi_n(z), \phi_n(z) \rangle_{\mathcal{L}_\theta} + \overline{M}_r \langle \psi_n(z), z^r \phi_n(z) \rangle_{\mathcal{L}_\theta} \right) \\
&= \mathbf{k}_n + \sum_{r\in\Omega} \left(M_r \int_{\mathbb{T}} \sum_{l=0}^{n} \frac{\psi_n^{(l)}(0)}{l!} z^{l+r} \sum_{l=0}^{n} \frac{\overline{\phi_n^{(l)}(0)}}{l!} z^{-l} \frac{dz}{2\pi i z} \right) \\
&\quad + \sum_{r\in\Omega} \left(\overline{M}_r \int_{\mathbb{T}} \sum_{l=0}^{n} \frac{\psi_n^{(l)}(0)}{l!} z^{l-r} \sum_{l=0}^{n} \frac{\overline{\phi_n^{(l)}(0)}}{l!} z^{-l} \frac{dz}{2\pi i z} \right) \\
&= \mathbf{k}_n + \sum_{r\in\Omega} \left(M_r \sum_{l=0}^{r-1} \frac{\psi_n^{(l)}(0)}{l!} \frac{\overline{\phi_n^{(l+r)}(0)}}{(l+r)!} + \sum_{l=r}^{n-r} \frac{\psi_n^{(l)}(0)}{l!} \left(M_r \frac{\overline{\phi_n^{(l+r)}(0)}}{(l+r)!} + \overline{M}_r \frac{\overline{\phi_n^{(l-r)}(0)}}{(l-r)!} \right) \right) \\
&\quad + \sum_{r\in\Omega} \left(\overline{M}_r \sum_{l=n-r+1}^{n-1} \frac{\psi_n^{(l)}(0)}{l!} \frac{\overline{\phi_n^{(l-r)}(0)}}{(l-r)!} \right) + \sum_{r\in\Omega} \overline{M}_r \frac{\overline{\phi_n^{(n-r)}(0)}}{(n-r)!}.
\end{aligned}$$

Using (20) and (27), we get

$$\tilde{\mathbf{k}}_n = \mathbf{k}_n + \sum_{r\in\Omega} \mathbf{W}_n^T \left((\mathbf{I}_n + \sum_{r\in\Omega} \mathbf{S}_n^r)^{-1} \right)^T \mathbf{Y}_n^r + \sum_{r\in\Omega} \overline{M}_r \frac{\overline{\phi_n^{(n-r)}(0)}}{(n-r)!}.$$

Conversely, assume $\mathbf{I}_n + \sum_{r\in\Omega} \mathbf{S}_n^r$ is nonsingular for every $n \geq 1$ and define $\{\psi_n\}_{n\geq 0}$ as in (22). For $0 \leq k \leq n-1$, we have

$$\begin{aligned}
\left\langle \sum_{l=0}^{n-r-1} \frac{\psi_n^{(l)}(0)}{l!} \frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!}, \phi_k(z) \right\rangle_{\mathcal{L}} &= \sum_{l=0}^{n-r-1} \left\langle \frac{\psi_n^{(l)}(0)}{l!} \frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!}, \phi_k(z) \right\rangle_{\mathcal{L}} \\
&= \sum_{l=0}^{n-r-1} \left\langle \frac{\psi_n^{(l)}(0)}{l!(l+r)!} \sum_{t=0}^{n-1} \frac{\phi_t(z)\overline{\phi_t^{(l+r)}(0)}}{\mathbf{k}_t}, \phi_k(z) \right\rangle_{\mathcal{L}} \\
&= \sum_{l=0}^{n-r-1} \frac{\psi_n^{(l)}(0)}{l!} \frac{\overline{\phi_k^{(l+r)}(0)}}{(l+r)!},
\end{aligned}$$

and, similarly,

$$\left\langle \sum_{l=j}^{n} \frac{\psi_n^{(l)}(0)}{l!} \frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!}, \phi_k(z) \right\rangle_{\mathcal{L}} = \sum_{l=r}^{n} \frac{\psi_n^{(l)}(0)}{l!} \frac{\overline{\phi_k^{(l-r)}(0)}}{(l-r)!}.$$

In addition,

$$\begin{aligned}\langle z^r\psi_n(z),\phi_k(z)\rangle_{\mathcal{L}_\theta} &= \int_{\mathbb{T}} z^r\psi_n(z)\overline{\phi_k(z)}\frac{dz}{2\pi iz}\\ &= \int_{\mathbb{T}} z^r\left(\sum_{l=0}^{n}\frac{\psi_n^{(l)}(0)z^l}{l!}\right)\left(\sum_{l=0}^{k}\frac{\overline{\phi_n^{(l)}(0)}z^{-l}}{l!}\right)\frac{dz}{2\pi iz}\\ &= \sum_{l=0}^{n-r-1}\frac{\psi_n^{(l)}(0)}{l!}\frac{\overline{\phi_k^{(l+r)}(0)}}{(l+r)!},\end{aligned}$$

and also

$$\langle\psi_n(z),z^r\phi_k(z)\rangle_{\mathcal{L}_\theta} = \sum_{l=r}^{n}\frac{\psi_n^{(l)}(0)}{l!}\frac{\overline{\phi_k^{(l-r)}(0)}}{(l-r)!}.$$

Thus, for $0\leq k\leq n-1$, and taking into account (24) and the previous equations, we have

$$\begin{aligned}\langle\psi_n(z),\phi_k(z)\rangle_{\mathcal{L}_\Omega} &= \langle\psi_n(z),\phi_k(z)\rangle_{\mathcal{L}} + \sum_{r\in\Omega}\left(M_r\langle z^r\psi_n(z),\phi_k(z)\rangle_{\mathcal{L}_\theta} + \overline{M}_r\langle\psi_n(z),z^r\phi_k(z)\rangle_{\mathcal{L}_\theta}\right)\\ &= \langle\phi_n(z),\phi_n(z)\rangle_{\mathcal{L}} - \sum_{r\in\Omega}M_r\left\langle\sum_{l=0}^{n-r-1}\frac{\psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l+r)}(z,0)}{(l+r)!},\phi_k(z)\right\rangle_{\mathcal{L}}\\ &\qquad - \sum_{r\in\Omega}\overline{M}_r\left\langle\sum_{l=r}^{n}\frac{\psi_n^{(l)}(0)}{(l)!}\frac{K_{n-1}^{(0,l-r)}(z,0)}{(l-r)!},\phi_k(z)\right\rangle_{\mathcal{L}}\\ &\quad + \sum_{r\in\Omega}M_r\left(\sum_{l=0}^{n-r-1}\frac{\psi_n^{(l)}(0)}{l!}\frac{\overline{\phi_k^{(l+r)}(0)}}{(l+r)!}\right) + \sum_{r\in\Omega}\overline{M}_r\left(\sum_{l=r}^{n}\frac{\psi_n^{(l)}(0)}{l!}\frac{\overline{\phi_k^{(l-r)}(0)}}{(l-r)!}\right)\\ &= 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\widetilde{\mathbf{k}}_n &= \langle\psi_n(z),\phi_n(z)\rangle_{\mathcal{L}_\Omega}\\ &= \langle\psi_n(z),\phi_n(z)\rangle_{\mathcal{L}} + \sum_{r\in\Omega}\left(M_r\langle z^r\psi_n(z),\phi_n(z)\rangle_{\mathcal{L}_\theta} + \overline{M}_r\langle\psi_n(z),z^r\phi_n(z)\rangle_{\mathcal{L}_\theta}\right)\\ &= \mathbf{k}_n + \sum_{r\in\Omega}\left(M_r\int_{\mathbb{T}}\sum_{l=0}^{n}\frac{\psi_n^{(l)}(0)}{l!}z^{l+r}\sum_{l=0}^{n}\frac{\overline{\phi_n^{(l)}(0)}}{l!}z^{-l}\frac{dz}{2\pi iz}\right)\\ &\quad + \sum_{r\in\Omega}\left(\overline{M}_r\int_{\mathbb{T}}\sum_{l=0}^{n}\frac{\psi_n^{(l)}(0)}{l!}z^{l-r}\sum_{l=0}^{n}\frac{\overline{\phi_n^{(l)}(0)}}{l!}z^{-l}\frac{dz}{2\pi iz}\right)\\ &= \mathbf{k}_n + \sum_{r\in\Omega}\Psi_n^T(0)\mathbf{Y}_n^r(0) + \sum_{r\in\Omega}\overline{M}_r\frac{\overline{\phi_n^{(n-r)}(0)}}{(n-r)!},\end{aligned}$$

which is different from 0 by assumption. Therefore, $\mathcal{L}_\Omega$ is quasi-definite. ☑

Notice that if $r_m = \min\{r : r \in \Omega\}$, then from Proposition 2.2 we conclude that if $n < r_m$ then $\mathbb{K}_{n-1}^{r_m}$ is the zero vector, $K_{n-1}^{(0,n-r_m)}(z,0) = 0$ and according to (22) we have $\psi_n(z) = \phi_n(z)$. This means that the only affected polynomials are those with degree $n \geqslant r_m$.

3. Finite moments perturbation through the inverse Szegő transformation

Let σ be a positive measure supported on the unit circle such that its corresponding moments $\{c_n\}_{n\in\mathbb{Z}}$ are real. Assume also that the perturbed measure $\tilde{\sigma}_\Omega$, defined by (17), is also positive and that M_r with $r \in \Omega$ is real, so that the moments associated with $\tilde{\sigma}_\Omega$ are also real. Our goal in this section is to determine the relation between the positive Borel measures α and $\tilde{\alpha}_\Omega$, supported in $[-1, 1]$, which are associated with σ and $\tilde{\sigma}_\Omega$, respectively, via the inverse Szegő transformation. This relation will be stated in terms of the corresponding measure and their sequences of moments.

Proposition 3.1. *Let σ be a positive nontrivial Borel measure with real moments supported in the unit circle, and let α be its corresponding measure supported in $[-1, 1]$, obtained through the inverse Szegő transformation. Let $\{c_n\}_{n\in\mathbb{Z}}$ and $\{\mu_n\}_{n\geq 0}$ be their corresponding sequences of moments. Assume that $\tilde{\sigma}_\Omega$, defined by (17) with $r \in \Omega$ and $M_r \in \mathbb{R}$, is positive. Then, the measure $\tilde{\alpha}_\Omega$, obtained by applying the inverse Szegő transformation to $\tilde{\sigma}_\Omega$, is given by*

$$d\tilde{\alpha}_\Omega = d\alpha + \frac{2}{\pi}\sum_{r\in\Omega} M_r T_r(x)\frac{dx}{\sqrt{1-x^2}}, \tag{29}$$

where $T_r(x) := \cos(r\theta)$ is the r-th degree Chebyshev polynomial of the first kind. Its corresponding sequence of moments is

$$\tilde{\mu}_n = \begin{cases} \mu_n, & \text{if } \quad 0 \leq n < r_m, \\ \mu_n + \frac{2}{\pi}\sum_{r\in\Omega} M_r B(n,r), & \text{if } \quad r_m \leq n, \end{cases} \tag{30}$$

with

- $B(n,r) = \frac{\pi r}{2}\sum_{k=0}^{[r/2]}\left(\frac{(-1)^k (r-k-1)!(2)^{r-2k}}{k!(r-2k)!}\right)$, *if* $\quad r+n-2k = 0$,
- $B(n,r) = 0$, *if $r+n$ is odd,*
- $B(n,r) = \frac{\pi r}{2}\sum_{k=0}^{[r/2]}\left(\frac{(-1)^k (r-k-1)!(2)^{r-2k}}{k!(r-2k)!}\prod_{i=1}^{(r+n-2k)/2}\frac{r+n-2k-(2i-1)}{r+n-2k-2(i-1)}\right)$, *if $r+n$ is even.*

Proof. Notice that, setting $z = e^{i\theta}$, $x = \cos\theta$, and taking into account that the inverse Szegő transformation applied to the normalized Lebesgue measure $d\theta/2\pi$ yields the Chebyshev measure of the first kind $\frac{dx}{\pi\sqrt{1-x^2}}$, the measure $\tilde{\alpha}_\Omega$ obtained by applying the inverse Szegő transformation to $\tilde{\sigma}_\Omega$ is given by

$$\begin{aligned} d\tilde{\alpha}_\Omega &= d\alpha + \sum_{r\in\Omega}\left(M_r(x+i\sqrt{1-x^2})^r + M_r(x+i\sqrt{1-x^2})^{-r}\right)\frac{dx}{\pi\sqrt{1-x^2}} \\ &= d\alpha + \sum_{r\in\Omega}\left(M_r(\cos r\theta + i\sin r\theta) + M_r(\cos r\theta - i\sin r\theta)\right)\frac{dx}{\pi\sqrt{1-x^2}} \\ &= d\alpha + \frac{2}{\pi}\sum_{r\in\Omega} M_r\frac{T_r(x)dx}{\sqrt{1-x^2}}. \end{aligned}$$

Notice that a measure that changes its sign in the interval $[-1,1]$ is added to $d\alpha$. Then, the moments associated with $\tilde{\alpha}_\Omega$ are given by

$$\tilde{\mu}_n = \int_{-1}^{1} x^n d\tilde{\alpha}_\Omega(x) = \mu_n + \frac{2}{\pi}\sum_{r\in\Omega} M_r\int_{-1}^{1} x^n\frac{T_r(x)dx}{\sqrt{1-x^2}}.$$

As a consequence, by the orthogonality of $T_r(x)$, we obtain for the n-th moments with $n\notin\Omega$

$$\tilde{\mu}_n = \begin{cases} \mu_n, & \text{if} \quad 0\le n < r_m, \\ \mu_n + \frac{2}{\pi}\sum_{r\in\Omega} M_r\int_{-1}^{1} x^n\frac{T_r(x)dx}{\sqrt{1-x^2}}, & \text{if} \quad r_m \le n. \end{cases} \tag{31}$$

Furthermore (see [8]), we have

$$T_r(x) = \frac{r}{2}\sum_{k=0}^{[r/2]}\frac{(-1)^k(r-k-1)!(2x)^{r-2k}}{k!(r-2k)!}, \quad r=1,2,3,...,$$

where $[r/2]=r/2$ if r is even and $[r/2]=(r-1)/2$ if r is odd. Therefore,

$$\begin{aligned} \int_{-1}^{1} x^n T_r(x)\frac{dx}{\sqrt{1-x^2}} &= \frac{r}{2}\int_{-1}^{1} x^n\sum_{k=0}^{[r/2]}\frac{(-1)^k(r-k-1)!(2x)^{r-2k}}{k!(r-2k)!}\frac{dx}{\sqrt{1-x^2}} \\ &= \frac{r}{2}\sum_{k=0}^{[r/2]}\frac{(-1)^k(r-k-1)!(2)^{r-2k}}{k!(r-2k)!}\int_{-1}^{1} x^{r+n-2k}\frac{dx}{\sqrt{1-x^2}}, \end{aligned}$$

and, since

$$\int_{-1}^{1} x^k\frac{dx}{\sqrt{1-x^2}} = \begin{cases} \pi, & \text{if} \quad k=0, \\ 0, & \text{if} \quad \text{k is odd}, \\ \left(\prod_{i=1}^{k/2}\frac{k-(2i-1)}{k-2(i-1)}\right)\pi, & \text{if} \quad \text{k is even}, \end{cases}$$

(31) becomes (30). ☑

From the previous proposition we can conclude that a perturbation of the moments c_r and c_{-r} with $r\in\Omega$, associated with a measure σ supported in the unit circle, results in a perturbation, defined by (30), of the moments μ_n, $n\geqslant r_m$, associated with a measure α supported in $[-1,1]$, when both measures are related through the inverse Szegő transformation.

4. Example

Let $\mathcal{L}$ be the Christoffel transformation of the normalized Lebesgue measure with parameter $\xi = 1$, i.e., $\langle p(z), q(z)\rangle_{\mathcal{L}} = \langle (z-1)p(z), (z-1)q(z)\rangle_{\mathcal{L}_\theta}$, and let $\Omega = \{1, 2\}$. Then,

$$\begin{aligned}\langle p(z), q(z)\rangle_{\mathcal{L}_{\{1,2\}}} = \langle (z-1)p(z), (z-1)q(z)\rangle_{\mathcal{L}_\theta} + M_1\langle zp(z), q(z)\rangle_{\mathcal{L}_\theta} + \overline{M}_1\langle p(z), zq(z)\rangle_{\mathcal{L}_\theta} \\ + M_2\langle z^2p(z), q(z)\rangle_{\mathcal{L}_\theta} + \overline{M}_2\langle p(z), z^2q(z)\rangle_{\mathcal{L}_\theta},\end{aligned} \tag{32}$$

i.e., the moments of order 1 and 2 are perturbed. Since the sequence $\{z^n\}_{n\geq 0}$ is orthogonal with respect to $\mathcal{L}_\theta$, the **MOPS** associated with $\langle (z-1)p(z), (z-1)q(z)\rangle_{\mathcal{L}_\theta}$ is given by (see [1])

$$\phi_n(z) = \frac{1}{z-1}\left(z^{n+1} - \frac{1}{n+1}\sum_{j=0}^{n} z^j\right), \quad n \geq 1,$$

or, equivalently,

$$\phi_n(z) = z^n + \frac{n}{n+1}\phi_{n-1}(z), \quad n \geq 1,$$

and its corresponding reversed polynomial is

$$\phi_n^*(z) = \frac{1}{1-z}\left(1 - \frac{1}{n+1}\sum_{j=0}^{n} z^{j+1}\right), \quad n \geq 1.$$

Furthermore, if $0 \leq s \leq n$, we have

$$\phi_n^{(s)}(0) = \frac{(s+1)!}{(n+1)},$$

$$\phi_n^{*(s)}(0) = \frac{s!(n+1-s)}{(n+1)},$$

$$\mathbf{k}_n = \frac{n+2}{n+1},$$

and if $0 \leq t, s \leq n-1$,

$$K_{n-1}^{(0,t)}(z,0) = \sum_{p=t}^{n-1} \frac{(t+1)!}{p+2}\phi_p(z),$$

$$K_{n-1}^{(s,t)}(0,0) = \sum_{p=\max\{s,t\}}^{n-1} \frac{(t+1)!(s+1)!}{(p+1)(p+2)}.$$

As a consequence, we have

$$a_{(s,l)_r} = \frac{(s+1)!(l+r+1)}{l!} \sum_{p=\max\{s,l+r\}}^{n-1} \frac{1}{(p+1)(p+2)},$$

$$c_{(s,l)_r} = \frac{(s+1)!(l-r+1)}{l!} \sum_{p=\max\{s,l-r\}}^{n-1} \frac{1}{(p+1)(p+2)},$$

$$b_{(s,l)_r} = \frac{(s+1)!}{l!} \left(M_r \sum_{p=\max\{s,l+r\}}^{n-1} \frac{l+r+1}{(p+1)(p+2)} + \overline{M}_r \sum_{p=\max\{s,l-r\}}^{n-1} \frac{l-r+1}{(p+1)(p+2)} \right).$$

We now proceed to obtain the **MOPS** associated with $\mathcal{L}_{\{1,2\}}$, denoted by $\{\psi_n\}_{n\geq 0}$. Notice that we have $c_{(s,n)_1} = \frac{(s+1)!}{n!(n+1)}$, $c_{(s,n)_2} = \frac{2(s+1)!}{n!(n+1)}$ if $0 \leq s \leq n-2$ and $c_{(n-1,n)_2} = \frac{n-1}{n(n+1)}$, and thus

$$\begin{aligned}
\mathbf{W}_n &= \begin{bmatrix} \phi_n^{(0)}(0) \\ \phi_n^{(1)}(0) \\ \vdots \\ \phi_n^{(n-2)}(0) \\ \phi_n^{(n-1)}(0) \end{bmatrix} - \overline{M}_1 n! \begin{bmatrix} c_{(0,n)_1} \\ c_{(1,n)_1} \\ \vdots \\ c_{(n-2,n)_1} \\ c_{(n-1,n)_1} \end{bmatrix} - \overline{M}_2 n! \begin{bmatrix} c_{(0,n)_2} \\ c_{(1,n)_2} \\ \vdots \\ c_{(n-2,n)_2} \\ c_{(n-1,n)_2} \end{bmatrix} \\
&= \frac{1-\overline{M}_1}{n+1} \begin{bmatrix} 1! \\ 2! \\ \vdots \\ (n-1)! \\ n! \end{bmatrix} - \frac{2\overline{M}_2}{n+1} \begin{bmatrix} 1! \\ 2! \\ \vdots \\ (n-1)! \\ n! \frac{n-1}{2n} \end{bmatrix}.
\end{aligned}$$

On the other hand, for $n \geq 2$ we have

$$\mathbb{K}_{n-1}^1(z,0) = \begin{bmatrix} 2M_1 \sum_{p=1}^{n-1} \frac{\phi_p(z)}{p+2} \\ 3M_1 \sum_{p=2}^{n-1} \frac{\phi_p(z)}{p+2} + \overline{M}_1 \sum_{p=0}^{n-1} \frac{\phi_p(z)}{p+2} \\ \vdots \\ \frac{nM_1}{(n-2)!} \sum_{p=n-1}^{n-1} \frac{\phi_p(z)}{p+2} + \frac{\overline{M_1}}{(n-3)!} \sum_{p=n-3}^{n-1} \frac{\phi_p(z)}{p+2} \\ \frac{\overline{M_1}}{(n-2)!} \sum_{p=n-2}^{n-1} \frac{\phi_p(z)}{p+2} \end{bmatrix},$$

$$\mathbf{S}_n^1 = \begin{bmatrix} M_1 a_{(0,0)_1} & b_{(0,1)_1} & \cdots & b_{(0,n-2)_1} & \overline{M}_1 c_{(0,n-1)_1} \\ M_1 a_{(1,0)_1} & b_{(1,1)_1} & \cdots & b_{(1,n-2)_1} & \overline{M}_1 c_{(1,n-1)_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_1 a_{(n-2,0)_1} & b_{(n-2,1)_1} & \cdots & b_{(n-2,n-2)_1} & \overline{M}_1 c_{(n-2,n-1)_1} \\ M_1 a_{(n-1,0)_1} & b_{(n-1,1)_1} & \cdots & b_{(n-1,n-2)_1} & \overline{M}_1 c_{(n-1,n-1)_1} \end{bmatrix},$$

and for $n \geq 3$,

$$\mathbb{K}_{n-1}^{2}(z,0) = \begin{bmatrix} 3M_2 \sum_{p=2}^{n-1} \frac{\phi_p(z)}{p+2} \\ 4M_2 \sum_{p=3}^{n-1} \frac{\phi_p(z)}{p+2} \\ \frac{5M_2}{2} \sum_{p=4}^{n-1} \frac{\phi_p(z)}{p+2} + \frac{\overline{M}_2}{2} \sum_{p=0}^{n-1} \frac{\phi_p(z)}{p+2} \\ \vdots \\ \frac{nM_2}{(n-3)!} \sum_{p=n-1}^{n-1} \frac{\phi_p(z)}{p+2} + \frac{\overline{M}_2}{(n-3)(n-5)!} \sum_{p=n-5}^{n-1} \frac{\phi_p(z)}{p+2} \\ \frac{\overline{M}_2}{(n-2)(n-4)!} \sum_{p=n-4}^{n-1} \frac{\phi_p(z)}{p+2} \\ \frac{\overline{M}_2}{(n-1)(n-3)!} \sum_{p=n-3}^{n-1} \frac{\phi_p(z)}{p+2} \end{bmatrix},$$

$$\mathbf{S}_n^2 = \begin{bmatrix} M_2 a_{(0,0)_2} & M_2 a_{(0,1)_2} & b_{(0,2)_2} & \cdots & b_{(0,n-3)_2} & \overline{M}_2 c_{(0,n-2)_2} & \overline{M}_2 c_{(0,n-1)_2} \\ M_2 a_{(1,0)_2} & M_2 a_{(1,1)_2} & b_{(1,2)_2} & \cdots & b_{(1,n-3)_2} & \overline{M}_2 c_{(1,n-2)_2} & \overline{M}_2 c_{(1,n-1)_2} \\ M_2 a_{(2,0)_2} & M_2 a_{(2,1)_2} & b_{(2,2)_2} & \cdots & b_{(2,n-3)_2} & \overline{M}_2 c_{(2,n-2)_2} & \overline{M}_2 c_{(2,n-1)_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ M_2 a_{(n-3,0)_2} & M_2 a_{(n-3,1)_2} & b_{(n-3,2)_2} & \cdots & b_{(n-3,n-3)_2} & \overline{M}_2 c_{(n-3,n-2)_2} & \overline{M}_2 c_{(n-3,n-1)_2} \\ M_2 a_{(n-2,0)_2} & M_2 a_{(n-2,1)_2} & b_{(n-2,2)_2} & \cdots & b_{(n-2,n-3)_2} & \overline{M}_2 c_{(n-2,n-2)_2} & \overline{M}_2 c_{(n-2,n-1)_2} \\ M_2 a_{(n-1,0)_2} & M_2 a_{(n-1,1)_2} & b_{(n-1,2)_2} & \cdots & b_{(n-1,n-3)_2} & \overline{M}_2 c_{(n-1,n-2)_2} & \overline{M}_2 c_{(n-1,n-1)_2} \end{bmatrix},$$

For illustrative purposes, we compute the first polynomials of the sequence:

- Degree one:

$$\begin{aligned} \psi_1(z) &= \phi_1(z) - \frac{\overline{M}_1 K_0^{(0,0)}(z,0)}{0!} \\ &= z + \frac{(1-\overline{M}_1)}{2}\phi_0(z), \end{aligned}$$

  since $\mathbb{K}_0^1(z,0) = 0$, $\mathbb{K}_0^2(z,0) = 0$ and $\phi_1(z) = z + \frac{1}{2}\phi_0(z)$.

- Degree two:

$$\psi_2(z)=\phi_2(z)$$
$$-\left(\frac{1-\overline{M}_1}{3}\begin{bmatrix}1!\\2!\end{bmatrix}^T-\frac{2\overline{M}_2}{3}\begin{bmatrix}1!\\\frac{1}{2}\end{bmatrix}^T\right)\left(\begin{bmatrix}\frac{M_1}{3}+1 & \frac{2\overline{M}_1}{3}\\ \frac{2M_1}{3} & \frac{\overline{M}_1}{3}+1\end{bmatrix}^{-1}\right)^T$$
$$\begin{bmatrix}\frac{2M_1}{3}\phi_1(z)\\ \overline{M}_1\left(\frac{\phi_0(z)}{2}+\frac{\phi_1(z)}{3}\right)\end{bmatrix}-\frac{2\overline{M}_1}{3}\phi_1(z)-\overline{M}_2\left(\frac{\phi_0(z)}{2}+\frac{\phi_1(z)}{3}\right)$$
$$=z^2+\frac{1-\overline{M}_1}{3}\left(2\phi_1(z)-A\begin{bmatrix}1!\\2!\end{bmatrix}^T\begin{bmatrix}\frac{\overline{M}_1}{3}+1 & -\frac{2M_1}{3}\\ -\frac{2\overline{M}_1}{3} & \frac{M_1}{3}+1\end{bmatrix}\begin{bmatrix}\frac{2M_1}{3}\phi_1(z)\\ \overline{M}_1\left(\frac{\phi_0(z)}{2}+\frac{\phi_1(z)}{3}\right)\end{bmatrix}\right)$$
$$-\overline{M}_2\left(\frac{\phi_0(z)}{2}+\frac{\phi_1(z)}{3}-\frac{2}{3}A\begin{bmatrix}1!\\\frac{1}{2}\end{bmatrix}^T\begin{bmatrix}\frac{\overline{M}_1}{3}+1 & -\frac{2M_1}{3}\\ -\frac{2\overline{M}_1}{3} & \frac{M_1}{3}+1\end{bmatrix}\begin{bmatrix}\frac{2M_1}{3}\phi_1(z)\\ \overline{M}_1\left(\frac{\phi_0(z)}{2}+\frac{\phi_1(z)}{3}\right)\end{bmatrix}\right),$$

since $\mathbb{K}_1^2(z,0)=0$, $\phi_2(z)=z^2+\frac{2}{3}\phi_1(z)$ and $A=\frac{1}{\det(\mathbf{I}_2+\mathbf{S}_2^1+\mathbf{S}_2^2)}=\frac{9}{|M_1+3|^2-4|M_1|^2}$.

- In general, the n-th degree polynomial is

$$\psi_n(z)=\phi_n(z)-\left(\frac{1-\overline{M}_1}{n+1}\begin{bmatrix}1!\\2!\\\vdots\\(n-1)!\\n!\end{bmatrix}^T-\frac{2\overline{M}_2}{n+1}\begin{bmatrix}1!\\2!\\\vdots\\(n-1)!\\n!(n-1)/2n\end{bmatrix}^T\right)\left[(\mathbf{I}_n+\sum_{r=1}^{2}\mathbf{S}_n^r)^{-1}\right]^T$$
$$\times\left(\sum_{r=1}^{2}\mathbb{K}_{n-1}^r(z,0)\right)-\frac{n\overline{M}_1}{n+1}\phi_{n-1}(z)-\overline{M}_2(n-1)\left(\frac{\phi_{n-2}(z)}{n}+\frac{\phi_{n-1}(z)}{n+1}\right)$$
$$=z^n+\frac{1-\overline{M}_1}{n+1}\left(n\phi_{n-1}(z)-\begin{bmatrix}1!\\2!\\\vdots\\(n-1)!\\n!\end{bmatrix}^T\left[(\mathbf{I}_n+\sum_{r=1}^{2}\mathbf{S}_n^r)^{-1}\right]^T\left(\sum_{r=1}^{2}\mathbb{K}_{n-1}^r(z,0)\right)\right)$$
$$-\overline{M}_2(n-1)\left(\frac{\phi_{n-1}(z)}{n+1}+\frac{\phi_{n-2}(z)}{n}\right)$$
$$+\frac{2\overline{M}_2}{n+1}\begin{bmatrix}1!\\2!\\\vdots\\(n-1)!\\n!\frac{n-1}{2n}\end{bmatrix}^T\left[(\mathbf{I}_n+\sum_{r=1}^{2}\mathbf{S}_n^r)^{-1}\right]^T\left(\sum_{r=1}^{2}\mathbb{K}_{n-1}^r(z,0)\right),$$

since $\phi_n(z)=z^n+\frac{n}{n+1}\phi_{n-1}(z)$.

On the other hand, assuming that the linear functional (32) is positive definite, the associated measure is

$$d\sigma=|z-1|^2\frac{d\theta}{2\pi}+2\mathfrak{Re}(M_1z)\frac{d\theta}{2\pi}+2\mathfrak{Re}(M_2z^2)\frac{d\theta}{2\pi},$$

and the corresponding moments are given by

$$\tilde{c}_n = \begin{cases} 2, & \text{if} \quad n = 0, \\ -1 + \overline{M}_1, & \text{if} \quad n = 1, \\ -1 + M_1, & \text{if} \quad n = -1, \\ \overline{M}_2, & \text{if} \quad n = 2, \\ M_2, & \text{if} \quad n = -2, \\ 0, & \text{in other case.} \end{cases}$$

Thus, the perturbed Toeplitz matrix is

$$\widetilde{\mathbf{T}} = \begin{bmatrix} 2 & -1+\overline{M}_1 & \overline{M}_2 & 0 & 0 & \cdots \\ -1+M_1 & 2 & -1+\overline{M}_1 & \overline{M}_2 & 0 & \cdots \\ M_2 & -1+M_1 & 2 & -1+\overline{M}_1 & \overline{M}_2 & \cdots \\ 0 & M_2 & -1+M_1 & 2 & -1+\overline{M}_1 & \cdots \\ 0 & 0 & M_2 & -1+M_1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

i.e., the first and second subdiagonals are perturbed. Furthermore, since $d\alpha = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}dx$ is the measure obtained applying the inverse Szegő transformation to $d\sigma = |z-1|^2\frac{d\theta}{2\pi}$, then according to (29) the measure supported in $[-1,1]$ is

$$d\tilde{\alpha} = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}dx + \frac{2}{\pi}M_1T_1(x)\frac{1}{\sqrt{1-x^2}}dx + \frac{2}{\pi}M_2T_2(x)\frac{1}{\sqrt{1-x^2}}dx. \tag{33}$$

Then, according to (30), the perturbed moments associated with the measure (33) are

$$\tilde{\mu}_n = \begin{cases} \mu_n, & \text{if} \quad n = 0, \\ \mu_n + 2M_1\prod_{i=1}^{\frac{n+1}{2}}\frac{n+1-(2i-1)}{n+1-2(i-1)}, & \text{if} \quad n \text{ is odd}, \\ \mu_n + 2M_2\left(2\prod_{i=1}^{\frac{n+2}{2}}\frac{n+2-(2i-1)}{n+2-2(i-1)} - \prod_{i=1}^{\frac{n}{2}}\frac{n-(2i-1)}{n-2(i-1)}\right), & \text{if} \quad n \text{ is even}, \end{cases}$$

where $\{\mu_n\}_{n\geq 0}$ are the moments associated with the measure $d\alpha = \frac{2}{\pi}\sqrt{\frac{1-x}{1+x}}$, and

$$\mu_n = \begin{cases} 2, & \text{if} \quad n = 0, \\ -2\prod_{i=1}^{\frac{n+1}{2}}\frac{n+1-(2i-1)}{n+1-2(i-1)}, & \text{if} \quad n \text{ is odd}, \\ 2\prod_{i=1}^{\frac{n}{2}}\frac{n-(2i-1)}{n-2(i-1)}, & \text{if} \quad n \text{ is even}. \end{cases}$$

Acknowledgements

The work of the second author was supported by Consejo Nacional de Ciencia y Tecnología of México, grant 156668. Both authors thank the anonymous referees for her/his valuable comments and suggestions. They contributed greatly to improve the presentation and contents of the manuscript.

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